Nonlinear Eigenvalue Problems; p-Laplace

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VIII Partial differential equations, optimal design and numerics
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Outline

PROBLEM SETTING

Let $\Omega \subset \mathbb{R}^2$ be a connected, bounded and open domain. The eigenvalue problem for Laplace Operator is

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The eigenvalues of the self adjoint, positive operator $-\Delta$ in Ω are denoted by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \cdots$$

The first eigenvalue has variational form. For any open $D \subset \Omega$, the first eigenvalue $\lambda_1(D)$ given by

$$\lambda_1(D) = \min_{\substack{u \in H_0^1(D) \\ u \neq 0}} \frac{\int_D |\nabla u(x)|^2 dx}{\int_D |u(x)|^2 dx}.$$

Problem A

- Given a bounded open set $\Omega \subset \mathbb{R}^2$, a partition of Ω is a family of disjoint, open and connected subsets $\{\Omega_i\}_{i=1}^n$ such that

$$\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n \subseteq \Omega, \qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j.$$

- By $\mathfrak{D}_n$ we mean the set of all n-partition of Ω.
- We are looking for a partition which minimize

$$I(\Omega_1, \cdots, \Omega_n) = \frac{1}{n}\sum_{i=1}^{n} \lambda_1(\Omega_i), \tag{1.2}$$

among all possible partitions.
- Such partition is called optimal partition.

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Problem B

- **Problem B:** For a any arbitrary partition $\mathfrak{D} = (\Omega_1, ..., \Omega_n) \in \mathfrak{D}_n$, we define

$$\Lambda(\mathfrak{D}) = \max_i \lambda_1(\Omega_i), \quad i = 1, \cdots, n.$$

- Define $\mathfrak{L}_\mathfrak{n}(\Omega)$ as follows:

$$\mathfrak{L}_\mathfrak{n}(\Omega) = \inf_{\mathfrak{D} \in \mathfrak{D}_\mathfrak{n}} \Lambda(\mathfrak{D})$$

- Known fact: $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$. This means if (Ω_1^*, Ω_2^*) be an optimal bi-partition then

$$\lambda_2(\Omega) = \lambda_1(\Omega_1^*) = \lambda_1(\Omega_2^*).$$

Conjecture by Caffarelli and Lin

For problem A, when n tends to infinity then for the optimal partition $\{\Omega^*\}_{i=1}^n$

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_1(\Omega_i^*) \simeq n\frac{\lambda_1(H)}{|\Omega|},$$

where H is the regular hexagon of area 1 in $\mathbb{R}^2$. Far from the boundary a tiling by regular hexagons of area $\frac{|\Omega|}{n}$ is asymptotically close to the optimal partition.
For problem B the conjecture is

$$\lim_{n\to\infty}\frac{\mathfrak{L}_n(\Omega)}{n} = \frac{\lambda_1(H)}{|\Omega|}.$$

Some references for optimal partition problem

- D. Bucur, G. Buttazzo, and A. Henrot, Existence results for some optimal partition problems. Adv. Math. Sci. Appl. 8 (1998), no. 2, 571–579.
- L. A. Caffarelli, F.-H. Lin, An optimal partition problem for eigenvalues. J. Sci. Comput. 31 (2007), no. 1-2, 5–18.
- B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini. *Nodal domains and spectral minimal partitions.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 26(1), (2009) 101–138.
- B. Helffer, *On spectral minimal partitions: a survey.* Milan J. Math. 78 (2010), no. 2, 575–590.
- F. Bozorgnia, *Optimal partitions for first eigenvalues of the Laplace operator.* NMPDE, 31 (2015) 923–949.
- B. Helffer and T. Hoffmann-Ostenhof. *A review on large k minimal spectral k-partitions and Pleijel's theorem.* In Spectral theory and partial differential equations, volume 640 of Contemp. Math., pages 39–57. Amer. Math. Soc., Providence, RI, 2015.
- B. Bogosel, D. Bucur, and I. Fragalà,*Phase Field Approach to Optimal Packing Problems and Related Cheeger Clusters.* Appl Math Optim (2018), 1–25.
- B. Bogosel, *Efficient algorithm for optimizing spectral partitions.* Applied Mathematics and Computation 333, (2018), 61–75.

Mathematical Background

Problems (A) can be written as minimization of

$$\sum_{i=1}^{n} \frac{\int_\Omega |\nabla u_i(x)|^2 dx}{\int_\Omega |u_i(x)|^2 dx},$$

Over the class of

$$\{(u_1, \ldots, u_n) : u_i \in H_0^1(\Omega), u_i(x) \cdot u_j(x) = 0, x \in \Omega, i \neq j\}.$$

- The functional is weakly lower semi-continuous
- The constraint is locally weakly compact
- Existence follows from direct methods in calculus of variation.
- Letting $\Omega_i = \{x \in \Omega : u_i(x) > 0\}$ we find a solution for Problem (A).

PROPERTIES OF OPTIMAL PARTITIONS

Theorem

There exists $(\Omega_1, \ldots, \Omega_n)$ minimizing the given functional in Problem (A). Furthermore, if $\phi_1, \cdots, \phi_n$ are corresponding eigenfunctions normalized in L_2, then, there exist $a_i \in \mathbb{R}$ such that the functions $u_i = a_i \phi_i$ verify in Ω the differential inequalities (in distributional sense)

- $-\Delta u_i \leq \lambda_1(\Omega_i) u_i, \quad$ *a.e, in* Ω,
- $-\Delta(u_i - \sum_{j \neq i} u_j) \geq \lambda_1(\Omega_i) u_i - \sum_{j \neq i} \lambda_1(\Omega_j) u_j$.

Here $\Omega_i = \{x \in \Omega : u_i(x) > 0\}$.

Note that the same theorem is true for problem (B) where $\lambda_1(\Omega_i), i = 1, \cdots, n$ is replaced by $\mathfrak{L}_{\mathfrak{n}}$.

L. A. Caffarelli, F.-H. Lin, An optimal partition problem for eigenvalues. J. Sci. Comput. 31 (2007), no. 1-2, pp. 5–18.

M. Conti, S. Terracini, and G. Verzini, An optimal partition problem related to nonlinear eigenvalues. J. of Funct. Anal. 198 (2003), no.1, pp. 160–196.

General case and extension

Let

$$\mathfrak{L}_{\mathfrak{n},\mathfrak{q}}(\Omega) = \inf_{\mathfrak{D}\in\mathfrak{D}_{\mathfrak{n}}} \Big(\frac{1}{n}\sum_{i=1}^{n} \lambda_1(\Omega_i)^q\Big)^{\frac{1}{q}},$$

- $q = 1$; $\mathfrak{L}_{\mathfrak{n},1}$: Problem (A),
- $q = \infty$; $\mathfrak{L}_{\mathfrak{n},\infty} = \mathfrak{L}_{\mathfrak{n}}$: Problem (B).

Extension to other operators:

- p-Laplace operator :

$$\inf_{\mathfrak{D}\in\mathfrak{D}_{\mathfrak{n}}} \frac{1}{n}\sum_{i=1}^{n} \lambda_1(p;\Omega_i),$$

- $p = 1$: Honeycomb conjecture
- $p = \infty$: Spherical packing problem
- Schrödinger operator $H = -\Delta + V$.

Eigenvalues of p-Laplace operator

- For $1 < p < \infty$, the first eigenvalue of the p-Laplace operator is given by

$$\lambda_1(p;\Omega) = \inf_{\substack{u\in W_0^{1,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} = \inf_{\substack{u\in W_0^{1,p}(\Omega)\\ u\neq 0}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}.$$

- The corresponding Euler-Lagrange equation is given by

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ which for $p = 2$, we have Laplace operator.

J. Benedikt, P. Girg, L. Kotrla, and P. Takáč, *Origin of the p-Laplacian and A. Missbach.* Electronic Journal of Differential Equations, 16, (2018), 1-17.

P. Lindqvist, Notes on the p-Laplace equation. Lecture notes. https://folk.ntnu.no/lqvist/p-laplace.pdf.

EIGENVALUES OF p-LAPLACE OPERATOR

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Higher Eigenvalues

First define Krasnoselskii genus of a set $A \subseteq W_0^{1,p}(\Omega)$ by

$$\gamma(A) = \min\{k \in \mathbb{N} | \exists f : A \to \mathbb{R}^k \setminus 0, f \text{ continuous and odd}\}.$$

For $k \in \mathbb{N}$ define

$$\Gamma_k := \{A \subseteq W_0^{1,p}(\Omega), \text{symmetric, compact and } \gamma(A) \geq k\}.$$

Then the eigenvalues of the p-Laplace are

$$\lambda_{k,p}(\Omega) = \min_{A \in \Gamma_k} \sup_{u \in A} \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p dx}. \tag{2.1}$$

Inverse power Algorithm for first eigenvalue

- Initials : $u_0, \lambda_0, \varepsilon$.
- Step k : Given $u^k \geq 0$ and $u^k = 0$ on $\partial\Omega$, scale by

$$\tilde{u}_k = \frac{u_k}{\|u_k\|_{L_p}}$$

set $\lambda_k = \int_\Omega |\nabla \tilde{u}^k(x)|^p dx$, then solve :

$$\begin{cases} -\Delta_p u = \lambda_k \tilde{u}_k^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- Set $\tilde{u}_{k+1} = \frac{u}{\|u\|_{L^p}}$ and calculate
  $\lambda_{k+1} = \int_\Omega |\nabla \tilde{u}_{k+1}(x)|^p dx$.
  if $|\lambda_{k+1} - \lambda_k| > \varepsilon$ **then**
  Set $k = k + 1$ and go to previous step;
  end
- Result: u_k, λ_k

Modification of algorithm

Let $u_0 \in L^p(\Omega)$ be as the first step in Algorithm 1 and define the sequence $\{\tilde{u}_k\}_{k=1}^{\infty}$ inductively according to

$$\tilde{u}_k = \frac{u_k}{\|u_k\|_{L^p(\Omega)}},$$

where u_k is the solution to

$$\begin{cases} -\Delta_p u_k = \tilde{u}_{k-1}^{p-1} & \text{in } \Omega, \\ \tilde{u}_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Note that in the equation (2.2) if we rewrite it in term of u_k then we have

$$\lambda_{k-1} = \frac{1}{\|\tilde{u}_{k-1}\|_{L^p(\Omega)}^{p-1}}.$$

References Inverse power method

- R. Biezuner, G. Ercole, E. Martins, Computing the first eigenvalue of the p-Laplacian via the inverse power method. Journal of Functional Analysis, (2009).
- M. Caliari, S. Zuccher, The Inverse Power Method for the p(x)-Laplacian Problem, J Sci Comput (2015).
- F. Bozorgnia, Convergence of Inverse Power Method for First Eigenvalue of p-Laplace Operator. Numerical Functional Analysis and Optimization (2016).
- R. Hynd, E. Lindgren, Approximation of the least Rayleigh quotient for degree p homogeneous functionals. Journal of Functional Analysis (2017).

Convergence

Lemma

Let λ_k and u_k be as above. Then

- $\lambda_k \leq \lambda_{k-1}$ *for every* $k \geq 1$.
- $\lim_{k\to\infty} \tilde{u}_k = u$ *where* u *is the first eigenfunction.*

Proof:

- Multiply the equation by u_k and integrate

$$\int_\Omega u_k \operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right)\,dx = \lambda_{k-1}\int_\Omega u_k \tilde{u}_{k-1}^{p-1}\,dx.$$

- Next

$$\int_\Omega |\nabla u_k|^p\,dx \leq \lambda_{k-1}\|u_k\|_{L^p(\Omega)}\|\tilde{u}_{k-1}\|^{p-1}_{L^p(\Omega)},$$

Notice that by definition $\|\tilde{u}_{k-1}\|_{L^p(\Omega)} = 1$ so

$$\|\nabla u_k\|^p_{L^p} \leq \lambda_{k-1}\|u_k\|_{L^p}. \tag{2.3}$$

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Notice that by definition $\|\tilde{u}_{k-1}\|_{L^p(\Omega)} = 1$ so

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Multiply the equation by $\tilde{u}_{k-1}$

$$\lambda_{k-1} = \int_\Omega |\nabla u_k|^{p-2} \nabla \tilde{u}_{k-1} \cdot \nabla u_k \, dx \leq \|\nabla \tilde{u}_{k-1}\|_{L^p} \|\nabla u_k\|_{L^p}^{p-1},$$

Since $\lambda_{k-1} = \|\nabla \tilde{u}_{k-1}\|_{L^p}^p$, we obtain

$$\|\nabla u_k\|_{L^p}^{p-1} \geq \lambda_{k-1}^{\frac{p-1}{p}}. \tag{2.4}$$

Inserting the inequality (2.4) into (2.3) we conclude

$$\|\nabla u_k\|_{L^p} \leq \lambda_{k-1}^{\frac{1}{p}} \|u_k\|_{L^p}.$$

Dividing both sides by $\|u_k\|_{L^p}$

$$\lambda_k \leq \lambda_{k-1}$$

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$$\lambda_k \leq \lambda_{k-1}$$

SECOND EIGENVALUE

- To extend the idea of power inverse for second eigenvalue!
- Note that λ_1 is isolated in the spectrum,

$$\lambda_2 = \inf\{\lambda : \text{ is eigenvalue and } \lambda > \lambda_1\}.$$

- Remind in the case $p = 2$ we have:

$$\mathfrak{L}_2 = \lambda_2 = \inf_{(\Omega_1,\Omega_2)\in\mathfrak{D}_2} \max(\lambda_1(\Omega_1), \lambda_1(\Omega_2))$$

Lemma

There exists $u \in W_0^{1,p}(\Omega)$ such that $(\{u_+ > 0\}, \{u_- > 0\})$ achieves infimum in $\mathfrak{L}_2$. Furthermore,

$$\lambda_1(\{u_+ > 0\}) = \lambda_1(\{u_- > 0\}).$$

F. Della Pietra, N. Gavitone, G. Piscitelli *On the second Dirichlet eigenvalue of some nonlinear anisotropic elliptic operators.* Bulletin des Sciences Mathématiques, 155, (2019), 10–32.

Second Eigenvalue

- Initialization: Set $k = 0$, choose initial $u_+^0 > 0$ and $u_-^0 > 0$ having disjoint supports and vanishing on the boundary, scale $u_\pm^0$ in $L^p(\Omega)$.
- Given $u^k = u_+^k - u_-^k$ where u_+^k and u_-^k are normalized in L^p, with disjoint supports, then obtain λ_+^k and λ_-^k by

$$\lambda_1^k(\Omega_1) = \int_{\Omega_1} |\nabla u_1^k(x)|^2 dx, \quad \lambda_1^k(\Omega_2) = \int_{\Omega_2} |\nabla u_2^k(x)|^2 dx,$$

- Solve

$$\begin{cases} -\Delta_p u = |u^k|^{p-2}\left(\lambda_+^k u_+^k - \lambda_-^k u_-^k\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

- Set u_+^{k+1} and u_-^{k+1} as positive and negative part of the solution of (2.5). Update Ω_+ and Ω_- as the supports of u_+^{k+1} and u_-^{k+1}.
- Stop if for a given tolerance ϵ the following holds:

$$|\lambda_1^{k+1}(\Omega^+) - \lambda_1^k(\Omega^+)| \leq \epsilon.$$

$$|\lambda_1^{k+1}(\Omega^-) - \lambda_1^k(\Omega^-)| \leq \epsilon.$$

- Set $k = k + 1$ and go to second step.

Second Eigenvalue

The main assumption is that domain Ω is symmetric such that
$$\|w_2^+\|_{L^p(\Omega)} = \|w_2^-\|_{L^p(\Omega)}.$$

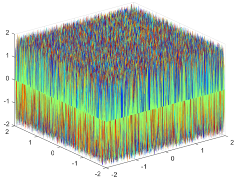
Lemma

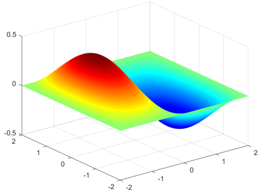
Let $\lambda_+^k(\Omega_+)$ and $\lambda_-^k(\Omega_-)$ be obtained by previous Algorithm. Then
$$\max\left(\lambda_+^k(\Omega_+), \lambda_-^k(\Omega_-)\right) \leq \max\left(\lambda_+^{k-1}(\Omega_+), \lambda_-^{k-1}(\Omega_-)\right),$$
for every $k \geq 1$.

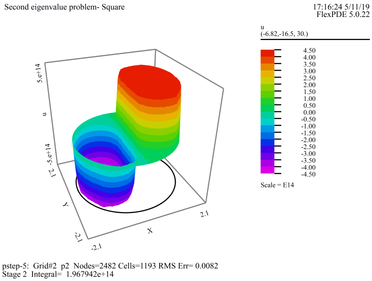
F. Bozorgnia, Approximation of the second eigenvalue of the p-Laplace operator in symmetric domains.
https://arxiv.org/abs/1907.13390

Second Eigenvalue

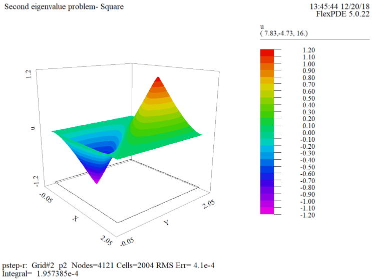
Let $\Omega = [-2,\ 2] \times [-2,\ 2]$. Then $\lambda_2 = 3.084251375340425$, Our approximate : $\lambda_2^{(20)} = 3.081432954134751$.





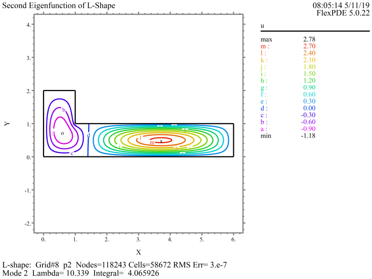

Second eigenvalue problem- Square
17:16:24 5/11/19
FlexPDE 5.0.22
u
(-6.82,-16.5, 30.)
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
-0.50
-1.00
-1.50
-2.00
-2.50
-3.00
-3.50
-4.00
-4.50
Scale = E14
5.e+14
u
-5.e+14
2.1
Y
X
2.1
-2.1
-2.1
pstep-5: Grid#2 p2 Nodes=2482 Cells=1193 RMS Err= 0.0082
Stage 2 Integral= 1.967942e+14

(c) p=1.5

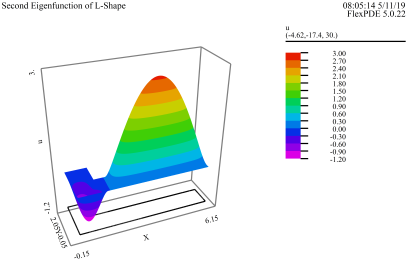

Second eigenvalue problem- Square
13:45:44 12/20/18
FlexPDE 5.0.22
u
(7.83,-4.73, 16.)
1.20
1.10
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
-0.70
-0.80
-0.90
-1.00
-1.10
-1.20
1.2
u
-1.2
-0.05
X
2.05
-0.05
Y
2.05
pstep-r: Grid#2 p2 Nodes=4121 Cells=2004 RMS Err= 4.1e-4
Integral= 1.957385e-4

(d) p=10

Second Eigenvalue



(e)



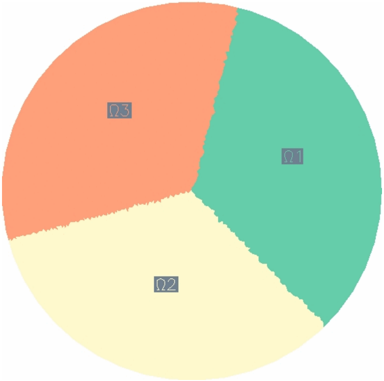
Algorithm 3 for $\mathfrak{L}_3$

We know that:

$$\mathfrak{L}_2 = \lambda_2.$$

Algorithm for the minimal 3-partition will be as follows.

- **Initialization:**
  Let $\mathcal{D}^0 = (\Omega_1^0, \Omega_2^0, \Omega_3^0)$ be a 3-partition of Ω.
- **Step (n):**
  For $n \geq 1$, we define the partition $D^n = (\Omega_1^n, \Omega_2^n, \Omega_3^n)$ by
  $\Omega_1^n = \Omega_3^{n-1}$,
  (Ω_2^n, Ω_3^n) is the nodal partition associated to the second eigenfunction of $-\Delta$ on $\mathrm{Int}(\Omega \setminus \Omega_1^n)$.


Ω3
Ω1
Ω2

Algorithm for n partitions

Given u_m^k with $\|u_m^k\|_{L_2} = 1$ then obtain $\lambda_1^k(\Omega_m)$. We iterate as

For $t = 0, 1, \cdots, k$

For $m = 1, \cdots, n$

For $i = 1, \cdots, n_x$

For $j = 1, \cdots, n_y$

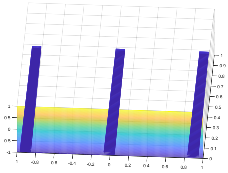
$$u_m^{(t+1)}(x_i, y_j) = \max\Big(\overline{u_m}^{(t)}(x_i, y_j) - \sum_{l \neq m} \overline{u_l}^{(t)}(x_i, y_j) - \lambda_1^k(\Omega_m)\frac{h^2}{4}u_m^{(t)}(x_i, y_j), 0\Big),$$

End

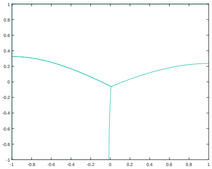
End

End

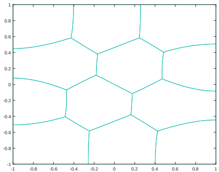
End

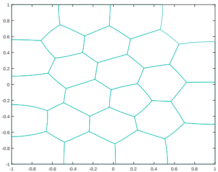


(g) Initial guess for $n = 3$



(h) $n = 3$





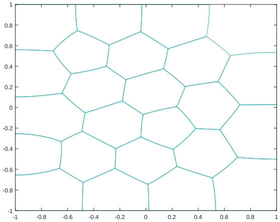


Figure: $n = 24$

Graph notation

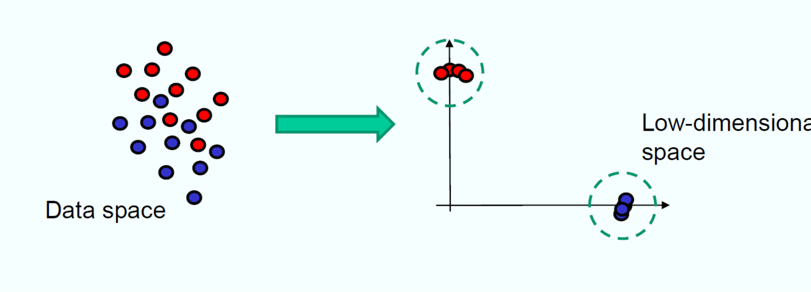
- Let $G = (V, E)$ be an undirected graph with vertex set $V = \{v_1, \cdots, v_n\}$.
- W denotes similarity or weight; each edge between two vertices v_i and v_j carries a non-negative weight $w_{ij} \geq 0$. The weighted adjacency matrix of the graph is the matrix $W = (w_{ij}) \quad i, j = 1, ..., n$.
- G is undirected we require $w_{ij} = w_{ji}$. The degree of a vertex $v_i \in V$ is defined as

$$d_i = \sum_{j \in V} w_{ij}.$$

- The degree matrix D is defined as the diagonal matrix with the degrees $d_1, ..., d_n$ on the diagonal.

Spectral clustering

- Given some data and a notion of similarity
- The task of partitioning the input data into maximally homogeneous groups (i.e. clusters)
- Given data points $v_1, \cdots, v_n$, pairwise affinities w_{ij}
- Find a low-dimensional embedding
- Project data points to new space



Cheeger Cut

- Given graph (V, E) and a subset of vertex $S \subset V$ the Cut(S, S^c) or (the perimeter $|\partial S|$) is defined by

$$\text{Cut}(S, S^c) := \sum_{i \in S, j \in S^c} w_{ij}$$

- Ratio cut and Normalized cut for a partition of V into C, C^c are defined as

$$Rcut(C, C^c) = \frac{cut(C, C^c)}{|C|} + \frac{cut(C, C^c)}{|C^c|}$$

$$NCut(C, C^c) = \frac{cut(C, C^c)}{vol(C)} + \frac{cut(C, C^c)}{vol(C^c)}$$

Note that the minimum is achieved if $|C| = |C^c|$.

Cheeger Cut

- Ratio Cheeger cut:

$$RCC(C, C^c) = \frac{cut(C, C^c)}{\min(|C|, |C^c|)}$$

- key point: The cut obtained by thresholding the second eigenvector of p-Laplace converges to optimal Cheeger cut as p tends to 1.
- Finding optimal ratio Cheeger cut $RCC^* = \min_{C \subset V} RCC$ is NP-hard problem.
- Tight relaxation:(Tomas Bühler, Matthias Hein, 2009)

$$\lambda_2(\Delta_1) = RCC^*$$

Graph p-Laplace

- Let $i \in V$. Depend on the choice of inner product

$$(\Delta_p^u f)_i = \sum_{j \in V} w_{ij}\, \phi_p(f_i - f_j),$$

-

$$(\Delta_p^n f)_i = \frac{1}{d_i} \sum_{j \in V} w_{ij}\, \phi_p(f_i - f_j).$$

- $\phi_p : \mathbb{R} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ as

$$\phi_p(x) = |x|^{p-1} sign(x).$$

- λ_p is an eigenvalue for Δ_p^u if there exists a function $v : V \to \mathbb{R}$ such that

$$(\Delta_p^u v)_i = \lambda_p \phi_p(v_i) \quad i = 1, \cdots n$$

Graph p-Laplace

- The variational characterization define similarly the functional $F_p : \mathbb{R}^V \to \mathbb{R}$

$$F_p(v) = \frac{Q_p(f)}{\|f\|_p^p}$$

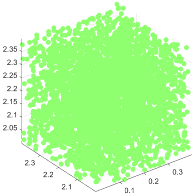
where

- $$Q_p(f) :=< f, \Delta_p^u f >= \frac{1}{2}\sum_{i,j} w_{ij}|f_i - f_j|^p$$

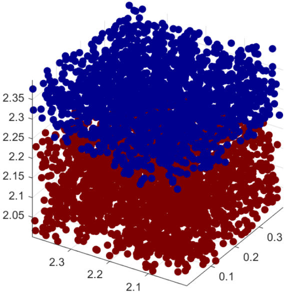
- The functional F_p has a critical point at $v \in R^V$ if and only if v is a p-eigenfunction of Δ_p^u . The corresponding eigenvalue λ_p is given as

$$\lambda_p = F_p(v)$$

Graph clustering



(a)



Thanks for your attention