On the Steklov eigenvalue and higher order generalizations

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Two problems related to the Steklov eigenvalue

- The composite membrane problem - Steklov version
  Joint work with Ki-Ahm Lee and Taehun Lee (Seoul National University)
- Higher order versions of the Steklov eigenvalue
  Joint work with Mariel Sáez (Santiago de Chile)

The Steklov eigenvalue

Let Ω be a smooth domain in $\mathbb{R}^n$ with boundary $\Sigma := \partial\Omega$.
Consider the eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \lambda u & \text{on } \Sigma. \end{cases}$$

Let λ_1 be the first non-zero eigenvalue. Note that

$$\lambda_1 = \inf_{\int_\Sigma u = 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Sigma u^2}$$

Motivation: the composite membrane problem

Build a membrane in Ω with varying densities such that has prescribed mass and frequency as small as possible.

- Chanillo, Grieser, Imai, Kurata, Ohnishi, 2000. Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes.
- Chanillo, Kenig, To, 2008. Regularity of the minimizers in the composite membrane problem in $\mathbb{R}^2$.
- Chanillo, Kenig, 2007. Weak uniqueness and partial regularity for the composite membrane problem.

The composite membrane problem - Steklov version

Build a membrane in Ω with varying density *concentrated at the boundary* $\Sigma = \partial\Omega$, such that has prescribed mass and frequency as small as possible.

More precisely, find:

- A density ρ such that

$$h \leq \rho \leq H, \quad \int_\Sigma \rho = m \quad \text{the total mass},$$

- A function u in $H^1(\Omega)$ with $\int_\Sigma \rho u = 0$,

which realize the realize the double infimum

$$\inf_\rho \inf_u \frac{\int_\Omega |\nabla u|^2}{\int_\Sigma \rho u^2}$$

that corresponds to minimizing the eigenvalue for

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \lambda \rho u & \text{on } \Sigma. \end{cases}$$

The free boundary

The best one can do is ρ taking two only values in Σ:

- h in $D \subset \Sigma$
- H in D^c

We get a *free boundary* between D and D^c.

Thus we set (u, D) unknowns and consider the eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u + \alpha \chi_D u = \lambda u & \text{on } \Sigma \end{cases}$$

where $D \subset \partial\Sigma$ is any measurable subset. Denote by $\lambda(D)$ the first eigenvalue and define

$$\Lambda(a) = \inf_{|D|=a} \lambda(D) \quad \text{optimal configuration}$$

Fractional version

Fix $s \in (0,1)$, consider the eigenvalue problem in $\Sigma \subset \mathbb{R}^n$

$$\begin{cases} (-\Delta)^s u + \alpha \chi_D u = \lambda u & \text{on } \mathbb{R}^n \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Sigma, \end{cases}$$

where $D \subset \Sigma$ is any measurable subset. Denote by $\lambda(D)$ the first eigenvalue and define

$$\Lambda(a) = \inf_{|D|=a} \lambda(D).$$

Theorem (G.-Lee-Lee)

Let (u, D) be the optimal configuration. Then D is a sublevel set of u.
Also, symmetry & symmetry breaking.

Ideas in the proof

- Bathtub lemma: $\{u < t\} \subset D \subset \{u \leq t\}$.
- Difficulty:
  - the regularity of the free boundary ∂D.
  - locally constant does not imply zero "fractional derivative".
- Weiss type monotonicity formula.
- Classification of blow up limits
  At singular points, these satisfy

$$(-\Delta)^s w = c_1 \chi_D - c_2 \chi_{D^c},$$

  and are homogeneous of degree $2s$.

Fourth order generalizations

Fourth order energy:

$$\mathcal{E}[u] = \frac{1}{2}\int_\Omega (\Delta u)^2 + \frac{2}{3}\int_\Sigma H|\tilde{\nabla} u|_h^2 \, d\sigma - \int_\Sigma A_{ij}\tilde{\nabla}^i u \tilde{\nabla}^j u \, d\sigma.$$

Boundary condition: $\partial_\nu u = 0$ on Σ

Third order boundary operator: conformal!!

$$B_3 u = -\partial_\eta \Delta u + 2\langle A_0, \tilde{D}^2 u\rangle - \frac{2}{3} H \tilde{\Delta} u + \frac{2}{3}\langle \tilde{\nabla} H, \tilde{\nabla} u\rangle.$$

Eigenvalue problem:

$$\begin{cases} \Delta^2 u = 0 \text{ in } \Omega, \\ \partial_\nu u = 0 \text{ on } \Sigma, \\ B_3 u = \mu u \text{ on } \Sigma. \end{cases}$$

Rayleigh-quotient

$$\mu_1 = \min_{\partial_\nu u = 0, \int_\Sigma u = 0} \frac{\mathcal{E}[u]}{\int_\Sigma u^2 \, d\sigma}$$

The Steklov eigenvalue in conformal geometry (Fraser-Schoen)

- Now Ω is a 2-surface with boundary $\Sigma = \partial\Omega$. Let λ_1 be the first Steklov eigenvalue for Ω.
- **Alternative formulation:** Let M be a surface with coordinates given by (first) Steklov eigenfunctions. Then M is a minimal surface in the unit ball B which meets ∂B orthogonally.
- **Theorem:** for genus γ with k boundary components, we have

$$\lambda_1|\Sigma| \leq 2(\gamma + k)\pi$$

  - Sharp for $\gamma = 1$, $k = 0$ (Weinstock).
  - Nor sharp for an annulus (explicit calculation).
- **Theorem:** To maximize.

Work in progress

Assume $|\Sigma| = 1$ (normalization).

- For the unit ball B^4,

$$\mu_m = 2m(m+1)(m+2), \quad \text{for} \quad m \geq 1,$$

  with multiplicity 4.
- For the annulus we have an increasing sequence of eigenvalues, value of μ_1 is explicit (numerics).
- In general, consider the embedding given by the Steklov eigenfunctions:

$$\mu_1 |\Sigma| \leq \frac{1}{2} \int_\Omega |\bar{H}|^2 \, dv + \int_\Sigma H \, d\sigma$$

  Here $\bar{H}$ represents the mean curvature of the embedding

Applications

- Liu-Wu. An Energetic Variational Approach for the CahnHilliard Equation with Dynamic Boundary Condition: Model Derivation and Mathematical Analysis
- Du-Liu-Ryham-Wang. A phase field formulation of the Willmore problem

Thank you!!