Feedback stabilization with delay boundary control of some unstable elliptic-parabolic systems

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Introduction

The aim of this work is to present some feedback controller to stabilize some parabolic-elliptic systems in 1-D.

In particular we are going to consider:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha v(x,t), & x \in (0,L),\ t > 0, \\ -v_{xx}(x,t) + \gamma v(x,t) = \beta u(x,t), & x \in (0,L),\ t > 0, \\ u(0,t) = 0,\ \ u(L,t) = 0, & t > 0, \\ v(0,t) = 0,\ \ v(L,t) = 0, & t > 0, \end{cases} \tag{1}$$

where $\alpha, \beta, \lambda \in \mathbb{R}$, $\gamma > 0$.

Stability analysis

Is necessary to know the stability properties of the systems when is not controlled. Easy calculations show us that the eigenvalues of the system are σ_n given by

$$\sigma_n = \frac{\beta\alpha}{\left(\frac{n\pi}{L}\right)^2 + \gamma} - \lambda - \left(\frac{n\pi}{L}\right)^2 \tag{2}$$

From here we can deduce that exist at most a finite number of unstable eigenmodes.

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Now we do a finite dimensional approach. Following the ideas of

- Coron and Trélat (2004) - Coron and Trélat (2006) . They deals with a semilinear heat and wave equation respectively.
- Prieur and Trélat (2018) - They work with a heat equation with delayed control.
- Guzmán et al. (2019) . In this paper they proposed a feedback delayed stabilization for a linear KS equation.

Main Result

Follow this framework we consider the delayed controlled parabolic-elliptic system:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha v(x,t), & x \in (0,L),\ t > 0, \\ -v_{xx}(x,t) + \gamma v(x,t) = \beta u(x,t), & x \in (0,L),\ t > 0, \\ u(0,t) = 0,\ \ u(L,t) = h(t-D) = h_D(t), & t > 0, \\ v(0,t) = 0,\ \ v(L,t) = 0, & t > 0, \end{cases} \tag{3}$$

With the time of delay $D > 0$.

The main idea of this work is prove the next result:

Theorem (1 Parada-Cerpa-Morris)

Consider the closed-loop system consisting of (3) with delayed Dirichlet boundary control. Then there exists a feedback delayed control $h_D(t)$ such that the controlled system is exponentially stabilizable, that is there exist $\mu > 0$ and $C > 0$ such that, for all $u_0(\cdot)$, $v_0(\cdot) \in H_0^1(0,L)$, with $u_0(0) = 0$

$$|h(t-D)| + \|(u,v)\|_{H_0^1(0,L)\times H_0^1(0,L)} \leq Ce^{-\mu t}\|(u_0,v_0)\|_{H_0^1(0,L)}$$

Spectral Decomposition

First we do a spectral decomposition.

Consider the time of delay $D > 0$. If we use change of variable and introducing the operators $F : L^2(0, L) \to H_0^1(0, L)$ and A we obtain:

$$w_t = Aw + a(\cdot)h_D(t) + b(\cdot)h_D^{'}(t), \qquad w(0, t) = w(L, t) = 0 \tag{4}$$

where:

$$a(x) = \left(-\lambda\frac{x}{L} + \alpha\beta F(x)\right) \qquad b(x) = -\frac{x}{L}$$

and F,A are defined by:

$$F(u) = v \;\; : \quad -v_{xx} + \gamma v = u \quad v(0) = v(L) = 0 \tag{5}$$

$$A := \partial_{xx} + \alpha\beta F(\cdot) - \lambda Id(\cdot) \tag{6}$$

With $D(A) = H^2(0, L) \cap H_0^1(0, L)$.

Note that A is self-adjoint and with compact inverse.

Spectral Decomposition

Let $(e_j) \subset H_0^1(0,L) \cap C^4([0,L])$ a Hilbert basis of eigenfunctions of A and (λ_j) the eigenvalues that satisfies:

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1 \qquad \lambda_j \to -\infty$$

With this, all solution $w(t,\cdot) \in H^2(0,L) \cap H_0^1(0,L)$ and thus

$$w(t,\cdot) = \sum_{j=1}^{\infty} w_j(t) e_j(\cdot)$$

and if we define $\nu_D(t) = h_D'(t)$ our controlled system is equivalent to:

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$$\begin{array}{rcl} h_D'(t) & = & \nu_D(t) \\ w_1'(t) & = & \lambda_1 w_1(t) + a_1 h_D(t) + b_1 \nu_D'(t) \\ & \vdots & \\ w_j'(t) & = & \lambda_j w_j(t) + a_j h_D(t) + b_j \nu_D'(t) \end{array} \tag{7}$$

where

$$a_j = \langle a(\cdot), e_j(\cdot) \rangle_{L^2} = \frac{1}{L} \int_0^L (-\lambda x + \alpha\beta F(x)) e_j(x) dx$$

$$b_j = \langle b(\cdot), e_j(\cdot) \rangle_{L^2} = -\frac{1}{L} \int_0^L x e_j(x) dx$$

Spectral Decomposition

Let $n \in \mathbb{N}\backslash\{0\}$ the number of positive eigenvalues and Π_1 the orthogonal projection to $\langle\{e_1, \cdots, e_n\}\rangle$ in $L^2(0, L)$ then

$$\forall k < n \quad \lambda_k < -\eta < 0$$

and let:

$$w^1 = \Pi_1 w = \sum_{j=1}^{n} w_j(t) e_j(\cdot)$$

Then using the matrices:

$$X_1(t) = \begin{pmatrix} h_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix} \quad B_1(t) = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix} \tag{8}$$

we can construct the next unstable finite dimensional system:

$$X_1'(t) = A_1 X_1(t) + B_1 \nu_D(t) \tag{9}$$

Stabilization of finite dimensional system

Consider an Artstein transformation:

$$Z_1(t) = X_1(t) + \int_{t-D}^{t} e^{(t-s-D)A_1} B_1 \nu(s) ds \tag{10}$$

We can transform the above system to

$$Z_1'(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \nu(t) \tag{11}$$

The invertibility of the Artstein transformation is follow from Prieur and Trélat (2018) and Bresch-Pietri et al. (2018) .

The Z_1 system is stabilizate if satisfies the Kalman condition.
It is sufficient to show the Kalman condition for (A_1, B_1).
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$$0 \neq \det(B_1, B_1 A_1, \cdots, B_1 A^n) = \prod_{j=1}^{n} (a_j + \lambda_j b_j) Vdm(\lambda_1, \cdots, \lambda_n)$$

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But $Vdm(\lambda_1, \cdots, \lambda_n) \neq 0$.

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So it is enough show that $a_j + \lambda_j b_j \neq 0$ for $j = 1, \cdots n$. Moreover we have that:

$$a_j + \lambda_j b_j = -e_j^{'}(L)$$

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In this context we have the next Lemma:

Lemma (1)

Suppose that $\gamma > 0$ and $\alpha\beta > 0$, then for all $j = 1, \cdots, n$, we have that $e_j^{'}(L) \neq 0$ where e_j is an eigenfunction of the operator A defined in (6).

Suppose for a moment that the above Lemma is true, then the systems

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Corollary (1)

$\forall D \geq 0$, $\exists K_1(D) \in \mathbb{R}^{1\times(n+1)}$ such that $A_2(D) = A_1 + e^{-DA_1} B_1 K_1(D)$ admits -1 has an eigenvalue of order $n+1$. Furthermore exists a symmetric positive definite matrix $P(D)$ such that:

$$P(D)A_2(D) + A_2(D)P(D) = -I_{n+1} \tag{12}$$

Stabilization of finite dimensional system

In virtue of the Corollary the function:

$$V(Z_1) = \frac{1}{2} Z_1^T P(D) Z_1 \tag{13}$$

is a Lyapunov function for the Z_1 system. So the feedback control $\nu(t) = K_1 Z_1$ stabilizate this system.

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We set:

$$\nu(t) = \begin{cases} 0 & \text{if } t \leq D \\ K_1 Z_1 & \text{if } t > D \end{cases} \tag{14}$$

Using the Artstein transformation we get:

$$\nu(t) = \begin{cases} 0 & \text{if } t \leq D \\ K_1(D) X_1(t) + K_1(D) \int_{\max(D,t-D)}^{t} e^{(t-D-s)A_1} B_1 \nu(s) ds & \text{if } t > D \end{cases} \tag{15}$$

and therefore the feedback control $\nu(t)$ makes $X_1(t)$ to go exponentially to zero as $t \to \infty$

Stabilization of finite dimensional system

In order the stability of the whole system we set:

$$V_D(t) = M(D)V_1(1) + M(D)\int_{(t-D,t)\cap(D,\infty)} V_1(s)ds - \frac{1}{2}\langle w(t), Aw(t)\rangle_{L^2(0,L)}$$
$$= \frac{M(D)}{2} Z_1(t)^T P(D) Z_1(t) + \frac{M(D)}{2}\int_{(t-D,t)\cap(D,\infty))} Z_1(s)^T P(D) Z_1(s) ds$$
$$- \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j w_j(t)^2 \qquad (16)$$

Where $M(D)$ is sufficiently large.

The next Lemmas tell us that V_D is a Lyapunov functional for whole system.

Stabilization of infinite dimensional system

Lemma (2)

Exists $C_2(D) > 0$ *such that:*

$$V_D(t) \geq C_2 \left(h_D(t)^2 + \|w(t)\|^2_{H^1_0(0,L)} \right) \tag{17}$$

For every $t \geq 0$

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Exist a constant $C_4(D) > 0$ *such that:*

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The functional V_D *decreases exponentially to* 0*.*

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with this Lemmas the Theorem 1 can be deduced directly.

Proof Lemma 1 (Sketch)

To conclude we have to prove the Lemma 1. We follow the idea from Cerpa (2014) .

Let $(e_j) \subset H_0^1(0,L) \cap C^4([0,L])$ the eigenfunctions of $A = \partial_{xx} + \alpha\beta F(\cdot) - \lambda Id(\cdot)$ so it possible to show that e_j is solution of the next four order homogeneous boundary problem:

$$\begin{gathered} e_j^{''''} - (\lambda + \lambda_j + \gamma)e_j^{''} + (\gamma(\lambda + \lambda_j) - \alpha\beta)e_j = 0 \\ e_j(0) = e_j(L) = e_j^{''}(0) = e_j^{''}(L) = 0 \end{gathered} \tag{19}$$

Then

$$e_j(x) = \sum_{i=1}^{4} C_i e^{\delta_i x} \tag{20}$$

where δ_i, $i = 1, 2, 3, 4$ are the roots of the polynomial:

$$x^4 - (\lambda + \lambda_j + \gamma)x^2 + (\gamma(\lambda + \lambda_j) - \alpha\beta) = 0$$

Proof Lemma 1 (Sketch)

Imposing the boundary conditions and adding $e_j'(L) = 0$, we obtain the next linear system.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{\delta_1 L} & e^{\delta_2 L} & e^{\delta_3 L} & e^{\delta_4 L} \\ \delta_1^2 & \delta_2^2 & \delta_3^2 & \delta_4^2 \\ \delta_1^2 e^{\delta_1 L} & \delta_2^2 e^{\delta_2 L} & \delta_3^2 e^{\delta_3 L} & \delta_4^2 e^{\delta_4 L} \\ \delta_1 e^{\delta_1 L} & \delta_2 e^{\delta_2 L} & \delta_3 e^{\delta_3 L} & \delta_4 e^{\delta_4 L} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

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Recall that $e_j(x) = \sum_{i=1}^{4} C_i e^{\delta_i x}$, so it is sufficient to prove that the unique solution of (21) is the null solution that is equivalent that $C_1 = C_2 = C_3 = C_4 = 0$.

Proof Lemma 1 (Sketch)

We know δ_i, $i = 1, 2, 3, 4$ are the roots:

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$$y^2 - (\lambda + \lambda_j + \gamma)y + (\gamma(\lambda + \lambda_j) - \alpha\beta) = 0 \tag{23}$$

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3. **The roots of** (23) **are** 0 **and other one positive (or negative).** In this case the roots of (22) are of the form 0, 0, A and $-A$ (or iA and $-iA$). Similar $e_j \equiv 0$.
4. **The roots of** (23) **are two negative.** This case is not possible.

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$$x^4 - (\lambda + \lambda_j + \gamma)x^2 + (\gamma(\lambda + \lambda_j) - \alpha\beta) = 0 \tag{22}$$

Let

$$y^2 - (\lambda + \lambda_j + \gamma)y + (\gamma(\lambda + \lambda_j) - \alpha\beta) = 0 \tag{23}$$

Then as $\alpha\beta > 0$ (23) only has real roots. So we have the next enlisted cases:

1. **The roots of** (23) **are reals and positive and therefore different.** In this case we have that the roots of (22) are of the form A, $-A$, B and $-B$, for $A, B > 0$ different. The unique solution of (21) is the null solution.
2. **The roots of** (23) **are different one positive and one negative.** Therefore the roots of (22) are of the form A, $-A$, iB and $-iB$. Similar, $e_j \equiv 0$.
3. **The roots of** (23) **are** 0 **and other one positive (or negative).** In this case the roots of (22) are of the form 0, 0, A and $-A$ (or iA and $-iA$). Similar $e_j \equiv 0$.
4. **The roots of** (23) **are two negative.** This case is not possible.

Therefore we can conclude that in all possible case we have that $e_j \equiv 0$, which is not possible because e_j is a non trivial eigenvalue of the operator A, which give us the Lemma 1.

Other Results

Recall that the controlled system is:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \lambda u(x,t) = \alpha v(x,t), & x \in (0,L),\ t>0, \\ -v_{xx}(x,t) + \gamma v(x,t) = \beta u(x,t), & x \in (0,L),\ t>0, \\ u(0,t)=0,\ u(L,t)=h(t), & t>0, \\ v(0,t)=0,\ v(L,t)=0, & t>0, \end{cases} \tag{24}$$

We consider some unstable cases where the instability is not too big.
This is the case if the parameters satisfy

$$\alpha\beta = \left[\frac{\left(\gamma(1-\delta_1) - \left(\frac{\pi}{L}\right)^2(1+\delta_1)\right)}{2\gamma\left(\left(\frac{\pi}{L}\right)^2+\gamma\right)} \right]^{-1} \left(\left(\frac{\pi}{L}\right)^2 + \delta_2\right), \qquad \gamma > \frac{\left(\frac{\pi}{L}\right)^2(1+\delta_1)}{1-\delta_1}$$

$$\lambda \in \left(\frac{\alpha\beta}{2\gamma}(1+\delta_1), \frac{\beta\alpha}{\left(\frac{\pi}{L}\right)^2+\gamma} - \left(\frac{\pi}{L}\right)^2 \right) \tag{25}$$

for some $\delta_1 \in (0,1)$ and $\delta_2 > 0$.
The next results were proved using the *Backstepping Method*. See Krstic and Smyshlyaev (2008).

Other Results

Define $\mathcal{T} = \{(x, y) \in 0 \leq y \leq x \leq L\}$.

Theorem (2 Parada-Cerpa-Morris)

Let $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$ *satisfy the conditions* (25). *There there exists* $k \in C^2(\mathcal{T})$ *such that the solutions of* (24) *with the control*

$$h(t) = -\int_0^L k(L, y)u(y, t)dy \tag{26}$$

satisfy

$$\|(u(\cdot, t), v(\cdot, t))\|_{L^2(0,L)\times L^2(0,L)} \leq R \cdot e^{(-2\lambda+\alpha\beta(1+\delta_1))t}\|u(\cdot, 0)\|_{L^2(0,L)}$$

for some $R > 0$. *Thus, this feedback law* (26) *exponentially stabilizes the origin.*

Other Results

Then, we considered the case where we can only measure the Neumann boundary condition of the elliptic solution, i.e., $v_x(0,t)$. We build the following observer:

Other Results

Then, we considered the case where we can only measure the Neumann boundary condition of the elliptic solution, i.e., $v_x(0,t)$. We build the following observer:

$$\begin{cases} \hat{u}_t - \hat{u}_{xx} + \lambda \hat{u} = \alpha \hat{v} + p_1(x)[v_x(0) - \hat{v}_x(0)], & x \in (0,L), t > 0, \\ -\hat{v}_{xx} + \gamma \hat{v} = \beta \hat{u}, & x \in (0,L),\ t > 0, \\ \hat{u}(0) = 0,\ \hat{u}(L) = h(t) + p_{10}[v_x(0) - \hat{v}_x(0)], & t > 0, \\ \hat{v}(0) = 0,\ \hat{v}(L) = 0 & , t > 0, \end{cases} \tag{27}$$

where $p(\cdot)$ and p_{10} are chosen appropriately.
And obtained the next result

Theorem (3 Parada-Cerpa-Morris)

Let $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$ satisfy conditions (25)*. There there exists $k \in C^2(\mathcal{T})$ such that the solutions of* (24)-(27) *with the control*

$$h(t) = -\int_0^L k(L,y)\hat{u}(y,t)dy \tag{28}$$

satisfy

$$\begin{aligned} &\|(u(\cdot,t) - \hat{u}(\cdot,t), v(\cdot,t) - \hat{v}(\cdot,t))\|_{L^2(0,L)\times L^2(0,L)} + \|(\hat{u}(\cdot,t), \hat{v}(\cdot,t))\|_{L^2(0,L)\times L^2(0,L)} \\ &\qquad \leq R \cdot e^{(-2\lambda + \alpha\beta(1+\delta_1))t} \left\{ \|u(\cdot,0) - \hat{u}(\cdot,0)\|_{L^2(0,L)} + \|\hat{u}(\cdot,0)\|_{L^2(0,L)} \right\} \end{aligned} \tag{29}$$

for some $R > 0$.

Remarks

- The condition $\alpha\beta > 0$ could be not necessary, but maybe we have to impose some conditions on L.
- Following the ideas here presented and for example Coron and Trélat (2006) ideas we can consider other kind of coupling.
- We can see that the backstepping result is more restrictive with the parameters involved the system.

Remarks

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- Following the ideas here presented and for example Coron and Trélat (2006) ideas we can consider other kind of coupling.
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Thanks for your Attention.

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