Singularities from Hamilton-Jacobi

Joint work with Piermarco Cannarsa & Albert Fathi

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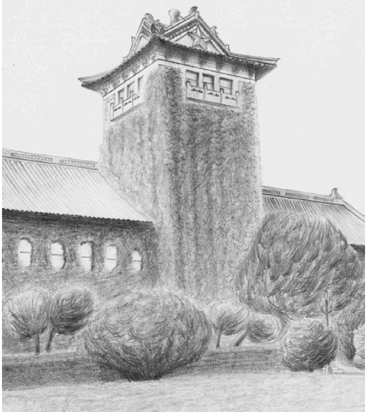

VIII Partial differential equations, optimal design and numerics

Centro de Ciencias de Benasque Pedro Pascual

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Introduction

Distance function and cut locus

- Let (M, g) be a complete C^2 Riemannian manifold and let d be the associated Riemannian distance. For any closed subset F, one can define the distance function d_F w. r. t. F by

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- Let γ be a geodesic starting at a point of F, we say y on γ is a cut point of F *along* γ if it is the first point on γ such that all the points on γ after y can be connected to some point in F by a geodesic segment $\tilde{\gamma}$ which is shorter than γ. The set $\mathrm{Cut}(F)$ of all such cut points is called the cut locus w.r.t. F.

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- It is clear that $\mathrm{Cut}(F)$ is essentially related to the nonsmooth part of the function d_F.
- It is also well known that d_F is a Lipschitz and locally semiconcave viscosity solution of the associated Hamilton-Jacobi equations.

Propagation of singularities

- We want to study the evolution of singularities of the viscosity solutions with respect to Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in M.$$

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- In the second part, we will introduce our recent results on the homotopic results on cut locus and singular sets of the viscosity solution in a general setting. We will also emphasize some important applications to Riemannian geometry.
- In the last part, we will give more remarks on the some expected results.

Literature on propagation of singularities

Reference on the propagation of singularities (possible incomplete)

- Albano-Cannarsa(1999,2000,2002), Albano(2002),
- Lieutier(2004)
- Bogaevsky(2006),
- Yu(2006,2007),
- Cannarsa-Mazzola−Sinestrari(2015),
- Strömberg(2013), Strömberg-Ahmadzdeh(2014),
- Cannarsa-Yu(2009),
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- Cannarsa-Yu(2009),
- Khanin-Sobolevski(2014)

Main reference of this talk:

1. Cannarsa-C-Zhang(2014), Cannarsa-C(2015), Cannarsa-C(2017), Cannarsa-C-Fathi(2017), Cannarsa-C-Wang-Yan (2019), Cannarsa-C-Mazzola-Wang (2019), Cannarsa-Chen-C (2019)
2. Preprints: Cannarsa-C-Fathi(2019)

A brief review on classical weak KAM theory

Hamilton-Jacobi equations

Let M be a C^2 smooth connected and compact manifold (without boundary) and $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. We consider the viscosity solutions of the stationary Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in M, \tag{HJ$_s$}$$

or the evolutionary one

$$D_t u(t,x) + H(x, D_x u(t,x)) = 0, \quad x \in M, t > 0. \tag{HJ$_e$}$$

Value function of Bolza Problem

By dynamical programming principle,

$$u(t,x) = \inf_{y\in M}\{u_0(y) + A_t(y,x)\}$$

is the unique viscosity solution of (HJ$_e$) with initial data $u(0,x) = u_0$, where

$$A_t(x,y) = \min_{\xi\in\Gamma^t_{x,y}} \int_0^t L(\xi(s),\dot{\xi}(s))ds \qquad (x,y\in M),$$

with

$$\Gamma^t_{x,y} = \{\xi\in W^{1,1}([0,t];M) : \xi(0) = x, \xi(t) = y\}$$

Here, $A_t(x,y)$ is called the fundamental solution with respect to (HJ$_e$), or generating function in the context of symplectic geometry.

Lax-Oleinik semigroups & weak KAM solutions

Let $u_0 \in C(M, \mathbb{R})$, for any $x \in M$, define

$$T_t^+ u_0(x) = \sup_{y \in M} \{u_0(y) - A_t(x, y)\},$$

$$T_t^- u_0(x) = \inf_{y \in M} \{u_0(y) + A_t(y, x)\}.$$

- This is also called Lasry-Lions regularization in PDEs. It is one kind of variational approximation process.

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- This is also called Lasry-Lions regularization in PDEs. It is one kind of variational approximation process.
- A continuous function u is said to be a weak KAM solution of (HJ$_s$) if u is a fixed point of the semigroup $\{T_t^-\}$ for all $t > 0$.

Dominated functions & Calibrated curves

- A function $u : M \to \mathbb{R}$ is said to be dominated by L iff, for each absolutely continuous arc $\gamma : [a, b] \to M$ with $a < b$, one has

$$u(\gamma(b)) - u(\gamma(a)) \leqslant \int_a^b L(\gamma(s), \dot{\gamma}(s))ds.$$

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- An absolutely continuous curve $\gamma : [a, b] \to M$ is said to be u-calibrated if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))ds + c[0](b - a).$$

Aubry sets

- The projected Aubry set w.r.t. u is defined by

$$\mathcal{I}(u) = \{x \in M : x = \gamma(0) \text{ for some } u\text{-calibrated curve } \gamma : \mathbb{R} \to M\}.$$

Aubry sets

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$$\mathcal{I}(u) = \{x \in M : x = \gamma(0) \text{ for some } u\text{-calibrated curve } \gamma : \mathbb{R} \to M\}.$$

- The α-limit set of a backward u-calibrated curve is contained in $\mathcal{I}(u)$.

Cut points & Singular points

Let u be a weak KAM solution of (HJ_s).

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- $\Sigma(u) \subset \mathrm{Cut}(u) \subset M \setminus \mathcal{I}(u)$, and $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)}$.

Regularity properties

Semiconcave functions

- Let $\Omega \subset \mathbb{R}^n$ be a convex open set, a function $u : \Omega \to \mathbb{R}$ is *semiconcave* if there exists a constant $C > 0$ such that
$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leqslant \frac{C}{2}\lambda(1-\lambda)|x-y|^2$$
for any $x, y \in \Omega$ and $\lambda \in [0, 1]$.

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- Equivalently, u is semiconcave with constant C if $u = \inf_\alpha u_\alpha$ with each u_α a C^2 functions whose Hessian (in the sense of distribution) is bounded above uniformly by $C\,\mathrm{Id}$.

Superdifferential

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We recall that, for any $x \in \Omega$, the closed convex sets

$$D^- u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geqslant 0 \right\},$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leqslant 0 \right\}.$$

are called the *subdifferential* and *superdifferential* of u at x, respectively.

Limiting differential

Let $u:\Omega\to\mathbb{R}$ be locally Lipschitz. We recall that a vector $p\in\mathbb{R}^n$ is called a *limiting differential* of u at x if there exists a sequence $\{x_n\}\subset\Omega\setminus\{x\}$ such that u is differentiable at x_k for each $k\in\mathbb{N}$, and

$$\lim_{k\to\infty} x_k = x \quad \text{and} \quad \lim_{k\to\infty} Du(x_k) = p.$$

The set of all limiting differentials of u at x is denoted by $D^*u(x)$.

Properties of superdifferentials

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a semiconcave function and let $x \in \Omega$. Then

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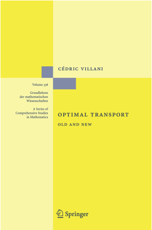
For more information on the semiconcavity, see, e.g.,



Tonelli Lagrangians

We concentrate on Tonelli systems.

A C^2-function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Tonelli Lagrangian* if the following assumptions are satisfied.

1. *Uniform convexity*: There exists a nonincreasing function $\nu : [0, +\infty) \to (0, +\infty)$ such that

$$L_{vv}(x, v) \geqslant \nu(|v|)I \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

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2. *Growth condition*: There exist two superlinear function $\theta,\overline{\theta}:[0,+\infty)\to[0,+\infty)$ and a constant $c_0>0$ such that
$$\overline{\theta}(|v|)\geqslant L(x,v)\geqslant\theta(|v|)-c_0\qquad\forall(x,v)\in\mathbb{R}^n\times\mathbb{R}^n.$$

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$$\overline{\theta}(|v|) \geqslant L(x, v) \geqslant \theta(|v|) - c_0 \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$
3. *Uniform regularity*: There exists a nondecreasing function $K : [0, +\infty) \to [0, +\infty)$ such that, for every multi-index $|\alpha| = 1, 2$,
$$|D^\alpha L(x, v)| \leqslant K(|v|) \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

An illustrative observation

Proposition

Let $x \in M$ and $u : M \to \mathbb{R}$ be a weak KAM solution of the H-J equation (HJ_s). *Then $p \in D^*u(x)$ if and only if there exists a unique C^2 backward calibrated curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ and $p = L_v(x, \dot{\gamma}(0))$.*

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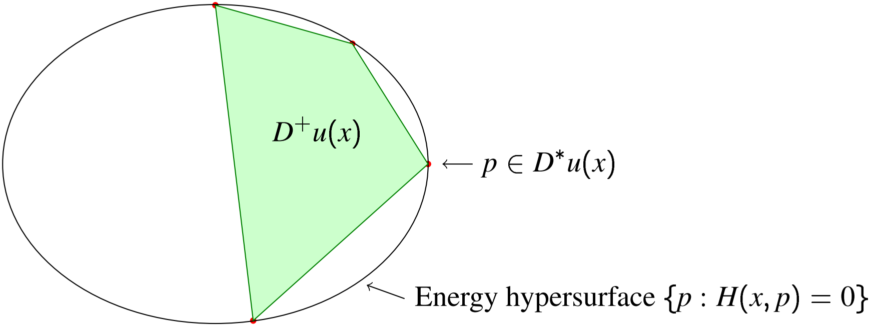

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- There exists $t_0 > 0$, the map $(t, y) \mapsto A_t(x, y)$ is locally semiconvex on $(0, t_0) \times M$.
- For each $\lambda > 0$, There exists $t_0 > 0$, the map $y \mapsto A_t(x, y)$ is convex on $B(x, \lambda t)$ with $t \in (0, t_0)$. The constant is C_2/t.

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- For each $\lambda > 0$, There exists $t_0 > 0$, the map $y \mapsto A_t(x,y)$ is convex on $B(x,\lambda t)$ with $t \in (0,t_0)$. The constant is C_2/t.
- If u is a weak KAM solution of (HJ_s), by the L-O representation formulae as a marginal function

$$u(x) = \inf_{y \in M} \{u(y) + A_t(y,x)\}, \quad x \in M, t > 0,$$

then u is semiconcave with constant, say C_1.

A $C^{1,1}$ argument

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- For each $\lambda > 0$, There exists $t_0 > 0$, the map $y \mapsto A_t(x,y)$ is also $C^{1,1}$ on $B(x, \lambda t)$ with $t \in (0, t_0)$.
- Moreover, for any $t \in (0, t_0]$,

$$
\begin{aligned}
D_y A_t(x,y) &= L_v(\xi(t), \dot{\xi}(t)), \\
D_x A_t(x,y) &= -L_v(\xi(0), \dot{\xi}(0)), \\
D_t A_t(x,y) &= -E_{t,x,y},
\end{aligned}
$$

where $\xi \in \Gamma^t_{x,y}$ is the unique minimizer of $A_t(x,y)$ and $E_{t,x,y}$ is the energy of the Hamiltonian trajectory $(\xi(s), p(s))$ with $p(s) = L_v(\xi(s), \dot{\xi}(s))$.

Connection to Lasry-Lions regularization

- For any $u \in BUC(\mathbb{R}^n, \mathbb{R})$ and any $t > 0$, we define

$$S_t^+ u(x) = \sup_{y \in \mathbb{R}^n} \{u(y) - \frac{1}{2t}|x-y|^2\},$$
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- For small t, $S_t^+ u$ is semiconvex and $S_t^- u$ is semiconcave since the kernel $\frac{1}{2t}|\cdot - y|^2$ is coercive, and $S_t^- \circ S_s^+$ or $S_t^+ \circ S_s^-$ makes u to be a $C^{1,1}$ function for small $0 < t < s$.

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- Here $L = \frac{1}{2}|v|^2$. In general study of this process can be found in [Bernard 2007][2], [Chen-C 2016][3] and [Chen-C-Zhang 2018][4].

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Intrinsic approach of propagation of Singularities

Propagation of Singularities

autonomous systems on $\mathbb{R}^n$

Singularities for arbitrary $t > 0$ [Cannarsa-C, 2017]

Suppose u is a weak KAM solution which is global Lipschitz and semiconcave. If $x \in \mathrm{Cut}\,(u)$, then any local maximizer of $u(\cdot) - A_t(x, \cdot)$ is contained in $\Sigma\,(u)$ for all $t > 0$. Moreover, There exists $t_0 > 0$ (t_0 is independent of x) such that, if $x \in \mathrm{Cut}\,(u)$, then the function

$$u(\cdot) - A_t(x, \cdot)$$

achieves a unique maximizer $y_{t,x}$ for all $t \in (0, t_0]$. Let the curve is defined by

$$\mathbf{y}(t) := \begin{cases} x & \text{if} \quad t = 0 \\ y_{t,x} & \text{if} \quad t \in (0, t_0], \end{cases} \tag{GC$_{loc}$}$$

then $\mathbf{y}(t) \in \Sigma\,(u)$ for all $t \in (0, t_0]$.

Sketch of the proof

Singularities for arbitrary $t > 0$

- For any $t > 0$ and $y_{t,x} \in \mathtt{arg\,max}_{loc}\{u(\cdot) - A_t(x,\cdot)\}$, suppose $y_{t,x}$ is a differentiable point of u. Thus

$$0 \in D^+\{u(\cdot) - A_t(x,\cdot)\}(y_{t,x}) = Du(y_{t,x}) - D^-\{A_t(x,\cdot)\}(y_{t,x}).$$

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- It follows that $A_t(x,\cdot)$ is differentiable at $y_{t,x}$ and

$$p_{t,x} = Du(y_{t,x}) = D_yA_t(x,y_{t,x}).$$

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$$p_{t,x} = Du(y_{t,x}) = D_yA_t(x,y_{t,x}).$$

- There exists two C^2 curves $\xi_{t,x} : [0,t] \to \mathbb{R}^n$ and $\gamma_x : (-\infty,t] \to \mathbb{R}^n$ such that $\xi_{t,x}(0) = x$, $\gamma_x(t) = \xi_{t,x}(t) = y_{t,x}$ and

$$p_{t,x} = L_v(\gamma_x(t),\dot{\gamma}_x(t)) = L_v(\xi_{t,x}(t),\dot{\xi}_{t,x}(t)).$$

Since $\xi_{t,x}$ and γ_x has the same endpoint condition at t, then they coincide on $[0,t]$. This leads to a contradiction since $x \in \mathrm{Cut}\,(u)$.

Sketch of the proof

Looking for a unique maximizer

- Let $x \in \mathrm{Cut}\,(u)$, denoted by $M_t(x) = \operatorname{arg\,max}_{loc}\{u(\cdot) - A_t(x, \cdot)\}$, the set-valued map $t \mapsto M_t(x) \subset \Sigma\,(u)$ is upper-semicontinuous.

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- Whether there exists a continuous selection of the set-valued map $t \mapsto M_t(x)$ is unclear!

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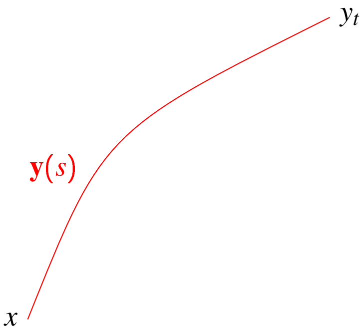
- Whether there exists a continuous selection of the set-valued map $t \mapsto M_t(x)$ is unclear!
- Since $u(\cdot)$ is semiconcave with constant C_1 and $A_t(x,\cdot)$ is convex on $B(x,\lambda t)$, $t \in (0,t_0)$, with constant C_2/t. Therefore $u(\cdot) - A_t(x,\cdot)$ is strictly concave on $B(x,\lambda t)$ if t satisfies

$$C_1 - C_2/t < 0.$$

Then we have a unique maximizer for $t < t_0 = C_2/C_1$.

A picture



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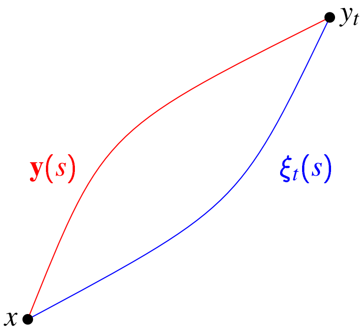

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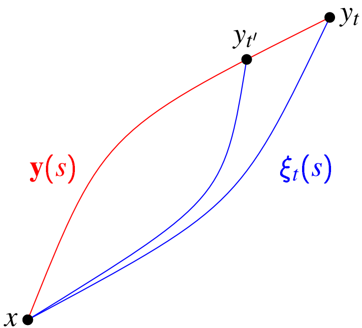

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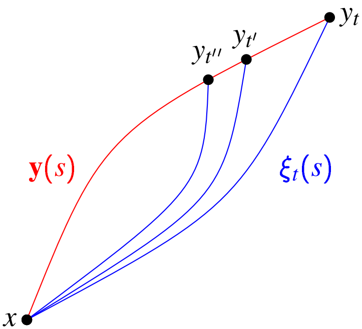

Global singular generalized characteristics

- The arc $\mathbf{y} : [0, t_0] \to M$ is Lipschitz (the constant is independent of x) and it is a generalized characteristic satisfying

$$\dot{\mathbf{y}}(t) \in \operatorname{co} H_p(\mathbf{y}(t), D^+u(\mathbf{y}(t))), \quad \text{a.e. } t \in [0, t_0].$$

  Moreover, $\dot{\mathbf{y}}^+(0) = H_p(x, p_0)$, where p_0 is the unique element of minimal energy: $H(x, p) \geqslant H(x, p_0)$ for all $p \in D^+u(x)$.

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- If M is compact, since t_0 is independent of x, then the local defined singular GCs can be extended to a global one.
- It is also true for non-compact M under standard Fathi-Maderna conditions using some local strategy!

Some remarks

The mechanism in principle

- The basic philosophy of our construction for propagation of singularities is that, under Tonelli-like conditions, if the solution has a representation form of inf-convolution, then the singularities can be interpreted by the associated sup-convolution!

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- compact or non-compact manifold;
- Time independent of not;
- Even an implicit representation form recent obtained by Wang-Wang-Yan and Cannarsa-C-Wang-Yan for the contact type H-J equations $H(x, u(x), Du(x)) = 0$!

The topology of cut locus

A homotopy

One can define a (continuous) homotopy $F : M \times [0, t_0] \to M$,

$$F(x, s) = y_{s,x},$$

which satisfies the following properties:

(a) for all $x \in M$, we have $F(x, 0) = x$;

(b) if $F(x, t) \notin \Sigma(u)$, for some $t > 0$, and $x \in M$, then the curve $s \mapsto F(x, s)$ is u-calibrating on $[0, t]$;

(c) if there exists a u-calibrating curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$, then $s \mapsto F(x, s) = \gamma(s)$, for every $s \in [0, \min(t, t_0)]$.

Homotopy equivalence

Theorem (Cannarsa-C-Fathi, 2017)

The inclusion $\Sigma(u) \subset \mathrm{Cut}\,(u) \subset \overline{\Sigma\,(u)} \cap (M \setminus \mathcal{I}(u)) \subset M \setminus \mathcal{I}(u)$ *are all homotopy equivalences.*

Corollary

For every connected component C *of* $M \setminus \mathcal{I}(u)$ *the three intersections* $\Sigma\,(u) \cap C$, $\mathrm{Cut}\,(u) \cap C$, *and* $\overline{\Sigma\,(u)} \cap C$ *are path-connected.*

The key point of the proof is that the cut time function $\tau(x)$ is upper semi-continuous, and the homotopy $G : (M \setminus \mathcal{I}(u)) \times [0, 1] \to M \setminus \mathcal{I}(u)$,

$$G(x, s) = F(x, s\alpha(x))$$

is the desired homotopy, where $\alpha : M \setminus \mathcal{I}(u) \to]0, +\infty[$ is a continuous function with $\alpha > \tau$ on $M \setminus \mathcal{I}(u)$.

Local path-conectedness

Theorem (Cannarsa-C-Fathi, 2017)

The spaces $\Sigma(u)$, *and* $\mathrm{Cut}(u)$ *are locally contractible, i.e. for every* $x \in \Sigma(u)$ *(resp.* $x \in \mathrm{Cut}(u)$*) and every neighborhood* V *of* x *in* $\Sigma(u)$ *(resp.* $\mathrm{Cut}(u)$*), we can find a neighborhood* W *of* x *in* $\Sigma(u)$ *(resp.* $\mathrm{Cut}(u)$*), such that* $W \subset V$*, and* W *in null-homotopic in* V.

Therefore $\Sigma(u)$, *and* $\mathrm{Cut}(u)$ *are locally path connected.*

The problem can be attacked by the local homotopy defined above for small t and Hahn-Baire's interpolation theorem (There exists a continuos function α, $u \leqslant \alpha \leqslant v$, with v lower-semicontinuous and u upper semicontinuous).

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Extensions to noncompact manifold

- Let M be any C^2 connected complete Riemannian manifold. Our purpose is to study certain topological properties of the cut locus the viscosity equations

$$D_t u + H(x, D_x u) = 0. \tag{HJ'$_e$}$$

[5] Cannarsa, P.; C.; Fathi, A.; *On the topology of the set of singularities of a solution to the Hamilton-Jacobi equation*. C. R. Math. Acad. Sci. Paris **355** (2017), no. 2, 176–180.

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- We can prove that $\Sigma_{t_0}(\hat{u})$ is locally contractible for any $u : M \to [-\infty, +\infty]$ such that $\hat{u}$, the so-called Lax-Oleinik evolution, is finite on $(0, t_0) \times M$.

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- This leads some very interesting applications to the distance functions on Riemannian manifold.

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Lax-Oleinik evolution

definition

- Let $u : M \to [-\infty, +\infty]$ be a function, we define

$$T_t^- u(x) = \inf_{y \in M} \{u(y) + A_t(y, x)\}.$$

The function $\hat{u}(t, x) = T_t^- u(x)$ is called the Lax-Oleinik evolution of u.

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- One can regard the representation of $T_t^- u$ as a marginal function! Thus, the expected regularity properties of $T_t^- u$ can be deduced by what of the associated fundamental solution.

Aubry set

Definition

Let $U : [0,T) \times M \to \mathbb{R}$, with $T \in (0,+\infty]$, be a viscosity solution, on $(0,T) \times M$, of the evolutionary Hamilton-Jacobi equation (HJ'$_e$). The Aubry set $\mathcal{I}_T(U)$ of U is the set of points $(t,x) \in (0,T) \times M$ for which we can find a curve $\gamma : [0,T) \to M$, with $\gamma(t) = x$ and

$$U(b,\gamma(b)) - U(a,\gamma(a)) = \int_a^b L(\gamma(s),\dot{\gamma}(s))\,ds,$$

for every $a < b \in [0,T)$.

Cut locus

- The set $\mathrm{Cut}_{t_0}(\hat{u})$, called cut locus of $\hat{u}$, of cut points of $\hat{u}$ is the set of points $(t,x)\in(0,t_0)\times M$ where no backward $\hat{u}$-characteristic ending at (t,x) can be extended to a $\hat{u}$-calibrating curve defined on $[0,t']$, with $t'>t$.

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- the point $(t,x) \in (0,t_0) \times M$ is in $\mathrm{Cut}_{t_0}(\hat{u})$ if and only if for any $\hat{u}$-calibrating curve $\delta : [a,b] \to M$, with $t \in [a,b]$ and $\delta(t) = x$, we have $t = b$.

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- We have

$$\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u}) \subset (0,t_0) \times M \setminus \mathcal{I}_{t_0}(\hat{u}).$$

Moreover, the set $\Sigma_{t_0}(\hat{u})$ is dense in $\mathrm{Cut}_{t_0}(\hat{u})$. Hence, we have

$$\Sigma_{t_0}(\hat{u}) \subset \mathrm{Cut}_{t_0}(\hat{u}) \subset \overline{\Sigma_{t_0}(\hat{u})}.$$

Local contractibility

The following local result is due to [Cannarsa-C-Fathi, 2018][6].

Theorem (Local contractibility)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Assume that the function $u : M \to [\infty, +\infty]$ is such that its Lax-Oleinik evolution $\hat{u}$ is finite at every point of $(0, t_0) \times M$, then the sets $\Sigma_{t_0}(\hat{u})$ and $\mathrm{Cut}_{t_0}(\hat{u})$ are locally contractible. In particular, they are locally path connected.

[6] Cannarsa, P.; Cheng, W.; Fathi, A. *Singularities of solutions of time dependent Hamilton-Jacobi equations. Applications to Riemannian geometry*, preprint, 2018.

Global homotopy equivalence

Main Result

Now we formulate our global result.

Theorem (Global homotopy equivalence)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Assume that the uniformly continuous function $U : [0,t] \times M \to \mathbb{R}$ is a viscosity solution, on $(0,t) \times M$, of the evolutionary Hamilton-Jacobi equation (HJ'$_e$). Then the inclusion $\Sigma_t(U) = \Sigma(U) \cap [(0,t) \times M] \subset [(0,t) \times M] \setminus \mathcal{I}_t(U)$ is a homotopy equivalence.

Functions Lipschitz in the large

definition

Definition

Let X be a metric space whose distance is denoted by d. A function $u : X \to \mathbb{R}$ is said to be Lipschitz in the large if there exists a constant $K < +\infty$ such that

$$|u(y) - u(x)| \leqslant K + Kd(x, y), \quad \forall x, y \in X.$$

When the inequality above is satisfied, we will say that u is Lipschitz in the large with constant K.

Functions Lipschitz in the large

Remarks

- Note that we do not assume in the definition above that u is continuous.

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- Note that we do not assume in the definition above that u is continuous.
- Obviously, when X is compact $u : X \to \mathbb{R}$ is Lipschitz in the large if and only if u is bounded.

Functions Lipschitz in the large

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- Note that we do not assume in the definition above that u is continuous.
- Obviously, when X is compact $u : X \to \mathbb{R}$ is Lipschitz in the large if and only if u is bounded.
- As is shown in [Fathi, 2018], the function $u : X \to \mathbb{R}$ is Lipschitz in the large if and only if there exits a (globally) Lipschitz function $\varphi : X \to \mathbb{R}$ such that

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Functions Lipschitz in the large

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- In particular, a Lipschitz in the large function $u : M \to \mathbb{R}$ is bounded from below by a Lipschitz function and therefore $\hat{u}$ is finite everywhere on $[0, +\infty) \times M$.

Global homotopy equivalence for Lipschitz in the large function

Theorem (Global homotopy equivalence for Lipschitz in the large function)

Assume $u : M \to \mathbb{R}$ is a Lipschitz in the large function. For every $T \in (0, +\infty]$ the inclusions $\Sigma_T(\hat{u}) \subset \mathrm{Cut}_T(\hat{u}) \subset [(0, T) \times M] \setminus \mathcal{I}_T(\hat{u})$ are homotopy equivalences.

It is not difficult to see that, for a function $u : M \to \mathbb{R}$ Lipschitz in the large with constant K, its lower semi-continuous regularization u^- is itself Lipschitz in the large with constant K. Therefore by the previous Proposition, without loss of generality we can prove Theorem above adding the assumption that u is lower semi-continuous.

Applications to Riemannian geometry

First Result

If C is a closed subset of the complete Riemannian manifold (M, g). As usual, the distance function $d_c : M \to [0, +\infty)$ to C is defined by

$$d_C(x) = \inf_{c \in C} d(c, x).$$

We will denote by $\Sigma^*(d_C)$ the set of points in $M \setminus C$ where d_C is not differentiable.

Theorem (locally contractibility of the singular set)

Consider the closed subset C of the complete Riemannian manifold (M, g). Then $\Sigma^*(d_C)$ is locally contractible.

Some historic remarks

- As a a first application, if we take $C = p$, the set $\Sigma^*(d_p)$ is nothing but the set of $q \in M$ such that there exists two distinct minimizing geodesics from p to q. The closure is known as the the cut locus $\text{Cut-locus}_{(M,g)}(p)$ of p for (M, g).

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- It is well-known that, for M compact, this cut locus $\text{Cut-locus}_{(M,g)}(p)$ is a deformation retract of $M \setminus \{p\}$, therefore it is locally contractible. However, even if there is an extensive literature on the cut locus, very little was known up to now the set of $q \in M$ such that there exists two distinct minimizing geodesics from p to q.

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- As Marcel Berger said[7]: The difficulty for all these studies is an unavoidable dichotomy for cut points: the mixture of points with two different segments and conjugate points.

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- As Marcel Berger said[7]: The difficulty for all these studies is an unavoidable dichotomy for cut points: the mixture of points with two different segments and conjugate points.
- Our methods permit to separate the study of these two sets.

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The set with non-unique minimal geodesic & Aubry set again

Definition

If (M, g) is a complete Riemannian manifold, we define the subset $\mathcal{U}(M, g) \subset M \times M$ as the set of $(x, y) \in M \times M$ such that there exists a unique minimizing g-geodesic between x and y. This set $\mathcal{U}(M, g)$ contains a neighborhood of the diagonal $\Delta_M \subset M \times M$.

The complement $\mathcal{NU}(M, g) = (M \times M) \setminus \mathcal{U}(M, g)$ is the set of points $(x, y) \in M$ such that there exists at least two distinct minimizing g-geodesics.

Definition

For a closed subset $C \subset M$, we define its Aubry set $A^*(C)$ as the set of points $x \in M \setminus C$ such that there exists a curve $\gamma : [0, +\infty) \to M$ parameterized by arc-length such that $d_C(\gamma(t)) = t$ and $x = \gamma(t_0)$ for some $t_0 > 0$.

Applications to Riemannian geometry

Results on $\mathcal{NU}(M,g)$

Theorem

For every complete Riemannian manifold (M,g), the set $\mathcal{NU}(M,g) \subset (M \times M) \setminus \Delta_M$ is locally contractible. Therefore the set $\mathcal{NU}(M,g)$ is locally path connected.

Theorem

If C is a closed subset of the complete Riemannian manifold (M,g), then the inclusion $\Sigma^*(d_C) \subset M \setminus (C \cup A^*(C))$ is a homotopy equivalence.

Theorem

For every compact connected Riemannian manifold M, the inclusion $\mathcal{NU}(M,g) \subset (M \times M) \setminus \Delta_M$ is a homotopy equivalence. Therefore the set $\mathcal{NU}(M,g)$ is path connected.

- The simplest Tonelli Hamiltonian $H_g = T^*M \to \mathbb{R}$ on the complete Riemannian manifold (M, g) is given by $H_g(x, p) = \frac{1}{2}|p|_x^2$. Its associated Lagrangian $L_g : TM \to \mathbb{R}$ is given by $L_g(x, v) = \frac{1}{2}|v|_x^2$.

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- If $C \subset M$, we define its (modified) characteristic function $\chi_C : M \to \{0, +\infty\}$:

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- If $C \subset M$, it is not difficult to see that the lower semi- continuous regularization of the characteristic function χ_C is precisely characteristic function $\chi_{\overline{C}}$, where $\overline{C}$ is the closure of C in M.

- Since d_C and d_C^2 shares the singularities $M \setminus \overline{C}$. Thus $\Sigma^*(d_C)$ coincides with the singular set with respect to d_C (outside the C).

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- Similar argument can be applied to $(M \times M, g \times g)$.

Remarks & possible extension

Analytic aspects I

we can summarize the basic idea to prove the global propagation of singularities as follows:

(i) We need a representation formulae for the viscosity solutions of certain problems in the form of inf-convolution.

(ii) We need the regularity properties of the associated fundamental solutions.

(iii) An argument using sup-convolution can be applied to get the result of propagation of singularities.

(iv) We need show that the arc obtained is a generalized characteristic on a time interval $[0, t_0]$ which can be extended to $+\infty$ if we can have some uniform property of t_0.

A1. Can technique points (i)-(iv) be applied to various type of problems?

Progress for Problem A1

- This method is successfully applied to the Dirichlet problem ([Cannarsa-C-Wang-Mazzola, SIMA, to apperear]).

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- This work is under preparation using the program above and recent works on certain contact type Hamilton-Jacobi equations initiated by Wang, Wang and Yan and the one in Lagrangian formalism in [Cannarsa-C-Wang-Yan, Springer INdAM series, 2019].
- In recent paper [Chen-C-Zhang, JDE, 2018], the discounted equation is also studied (but not the global case).

- We consider the equation (HJ_c) using the Herglotz' generalized variational principle[8] which is a suitable Lagrange formalism for (HJ_c).

[8] Cannarsa, P.; Cheng, W.; Wang, K.; Yan, J. *Herglotz' generalized variational principle and contact type Hamilton-Jacobi equations*, Springer INdAM volume, *Trends in Control Theory and Partial Differential Equations*, 2019

- We consider the equation (HJ_c) using the Herglotz' generalized variational principle[8] which is a suitable Lagrange formalism for (HJ_c).
- Herglotz' generalized variational principle for (HJ_c) is: Let $x, y \in \mathbb{R}^n$, $t > 0$ and $u_0 \in \mathbb{R}$. Set

$$\Gamma^t_{x,y} = \{\xi \in W^{1,1}([0,t], \mathbb{R}^n) : \xi(0) = x,\ \xi(t) = y\}.$$

We consider a variational problem

$$\text{Minimize} \quad u_0 + \inf \int_0^t L(\xi(s), u_\xi(s), \dot{\xi}(s))\, ds, \tag{0.1}$$

where the infimum is taken over all $\xi \in \Gamma^t_{x,y}$ such that the Carathéodory equation

$$\dot{u}_\xi(s) = L(\xi(s), u_\xi(s), \dot{\xi}(s)), \quad a.e.\ s \in [0,t], \tag{0.2}$$

admits an absolutely continuous solution u_ξ with initial condition $u_\xi(0) = u_0$.

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Theorem (Hong-C-Hu-Zhao, 2019)

Suppose L satisfies condition (L1)-(L3) *and* (L6) *and H is the associated Hamiltonian, and* (HJ_c) *has a Lipschitz viscosity solution $u(x)$, then the following representation formula holds*

$$u(x) = \inf_{\xi \in \mathcal{A}^*_{\infty,x}} \int_{-\infty}^{0} e^{\int_s^0 L_u(\xi,\dot{\xi},u_\xi)\,d\tau} (L(\xi,\dot{\xi},u_\xi) - u_\xi \cdot L_u(\xi,\dot{\xi},u_\xi))\,ds, \tag{0.3}$$

where u_ξ satisfies (0.2) *with $u_\xi(0) = u(\xi(0)) = u(x)$ for all $t > 0$. Moreover, the infimum in* (0.3) *can be achieved.*

This representation formula allows us to study the problem of singularities for equation (HJ_c).

For more general results, see
Cannarsa, P.; Cheng, W.; Jin, L.; Wang, K.; Yan, J. *Herglotz' variational principle and Lax-Oleinik evolution*, preprint, arXiv:1907.05769, 2019.

Analytic aspects II

A2. What is the essential conditions for a Hamiltonian H having uniqueness property?

A3. Can we drop the uniformness requirement of such t_0 to obtain a global result?

A4. Can one improve the program (i)-(iv) using a $C^{1,\alpha}$ ($\alpha \in (0, 1)$) argument for certain problems involving state constraint?

Analytic aspects III

The dynamical explanation of Lasry-Lions regularization in classical weak KAM setting first given by Pernard. The relation among Lasry-Lions regularization, Lax-Oleinik semigroup and generalized characteristics is first studied in [Chen-C, Sci. China Math., 2016] and then in [Chen-C-Zhang, JDE, 2018] for the system with a positive discount factor.

A5. How about the Lasry-Lions regularization for state constraint type problems?

Analytic aspects III

In [Cannarsa-Yu. JEMS, 2009], local propagation results for arbitrary pair (u, H) is studied. They also studied partial differential inclusion on the propagation of singularities.

A6. How about the intrinsic nature on the problem of global propagation of singularities on a pair (u, H), especially at a critical point?

A7. How about the intrinsic nature on the partial differential inclusions of generalized characteristics?

Dynamic, topological and geometric aspects I

In [Cannarsa-Chen-C, JDE, 2019], an interesting result is the relations between the ω-limit set of the relevant singular semiflow on $\mathbb{T}^n$ and the Aubry set. It is possible that the singularities of a weak KAM solution evolute along the generalized characteristics approaching the Aubry set.

B1. What is the dynamical nature of the invariant measures produced by the semiflow ϕ_t?

B2. How about the dynamic and topological structure of the supports of such invariant measures produced by the semiflow ϕ_t?

B3. Are there some finer properties on $\mathbb{T}^2$?

Dynamic, topological and geometric aspects II

B4. What is the nature of the existence or non-existence of the critical points?

B5. Is there a curvature condition characterizing the non-existence of critical point even for the classical mechanical systems, like what used in Cheeger-Gromoll's splitting theorem in Riemannian geometry?

B6. Let u be a weak KAM solution with respect to a Tonelli Hamiltonian H. Invoking problem **A6**, at a critical point with respect to u, how should we change H to understand the further propagation of singularities of u? Is this a way to solve problem **A7**?

Dynamic, topological and geometric aspects III

The propagation of singularities on Mather's barrier function is studied in [Cannarsa-C-Zhang, CMP, 2014] and the relation between critical points of Mather's barrier function and the homoclinic orbits for Aubry sets is studies in [Cannarsa-C, Nonlinearity, 2015].

B7. Can we get more information, by using the intrinsic kernel in the process of Lasry-Lions regularization for the Mather's barrier function, to obtain the dynamical results from the critical points of the barrier functions?

Dynamic, topological and geometric aspects IV

Recalling the results in [Cannarsa-Peirone, TAMS, 2001], for the distance function d_F with respect to a closed subset $F \subset \mathbb{R}^n$, some amazing results on the asymptotic properties of the unbounded component of $\mathrm{Sing}\,(d_F)$ were obtained.

B8. What is the analogy of these results and how about the extensions for weak KAM solutions?

A reference for more detailed explanation of the problems raised here is

Cannarsa, P.; Cheng, W., On and beyond propagation of singularities of viscosity solutions, arXiv:1805.11583, 2018.

Gracias por su atención!