Existence, uniqueness and numerical investigation of segregation models

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VIII Partial differential equations, optimal design and numerics
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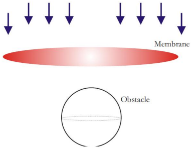
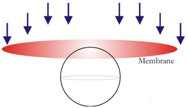
Outline:

- One Phase Obstacle
- Description of model
- Related Systems
- Existence and Uniqueness
- Analysis and asymptotic behaviour of systems
- Numerical schemes.


Membrane
Obstacle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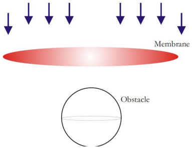

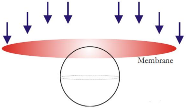

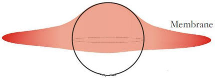

A. Petrosyan, H. Shahgholian, Nina Uraltseva,*Regularity of free boundaries in obstacle-type problems.* Grad. Stud. Math., vol. 136, AMS (2012).

Models in spatial segregation:

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3- Singularly perturbed elliptic systems

Adjacent segregation model **Problem (A):**

- Let m be a fixed integer. We call the m-tuple $U = (u_1, \cdots, u_m) \in (H^1(\Omega))^m$, *pairwise segregated states* if
$$u_i(x) \cdot u_j(x) = 0, \text{ a.e. for } \quad i \neq j, \ x \in \Omega.$$
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- Let $\Omega \subset \mathbb{R}^d$ be a connected, bounded domain with smooth boundary.
- The density of i-th component $u_i(x) : i = 1, \cdots, m$ with the internal dynamic is prescribed by f_i.
- The steady-states of m competing components in Ω is given by

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j \neq i}^{m} a_{ij} \ u_j^\varepsilon(x) + f_i(x, u_i^\varepsilon(x)) & \text{in } \Omega \\ u_i \geq o & \text{in } \Omega \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega. \end{cases} \quad (1)$$

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- The boundary values ϕ_i are non-negative and have disjoint supports on the boundary, i.e,
$$\phi_i \cdot \phi_j = 0 \quad \text{on} \quad \partial\Omega.$$

L. Caffarelli, F. Lin, *Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries*, J. Amer. Math. Soc. **21,** no. 3, 847–862, (2008).

M. Conti, S. Terracini, and G. Verzini, *Asymptotic estimate for spatial segregation of competitive systems*, Advances in Mathematics. **195,** 524-560, (2005).

An optimal partition problem

- Given Ω we are looking for m-partition $(\Omega_1, \Omega_2, \cdots \Omega_m)$ such that it minimize the following

$$\inf_{(\Omega_1, \Omega_2, \cdots, \Omega_m)} \sum_{i=1}^{m} \lambda_1(\Omega_i).$$

- Here $\lambda_1(D)$ is the first eigenvalue of $-\Delta$ in D with zero boundary condition.
- It can be reformulate as

$$\text{Minimize } E(u_1, \cdots, u_m) = \sum_{i=1}^{m} \int_\Omega |\nabla u_i|^2 \, dx,$$

over the set

$$K = \{(u_1, \ldots, u_m) \in (H_0^1(\Omega))^m : u_i \cdot u_j = 0 \text{ for } i \neq j, \ \|u_i\|_{L^2(\Omega_i)} = 1\}.$$

An Optimal Partition problem

If $(u_1, u_2, \cdots, u_m)$ minimizes E on K and

$$\Omega_i = \{x \in \Omega : u_i > 0\}$$

is a good candidate to be an optimal partition.
To penalization the condition $u_i \cdot u_j = 0$

$$E^\varepsilon = \sum_{i=1}^{m} \int_\Omega |\nabla u_i|^2 + \frac{1}{\varepsilon} \int_\Omega \sum_{j<i} u_i^2 u_j^2 \, dx$$

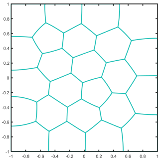
Over the set over the set

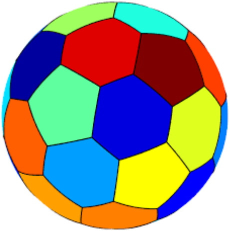
$$K' = \{(u_1, \ldots, u_m) \in (H_0^1(\Omega))^m : \|u_i\|_{L^2(\Omega_i)} = 1\}.$$

The minimizer satisfies

$$\begin{cases} -\Delta u_i^\varepsilon = \lambda_i u_i^\varepsilon - \frac{1}{\varepsilon} u_i^\varepsilon \sum_{j \neq i}^{m} (u_j^\varepsilon)^2 & \text{in } \Omega \\ u_i^\varepsilon \geq 0 & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Some references for numerics optimal partition problem





D. Bucur, G. Buttazzo, and A. Henrot, *Existence results for some optimal partition problems.* Adv. Math. Sci. Appl. 8 (1998), no. 2, 571–579.

B. Bourdin, D. Bucur, and É. Oudet. *Optimal partitions for eigenvalues.* SIAM J. Sci. Comput.31(2009), 4100–4114.

B. Helffer, *On spectral minimal partitions: a survey.* Milan J. Math. 78 (2010), no. 2, 575–590

F. Bozorgnia, *Optimal partitions for first eigenvalues of the Laplace operator.* NMPDE, 31 (2015) 923-949.

B. Bogosel, D. Bucur, and I. Fragalà,*Phase Field Approach to Optimal Packing Problems and Related Cheeger Clusters.* Appl Math Optim (2018), 1-25.

Adjacent segregation model (B)

Problem (B): Consider the following minimization problem

$$\text{Minimize } E(u_1, \cdots, u_m) = \int_\Omega \sum_{i=1}^{m} \left(\frac{1}{2} |\nabla u_i|^2 + f_i u_i \right) dx,$$

over the set

$$K = \{(u_1, \ldots, u_m) \in (H^1(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0 \text{ in } \Omega, \text{ for } i \neq j,\ u_i = \phi_i \text{ on } \partial\Omega\}.$$

Here $\phi_i \cdot \phi_j = 0$, $\phi_i \geq 0$ on the boundary $\partial\Omega$. Also we assume that f_i is uniformly continuous and $f_i(x) \geq 0$.

F. Bozorgnia, A Arakelyan, *Numerical algorithms for a variational problem of the spatial segregation of reaction-diffusion systems.* Applied Mathematics and Computation 219, (2013) 8863-8875.

M. Conti, S. Terracini, and G. Verzini, *A varational problem for the spatial segregation of reaction-diffusion systems*, Indiana Univ. Math. J. **54**, no 3, (2005) 779–815.

Different cases for minimization Problem (B)

- $m = 1$: One phase Obstacle problem

$$\text{Minimize } E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + f\, u \right) dx,$$

over the admissible set $K = \{u \in H^1(\Omega) : u \geq 0, u = \phi \quad \text{on} \quad \partial\Omega\}$.

- $m = 2$: Two-phase membrane problem

$$E(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + f_1 \max(v, 0) - f_2 \min(v, 0) \right) dx,$$

over

$$K = \{v \in H^1(\Omega),\ v = g \text{ on } \partial\Omega,\ \ g \text{ changes sign on } \partial\Omega.\}$$

- Minimizer solves

$$\begin{cases} \Delta u = f_1 \chi_{\{u>0\}} - f_2 \chi_{\{u<0\}} & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

- In $E(v)$ set $u_1 = v^+ = \max(v, 0) \quad u_2 = v^- = \max(-v, 0)$ then

$$E(v) = E(u_1, u_2) = \int_\Omega \left(\frac{1}{2}(|\nabla u_1|^2 + |\nabla u_2|^2) + f_1 u_1 + f_2 u_2 \right) dx.$$

G.S . Weiss, *An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary.* Interfaces Free Bound 2001, 3:121-128.

Segregation at distance

- System has similarity with system in Problem (A)
- But: Annihilation of coefficients for $u_1(x)$ is based on values on u_2 in full neighborhood so $\rightarrow$ we have to prescribe u_1 and u_2 in a neighborhood of Ω.

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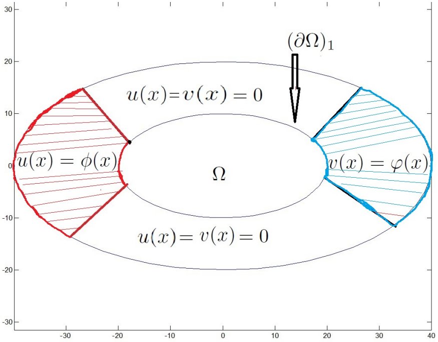

The Model of segregation at distance

The model is described by the following system

$$\begin{cases} -\Delta u_i^\varepsilon(x) = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j \neq i} H(u_j^\varepsilon)(x) & x \in \Omega, \\ u_i(x) = \phi_i(x) & x \in (\partial\Omega)_1, \\ i = 1, \cdots, m. \end{cases} \tag{2}$$

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where

$$H(u_j^{\varepsilon})(x) = \int_{B_1(x)} u_j^{\varepsilon}(y) dy,$$

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Assumptions: $\phi_i(x)$ for $i = 1, \cdots, m$ are non-negative $C^{1,\alpha}$ functions such that have disjoint supports in distance more than two

$$(supp\, \phi_i(x))_1 \cap (supp\, \phi_j(x))_1 = \emptyset.$$

L. Caffarelli, S. Patrizi, and V. Quitalo, *On a long range segregation model.* J. Eur. Math. Soc. 19,(2017) 3575-3628.

F. Bozorgnia, *Uniqueness result for long range spatially segregation elliptic system. Acta Applicandae Mathematicae, (2017), 1-14.*

A class of Singular Perturbed Elliptic system:

- The m-tuple $U = (u_1, \cdots, u_m)$ are called mutually segregated if

$$\prod_{j=1}^{m} u_j(x) = 0 \quad x \in \Omega.$$

- Consider the following system,

$$\begin{cases} \Delta u_i^\varepsilon = \frac{A_i(x)}{\varepsilon} F(u_1^\varepsilon, \cdots, u_m^\varepsilon) & \text{in } \Omega, \\ u_i^\varepsilon \geq 0, & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where

$$F(u_1, \cdots, u_m) = \prod_{j=1}^{m} u_j^{\alpha_j}, \quad \alpha_i \geq 0.$$

- (A1) ϕ_i are non-negative $C^{1,\alpha}$ and $\prod_{i=1}^{m} \phi_i = 0$ on $\partial\Omega$.
- (A2) The functions $A_i(x)$ are smooth, nonnegative and

$$A_i(x) \leq \sum_{j \neq i} A_j(x)$$

.

Aim: Existence, Uniqueness and numerical simulation for Systems (1), (2)and (3) for fixed ε and the limit as ε tends to zero.

Modeling

- The system (3) and the limiting system for $\epsilon \downarrow 0$ appear in theory of flames and are related to a model called Burke-Schumann approximation.
- Oxidizer and reactant mix on a thin sheet and the flame precisely occurs there.
- Introduce a large parameter called Damköhler number, denoted by D_a, which is the parameter measuring the intensity of the reaction
- Then, the a chemical reaction is described by

$$\text{Oxidizer} + \text{ Fuel} \rightarrow \text{Products}.$$

Let Y_O and Y_F, respectively, denote the mass fraction of the oxidizer and the fuel:

$$\begin{cases} -\Delta Y_O + v(x).\nabla Y_O = D_a\, Y_O\, Y_F & \text{in } \Omega, \\ -\Delta Y_F + v(x).\nabla Y_F = D_a\, Y_O\, Y_F & \text{in } \Omega, \end{cases}$$

with given incompressible velocity field v and a Dirichlet boundary condition on $\partial\Omega$.

L. Caffarelli and J. Roquejoffre, *Uniform Hölder estimate in a class of elliptic systems and applications to singular limits in models for diffusion flames,* Arch. Ration. Mech. Anal. **183,** no. 3, (2007) 457–487.

F. Bozorgnia, M. Burger, *On a Class of Singularly Perturbed Elliptic Systems with Asymptotic Phase Segregation.* arXiv(2019):1901.08750.

Existence and Uniqueness

Theorem

For each $\varepsilon > 0$*, there exist a unique positive solution* $(u_1^\varepsilon, \cdots, u_m^\varepsilon)$ *of system in (1), (2) and (3).*

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Sketch of the Proof(for System 3)

- Consider the harmonic extension u_i^0 for $i = 1, \cdots, m$ given by

$$\begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega, \\ u_i^0 = \phi_i & \text{on } \partial\Omega, \end{cases} \tag{4}$$

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$$\begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega, \\ u_i^0 = \phi_i & \text{on } \partial\Omega, \end{cases} \tag{4}$$

- Given u_i^k, consider the solution of the following linear system for system (1)

$$\begin{cases} \Delta u_i^{k+1} = \frac{A_i(x)}{\varepsilon} \frac{u_1^k \cdots u_{i-1}^k u_i^{k+1} u_{i+1}^k \cdots u_m^k + u_1^{k+1} \cdots u_{i-1}^{k+1} u_i^{k+1} u_{i+1}^k \cdots u_m^k}{2} & \text{in } \Omega, \\ u_i^{k+1}(x) = \phi_i(x) & \text{on } \partial\Omega. \end{cases} \tag{5}$$

- The following inequalities hold:

$$u_i^0 \geq u_i^2 \geq \cdots \geq u_i^{2k} \geq \cdots \geq u_i^{2k+1} \geq \cdots \geq u_i^3 \geq u_i^1, \quad \text{in } \Omega,$$

which implies

$$u_i^{2k} \to \overline{u}_i \quad \text{and} \quad u_i^{2k+1} \to \underline{u}_i \quad \text{uniformly in } \Omega.$$

Existence and Uniqueness

- We have : $\overline{u}_i \geq \underline{u}_i$.

Existence and Uniqueness

- We have : $\overline{u}_i \geq \underline{u}_i$. We will show that in fact the equality holds. To do this, first consider the equations for the m^{th}

$$\begin{cases} \Delta \overline{u}_m = \frac{A_m(x)}{2\varepsilon}\, \overline{u}_m \left(\overline{u}_1 \cdots \overline{u}_i\, \overline{u}_{i+1} \cdots \overline{u}_{m-1} + \underline{u}_1 \cdots \underline{u}_i \underline{u}_{i+1} \cdots \underline{u}_{m-1} \right) & \text{in } \Omega, \\ \Delta \underline{u}_m = \frac{A_m(x)}{2\varepsilon}\, \underline{u}_m \left(\underline{u}_1 \cdots \underline{u}_i \underline{u}_{i+1} \cdots \underline{u}_{m-1} + \overline{u}_1 \cdots \overline{u}_i\, \overline{u}_{i+1} \cdots \overline{u}_{m-1} \right) & \text{in } \Omega, \\ \overline{u}_m = \underline{u}_m = \phi_m(x) & \text{on } \partial\Omega, \end{cases} \tag{6}$$

  which implies

$$\overline{u}_m = \underline{u}_m.$$

- The argument is repeated backward which shows equality for every i.
- Assume there exist another positive solution $(w_1, \cdots, w_n)$, then by induction:

$$u_i^{2k+1} \leq w_i \leq u_i^{2k}, \quad \text{for } \ k \geq 0, \tag{7}$$

  which shows

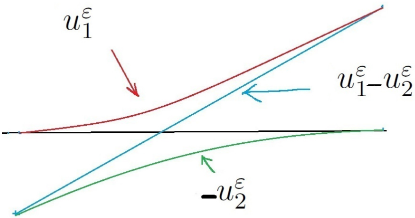
$$u_i = w_i.$$

Goal: Analyze of Problem (A) as $\varepsilon \to 0$ in first model

Assume $a_{ij} = 1, f_i(x, u_i) = 0$. The case of two components $m = 2$:

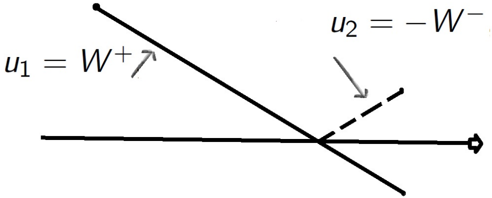
$$\begin{cases} \Delta u_1^\varepsilon = \frac{1}{\varepsilon} u_1^\varepsilon(x) u_2^\varepsilon(x) & \text{in } \Omega \\ \Delta u_2^\varepsilon = \frac{1}{\varepsilon} u_2^\varepsilon(x) u_1^\varepsilon(x) & \text{in } \Omega \\ + \text{ Boundary conditions.} \end{cases}$$

Easy fact: $\Delta(u_1^\varepsilon - u_2^\varepsilon) = 0, \quad \forall \varepsilon$. This remains true when ε tends to zero.



Theorem Let W be harmonic with the boundary data $\phi_1 - \phi_2$. Let $u_1 = W^+, u_2 = -W^-$, then the pair(u_1, u_2) is the limit configuration of any sequences $(u_1^\varepsilon, u_2^\varepsilon)$ and:

$$\| u_i^\varepsilon - u_i \|_{H^1(\Omega)} \leq C(\varepsilon)^{1/6} \text{ as } \varepsilon \to 0, \quad i = 1, 2.$$



M. Conti, S. Terracini, and G. Verzini, *Asymptotic estimate for spatial segregation of competitive systems*, Advances in Mathematics. **195,** 524-560, (2005).

Goal: study the system as $\varepsilon \to 0$ in model 1

Theorem1[CTV]:
Let $U^\varepsilon = (u_1^\varepsilon, ..., u_m^\varepsilon)$ be the solution of system at fixed ε. Let $\varepsilon \to 0$, then there exists $U \in (H^1(\Omega))^m$ such that for all $i = 1, \cdots, m$:

1. up to a subsequences, $u_i^\varepsilon \to u_i$ strongly in $H^1(\Omega)$,
2. $u_i \cdot u_j = 0$ if $i \neq j$ a.e in Ω,
3. $\Delta u_i = 0$ in the set $\{u_i > 0\}$.
4. Let x belongs to interface such that $m(x) = 2$ then

$$\lim_{y \to x} \nabla u_i(y) = - \lim_{y \to x} \nabla u_j(y) \quad \text{Free boundary condition.}$$

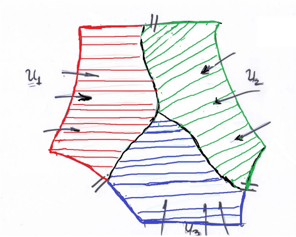


Figure:

Relation between problem A and B for $m = 3$

- The case $m = 3$: Uniqueness of the limiting configuration as ε tends to zero on a planar domain, with appropriate boundary conditions

$$-\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j \neq i}^{m} u_j^\varepsilon(x) \quad i = 1, 2, 3,$$

- Moreover the limiting configuration minimizes

$$\text{Minimize } E(u_1, u_2, u_3) = \int_\Omega \sum_{i=1}^{3} \left(\frac{1}{2} |\nabla u_i|^2 \right) dx,$$

among all segregated states $u_i \cdot u_j = 0$ a.e. with the same boundary conditions.

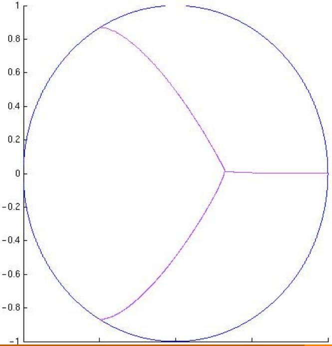
M. Conti, S. Terracini, and G. Verzini, *Uniqueness and least energy property for solutions to strongly competing systems.* Interfaces and Free Boundaries 8 (2006), 437-–446.

Examples for the first model

- Let $\Omega = B_1, m = 3$. The boundary values ϕ_i for $i = 1, 2, 3$ are

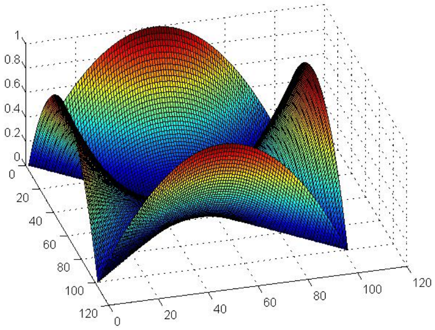
$$\phi_1(1,\Theta) = \begin{cases} |\sin(\frac{3}{2}\Theta)| & 0 \le \Theta \le \frac{2\pi}{3} \\ 0 & \text{elsewhere} \end{cases} \quad \phi_2(1,\Theta) = \begin{cases} |\sin(\frac{3}{2}\Theta)| & \frac{2\pi}{3} \le \Theta \le \frac{4}{3} \\ 0 & \text{elsewhere.} \end{cases}$$

$$\phi_3(1,\Theta) = \begin{cases} 4|\sin(\frac{3}{2}\Theta)| & \frac{4\pi}{3} \le \Theta \le 2\pi, \\ 0 & \text{elsewhere.} \end{cases}$$

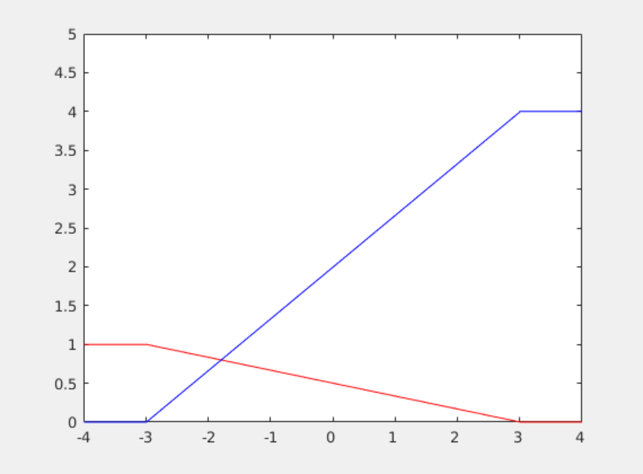


Example

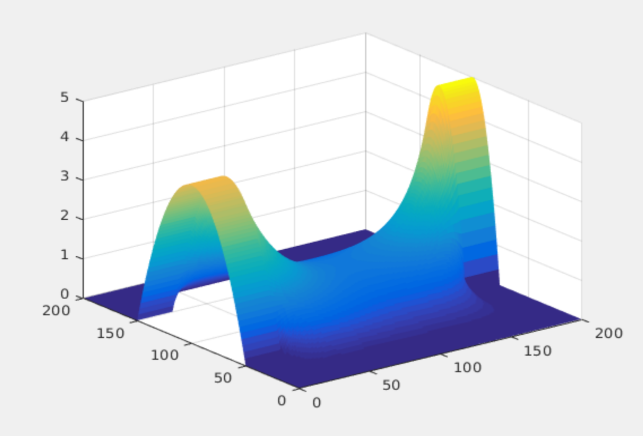
- we applied second method with $\Omega = [0,1] \times [0,1]$ $,\phi_1 = 1 - x^2, \phi_2 = 1 - y^2, \phi_3 = 1 - x^2, \phi_4 = 1 - y^2$



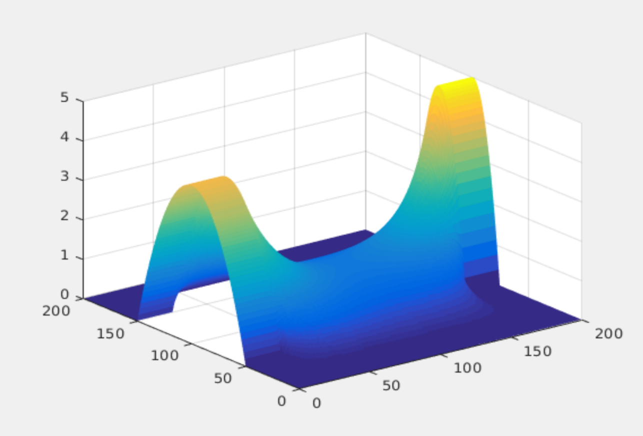
1D segregation example



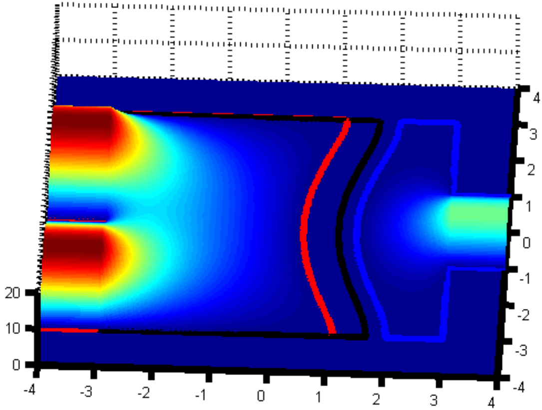
2D segregation example



2D segregation example



Relation between interfaces in model (1) and (2)



Examples for Singular Perturbed system

Let $\Omega = B_1, m = 3$. The boundary values ϕ_i for $i = 1, 2, 3$ are defined by

$$\phi_1(1,\Theta) = \begin{cases} |\sin(\frac{3}{2}\Theta)| & 0 \le \Theta \le \frac{4\pi}{3}, \\ 0 & \text{elsewhere}, \end{cases} \qquad \phi_2(1,\Theta) = \begin{cases} |\sin(\frac{3}{2}\Theta)| & \frac{2\pi}{3} \le \Theta \le 2\pi, \\ 0 & \text{elsewhere}. \end{cases}$$

$$\phi_3(1,\Theta) = \begin{cases} |\sin(\frac{3}{2}\Theta)| & \frac{4\pi}{3} \le \Theta \le 2\pi + \frac{2\pi}{3}, \\ 0 & \text{elsewhere}. \end{cases}$$

Here the boundary conditions satisfy

$$\phi_1 \cdot \phi_2 \cdot \phi_3 = 0.$$

Example for the singular perturbed system

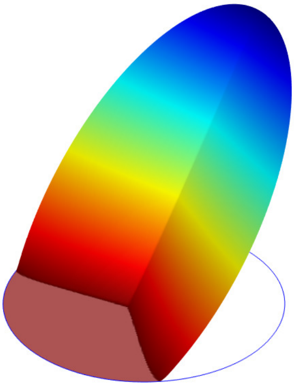


Figure: surface of u_1

Example for the singular perturbed system

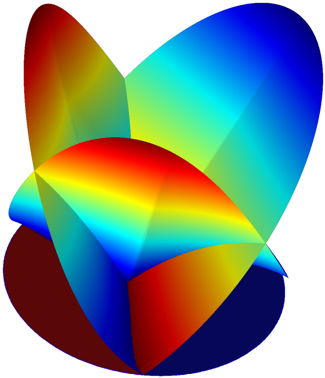


Figure: $u_1 + u_2 + u_3$.

Thanks for your attention