Fluid-Structure models arising in blood-flow models

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Outline

1. Introduction
2. Linearized system
3. Main results
4. Future direction of work
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**Motivation**: Blood flow in large arteries. Viscous fluid interacts with a thin elastic structure located on one part of the fluid domain.

The fluid domain depends on the structure displacement. We have a free boundary value problem.
The reference configuration:

\[
\Omega = \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1 \in (0, L), \sqrt{z_2^2 + z_3^2} \leq 1 \right\}
\]

\(\Gamma_s\) is the lateral boundary, which is deformable. \(\Gamma_{in}\) and \(\Gamma_{out}\) are inflow and outflow boundaries respectively.

Current configuration: Let \(\vec{d}(t, \cdot)\) displacement of the shell from the reference configuration \(\Gamma_s\). Displacement is only in the radial direction. Thus \(\vec{d}(t, z_1, \theta) = \eta(t, z_1, \theta)e_r(\theta)\).

\[
\Omega_{\eta(t)} = \left\{ (z_1, x, y) \in \mathbb{R}^3 \mid z_1 \in (0, L), \sqrt{x^2 + y^2} \leq 1 + \eta(t, \cdot) \right\},
\]

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\]
Governing equations

- **Fluid equation**: The fluid is Newtonian, viscous and incompressible. The fluid velocity $u$ and pressure $p$ satisfy
  \[
  \rho_f \left( \partial_t u + u \cdot \nabla u \right) - \text{div} \sigma(u, p) = 0, \quad \text{div} \, u = 0 \quad \text{in} \ (0, T) \times \Omega_{\eta(t)},
  \]
  where $\sigma(u, p) = (\nabla u + \nabla u)^\top - pl$.

- **Boundary conditions**:
  \[
  \sigma(u, p)n = 0 \quad \text{on} \ \Gamma_{\text{in}} \cup \Gamma_{\text{out}}.
  \]

- **Structure equation**: $\eta$ satisfies viscoelastic cylindrical nonlinear Koiter shell equation:
  \[
  \partial_{tt}\eta + \mathcal{L}_{\text{mem}} \eta + \Delta_s^2 \eta - \beta_2 \Delta_s \partial_t \eta = \mathcal{H}(u, p, \eta) \quad \text{on} \ \Gamma_s,
  \]

- **
  \[
  \eta = \frac{\partial \eta}{\partial n} = 0 \quad \text{on} \ \partial \Gamma_{\text{in}} \cup \partial \Gamma_{\text{out}}.
  \]
Interface conditions

Coupling between the fluid and the structure is expressed through the kinematic and dynamic lateral boundary conditions:

• Continuity of the velocity (the no-slip condition) at the interface $\Gamma_{\eta}$

$$u = \partial_t \eta e_r \text{ on } \Gamma_{\eta}$$

• Balance of the contact forces at the interface

$$\mathcal{H}(u, p, \eta) = -J(\sigma(u, p)\tilde{n})|_{\Gamma_{\eta}} \cdot e_r,$$

$\tilde{n}$ is the unit normal to $\Gamma_{\eta}$.

Goal: To study existence and uniqueness of strong solutions in $L^2$ framework.
State of the art

- **Strong Solution and 2D/1D model:**
  - Structure equation: \( \partial_{tt}\eta + \alpha \partial_{xxxx}\eta - \beta \partial_{xx}\eta - \gamma \partial_{txx}\eta = H. \)
  - fluid boundary conditions at the inlet and outlet.

- **Local in time existence:** Lequeurre (11 and 13), Casanova (18), Grandmont, Hilariet and Lequeurre (18), Badra and Takahashi (19), Djebour and Takahashi (19), ...
  - \( \gamma = 0 \) and periodic boundary condition at the inlet/outlet.
  - \( \gamma > 0 \), Dirichlet / pressure boundary conditions.

- **Global in time existence:** Grandmont and Hilariet (2016), \( \gamma > 0 \), \( \alpha > 0 \) and periodic boundary conditions.

**Our result:** Local in time existence with \( \gamma > 0 \), \( \alpha > 0 \) and Neumann boundary condition at the inlet/outlet.
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Monolithic approach:

- Rewrite the system in the fixed domain: Lagrangian or Geometric change of variables.
- System in fixed domain

\[ z'(t) = \mathcal{A}_{FS} z(t) + \mathcal{N}(z), \quad z(0) = z_0. \]

- Linearized FSI system in a suitable space \( \mathcal{X} \):

\[ z'(t) = \mathcal{A}_{FS} z(t) + f(t), \quad z(0) = z_0. \]

- Regularity of linear system.

- Fixed point argument. (local in time or global in time for small initial data)
Linearized problem in 2D/1D setting

\[ \Omega = (0, L) \times (0, 1), \Gamma_s = (0, L) \times \{1\}, \Gamma_{in} = \{0\} \times (0, 1) \text{ and } \Gamma_{out} = \{L\} \times (0, 1) \]

\[ \begin{cases} \partial_t u - \Delta u + \nabla p = f, \text{div} u = 0, & \text{in } \Omega, \\ u = \partial_t \eta e_2 & \text{on } \Gamma_s, \\ \sigma(u, p)n = 0 & \text{on } \Gamma_{in} \cup \Gamma_{out}, \\ \partial_{tt} \eta + \partial_{xxxx} \eta - \partial_{txx} \eta = p|_{\Gamma_s} + h & \text{in } \Gamma_s \end{cases} \]

- The linear fluid-structure operator generates an analytic semigroup.
- The fluid operator (with homogeneous BC) and the structure operator generates analytic semigroup.
- The coupling can be seen as compact perturbation.
• Remove pressure from the fluid and structure equation.
• Use Leray projector to remove the pressure from fluid equation.
  \[ \partial_t P u = A_F P u + B \partial_t \eta. \]
• The pressure can be written as
  \[ \Delta p = 0, \quad \frac{\partial p}{\partial n} = -\partial_{tt} \eta + \Delta u \cdot n \text{ on } \Gamma_s, \quad p = \varepsilon(u)n \cdot n \text{ on } \Gamma_{in/out}. \]

  Thus \( p = N_0(\partial_{tt} \eta) + N_1(u). \)
• The structure equation becomes:
  \[ (I + \gamma_s N_0) \partial_{tt} \eta - A_s = \gamma_s N_1(u). \]
• The operator \((I + \gamma_s N_0)\) is known as “added mass” operator and is invertible in \(L^2(\Gamma_s)\).
The fluid-structure operator

The system can be written as

\[
\frac{d}{dt} \begin{pmatrix} P u \\ \eta_1 \\ \eta_2 \end{pmatrix} = A_{FS} \begin{pmatrix} P u \\ \eta_1 \\ \eta_2 \end{pmatrix} + \text{source term}.
\]

\[
A_{FS} = \begin{pmatrix} I & 0 & B \\ 0 & 0 & I \\ (I + \gamma_s N_0)^{-1} & N_1(u) & -\Delta^2 & \Delta \end{pmatrix}
\]

- \( \mathcal{X} = L^2_\sigma(\Omega) \times H^2(\Gamma_s) \times L^2(\Gamma_s) \)
- \( \mathcal{D}(A_{FS}) \sim H^{3/2+\epsilon_0} \times H^4(\Gamma_s) \times H^2(\Gamma_s) \)
- Loss of regularity for fluid due to mixed boundary condition and the angle of Dirichlet-Neumann junction is \( \pi/2 \).
- Study the weak form of \( N_1(u) \) and to show it is a compact operator.
- \((u, \eta_1, \eta_2) \in L^2(0, T; \mathcal{D}(A_{FS})) \cap H^1(0, T; \mathcal{X})\).
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Main result

Theorem (DM, J.-P. Raymond, A. Roy)

Let \( \eta(0) = 0, \ (u_0, \partial_t \eta(0)) \in H^1(\Omega) \times H^1(\omega) \) with compatibility conditions. Then there exists a \( T > 0 \), depending only on the initial data such that the system admits a strong solution

\[
u \in L^2(0, T; H^{3/2+\varepsilon_0}(\Omega_{\eta(\cdot)})) \cap H^1(0, T; L^2(\Omega_{\eta(\cdot)})) \cap C([0, T]; H^1(\Omega_{\eta(\cdot)})),
\]

\[
p \in L^2(0, T; H^{1/2+\varepsilon_0}(\Omega_{\eta(\cdot)})), \quad \text{div} \ \sigma(u, p) \in L^2(\tilde{Q}_T),
\]

\[
\eta \in L^2(0, T; H^4(\omega)) \cap H^2(0, T; L^2(\omega)),
\]

\[
1 + \eta(t, \cdot) > 0, \quad t \in [0, T]
\]

for some \( \varepsilon_0 \in (0, 1/2) \).
$L^p - L^q$ regularity

- We look for solutions of fluid and structure in $L^p(0, T; L^q)$.
- The idea is the same: $L^p - L^q$ regularity of fluid and structure with compactness of the fluid-structure coupling.
- $L^p - L^q$ regularity is no longer characterised by analyticity of the linear semigroup. We need to show $\mathcal{R}$-sectoriality of the resolvent operator.

**Theorem (DM, T. Takahashi)**

The reference domain is smooth. Let us assume that $\frac{1}{p} + \frac{n}{2q} < \frac{3}{2}$. For suitable initial data with compatibility conditions, we have local in time existence of strong solutions:

- $u \in L^p(0, T; W^{2,q}) \cap W^{1,p}(0, T; L^q)$
- $\eta \in L^p(0, T; W^{4,q}) \cap W^{2,p}(0, T; L^q)$. 
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Future direction of work

- In the 3D case, can we remove the viscosity of the structure.
- Wave or damped wave.
- Global existence in 2D, without the damping term.
- Other fluid models: Compressible Navier-Stokes-Fourier.
Thank you very much.