Greedy Algorithm for Parameter Dependent Operator Lyapunov Equations.
Application to Control Problems

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VIII PDEs, optimal design and numerics
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Numerical examples
We consider a family of parameter-dependent operator Lyapunov equations

\[ A_\nu P_\nu + P_\nu A^{\ast}_\nu = -Q_\nu \]  

(OLE\(\nu\))

- \(\nu\) – a parameter ranging over compact set \(\mathcal{N} \subseteq \mathbb{R}^d\)
- \(A_\nu\) – an unbounded operator on a Hilbert space \(X\)
- \(Q_\nu\) – a bounded operator on \(X\), \(Q_\nu \geq 0\)
- \(P_\nu\) – the solution

**Problem**
Find the efficient algorithm for solving (OLE\(\nu\)) for a wide range of parameters.
Assumptions

For each $\nu$

- $D(A_\nu)$ is dense in $X$
- the operator $A_\nu$ is closed and stable

Then there exists a unique nonnegative solution $P \in \mathcal{L}(X)$

$$P_\nu = \int_0^\infty e^{tA_\nu} Q_\nu e^{tA^*_\nu} dt$$

Different methods for computing the solution.

- Bartels, Stewart *Comm. ACM*, 1972. - the Schur decomposition
- Saad (1990) - Krylov subspace methods
- Simoncini *SIAM Rev.*, 2016. - iterative methods

Computational expensive.

Can we construct the solution manifold

$$\mathcal{P} = \{P_\nu : \nu \in K\}$$

without applying the above methods for each new value of $\nu$?
The idea

To determine a finite number of values of $\nu$ that yield the best possible approximation of the solution manifold $\mathcal{P}$


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To determine a finite number of values of \( \nu \) that yield the best possible approximation of the solution manifold \( \mathcal{P} \)

In order to achieve this goal we rely on greedy algorithms and reduced bases methods for parameter dependent PDEs or abstract equations in Banach spaces.

The pure greedy method

\( X \) – a Banach space \( K \subset X \) – a compact subset.

- The method approximates \( K \) by a series of finite dimensional linear spaces \( V_n \) (a linear method).
- **Offline** procedure generates approximation subspace within given precision error; **Online** routine calculates approximations for any element in \( K \).

The algorithm

**The first step** Choose \( x_1 \in K \) such that

\[
\|x_1\|_X = \max_{x \in K} \|x\|_X.
\]

**The general step** Having found \( x_1..x_n \), denote \( V_n = \text{span}\{x_1, \ldots, x_n\} \).

Choose the next element

\[
x_{n+1} := \arg \max_{x \in K} \text{dist}(x, V_n).
\] (1)

**The algorithm stops** when \( \sigma_n(K) := \max_{x \in K} \text{dist}(x, V_n) \) becomes less than the given tolerance \( \varepsilon \).
In order to estimate the efficiency of the (weak) greedy algorithm we compare its approximation rates $\sigma_n(K)$ with the best possible one.

**The Kolmogorov $n$ width, $d_n(K)$**

- measures how well $K$ can be approximated by a subspace in $X$ of a fixed dimension $n$.

\[
d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{y \in Y} \|x - y\|_X.
\]

Thus $d_n(K)$ represents optimal approximation performance that can be obtained by a $n$-dimensional linear space. The greedy approximation rates have same decay as the Kolmogorov widths.

**Theorem**

(Cohen, DeVore ’15) \(^3\)

For any $\alpha > 0, C_0 > 0$

\[
d_n(K) \leq C_0 n^{-\alpha} \implies \sigma_n(K) \leq C_1 n^{-\alpha}, \quad k \in \mathbb{N},
\]

where $C_1 := C_1(\alpha, C_0, \gamma)$.

Performance obstacles

- The set \( K \) in general consists of infinitely many vectors.

- In practical implementations the set \( K \) is often unknown (e.g. it represents the family of solutions to parameter dependent problems).
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Practical realisations depends crucially on an existence of an appropriate surrogate!
Implementation: Residual Analysis

Knowing $P_1$ how to measure

$$\text{dist}(P_1 - P_\nu)$$

without knowing $P_\nu$?

Check residual

$$R_\nu(P_1) := A_\nu P_1 - P_1 A_\nu + B_\nu B_\nu^*$$
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**Theorem**

Suppose that

1) $A_\nu$ is sectorial, i.e it is a generator of an analytical semigroup ;
2) $D(A_{\nu_1}) = D(A_{\nu_2})$ and $D(A_{\nu_1}^*) = D(A_{\nu_2}^*)$ for $\nu_1, \nu_2 \in \mathcal{N}$.

Then

\[ \|R_\nu\| \sim \|P_1 - P_\nu\| \]
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$$||R_\nu|| \sim ||P_1 - P_\nu||$$

Tricky part - functional setting (norms in which spaces?)

Result in finite dimensional setting

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Then

$$\|R_\nu\|_{\mathcal{L}(X_1^d, X_{-1})} \sim \|P_1 - P_\nu\|_{\mathcal{L}(X_1^d, X)}$$
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2) $D(A_{\nu_1}) = D(A_{\nu_2})$ and $D(A_{\nu_1}^*) = D(A_{\nu_2}^*)$ for $\nu_1, \nu_2 \in \mathbb{N}$.

Then

$$||R_\nu||_{\mathcal{L}(X^d_{-1}, X)} \sim ||P_1 - P_\nu||_{\mathcal{L}(X^d_1, X)}$$

Collateral result:

**Theorem**

Lyapunov operator $L_A(P) = AP + PA^*$ is a bounded and coercive operator from $\mathcal{L}(X^d_1, X)$ to $\mathcal{L}(X^d_1, X_{-1})$. 
Control problem

Consider the control system

\[
\begin{cases}
\frac{d}{dt} x(t) &= Ax(t) + Bu(t), \quad 0 \leq t \leq T \\
x(0) &= x_0
\end{cases}
\]

where \( B \) is an admissible control operator.

Suppose that \( x_T \) is a reachable state.

Then the optimal norm control \( \hat{u} \) is of the type

\[
\hat{u} = B^* e^{(T-t)A^*} \phi_T
\]

for some vector \( \phi_T \) which corresponds to initial datum of the adjoint equation.

In addition, the following equation holds

\[
x_T - e^{tA} x_0 = \Lambda_T \phi_T,
\]

where \( \Lambda_T \) is the Gramian operator

\[
\Lambda_T = \int_0^T e^{tA} B B^* e^{tA} dt
\]

The minimal control energy is given by

\[
\|\hat{u}\|^2 = \Lambda_T \phi_T \cdot \phi_T.
\]
For dissipative systems $\Lambda_T$ can be well approximated by the infinite time Gramian operator.

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which is the solution to (OLE) with $Q = BB^*$

Solving for $\Lambda_\infty$ is much easier than constructing $\Lambda_T$ (which satisfies differential Lyapunov equation).

But we even want to avoid solving for $\Lambda_\infty$!
For dissipative systems $\Lambda_T$ can be well approximated by the infinite time Gramian operator.

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Solving for $\Lambda_{\infty}$ is much easier than constructing $\Lambda_T$ (which satisfies differential Lyapunov equation).

But we even want to avoid solving for $\Lambda_{\infty}$!

We introduce parameter dependence

\[
\begin{cases}
\frac{d}{dt} x_{\nu}(t) = A_{\nu} x_{\nu}(t) + B_{\nu} u_{\nu}(t), & 0 \leq t \leq T \\
x_{\nu}(0) = x_{0,\nu}
\end{cases}
\]

We apply the greedy algorithm for solving (approximately) $\Lambda_{\infty,\nu}$

The algorithm is independent of $x_0, x_T$ and $T$!
Example 1: 1D Heat Equation

\[
\begin{aligned}
\frac{\partial}{\partial t} z - \nu \Delta z &= 0 \quad \text{in} \quad (0, 1) \times (0, T), \\
z(0, t) &= 0, \\
z(x, 0) &= z_0.
\end{aligned}
\]

The parameter \( \nu \) ranges within \( \mathcal{N} = [0.7, 1300] \)

The greedy algorithm has been applied with

- discretized system of dimension \( N = 40 \),
- \( \epsilon = 0.01 \),
- uniform discretization of \( \mathcal{N} \) in \( l = 100 \).

The offline algorithm stops after only one iteration in approximately 0.06 seconds!
Example 1: 1D Heat Equation

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\begin{align*}
\frac{\partial}{\partial t} z - \nu \Delta z &= 0 \quad \text{in} \quad (0, 1) \times (0, T), \\
z(0, t) &= 0, \quad z(1, t) = u_{\nu}(t), \\
z(x, 0) &= z_0.
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By change of variables:

\[
A_{\nu} = \nu A \quad \Rightarrow \quad \Lambda_{\infty, \nu} = \nu \Lambda_{\infty}
\]

(Holds just for \( T = \infty \))
Example 1: 1D Heat Equation - Online part

We aim to steer the system

- from $z_0 = 0$ to $z_1 = \sin(\pi x)$
- in time $T = 0.1$
- for $\nu = 23$

Calculation of the approximate Gramian is rather straightforward. It is applied for construction of the optimal control. It drives the system to final state $z^1$ within the error $|z^1 - z(T)| = 3.77 \times 10^{-5}$.

Figure: Evolution of a) the approximate control and b) the solution of semi-discretized example problem.
Example 2: Anisotropic 2D Heat Equation

\[
\frac{\partial}{\partial t} z - \Delta_\nu z = 0 \quad \text{in} \quad (0, 1)^2 \times (0, T),
\]
\[
z(x, t) = v_0(x, t), \quad \text{for} \quad x \in \partial([0, 1]^2)
\]
\[
z(x, 0) = 0
\]

\[
\Delta_\nu = \frac{\partial^2}{\partial x_1^2} + (1 + \nu) \frac{\partial^2}{\partial x_2^2}, \quad \nu \in \mathcal{N} = [0, 1]
\]

\[
v_0(x, t) = \begin{cases} 
  u_\nu(t), & x_1 = 1 \\
  0, & \text{otherwise}
\end{cases}
\]

The greedy algorithm has been applied for the discretized system of dimension \( N = 400 \) with \( \epsilon = 0.05 \), and the uniform discretization of \( \mathcal{N} \) in \( l = 40 \).

The offline algorithm stops after 12 iterations, choosing 12 parameter values out of 40 in a zigzag manner.
Example 2: Anisotropic 2D Heat Equation-cont.

We aim to steer the system
- from $z_0 = 0$ to $z_1 = \sin(\pi x) \ast \sin(\pi x_2)$
- in time $T = 1$
- for $\nu = 0.1$

$\Lambda_{\infty,\nu}$ is approximated by a suitable linear combination of $\Lambda_{\infty,i}, i = 1..12$.

Elapsed time is 0.21 s and the error is $|z^1 - z(T)| = 2.0 \times 10^{-4}$.

![Graph of $u(t)$](image1)

![Graph of states $z(T)$ and $z^1$](image2)

**Figure:** a) Evolution of the approximate control and b) the states $z(T)$ (dashed) and $z^1$. 
Conclusion

Done:
- Greedy algo for solving parameter dependent OLE
- Provides approximation of infinite time control Gramians (*independent of initial and final data, and final time!*)
- Enables construction of optimal controls for dissipative systems

Further work:
- **Differential** Lyapunov equation
- It would provides approximation of *finite time* control Gramians
- Enables construction of optimal controls for *non-dissipative* systems
Conclusion

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- Greedy algo for solving parameter dependent OLE
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Further work:

- Differential Lyapunov equation
- It would provides approximation of finite time control Gramians
- Enables construction of optimal controls for non-dissipative systems

Thanks for your attention!