Dynamics and control for the “Guidance by repulsion” model

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VIII Partial differential equations, optimal design and numerics

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1. The guidance-by-repulsion ODE model

2. Control of the guidance-by-repulsion model

3. Long-time behavior of the model

4. Numerical Simulations on optimal control strategies

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Motivation: Shepherd dogs and sheep

The number of individuals is small, yet the interaction dynamics and control strategies is complex

We consider the "guidance by repulsion" model based on the two-agents framework: the driver tries to drive the evader.

The drivers want to control the evaders:

1. Gathering of the evaders,
2. Driving the evaders into a desired area.

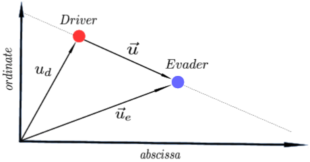


Figure: Picture of Border Collie [from Wikipedia] and the diagram of the model

Motivation: "Guidance by repulsion" model

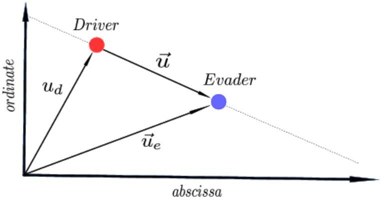
R. Escobedo, A. Ibañez and E.Zuazua, Optimal strategies for driving a mobile agent in a "guidance by repulsion" model, Communications in Nonlinear Science and Numerical Simulation, 39 (2016), 58-72.

[R. Escobedo, A. Ibañez, E. Zuazua, 2016] suggested a **guidance by repulsion** model based on the two-agents framework: *the driver*, which tries to drive the *evader*.

1. The driver follows the evader but cannot be arbitrarily close to it (because of chemical reactions, animal conflict, etc).
2. The evader moves away from the driver but doesn't try to escape beyond a not so large distance.
3. The driver is faster than the evader.
4. At a critical short distance, the driver can display a **circumvention maneuver** around the evader, forcing it to change the direction of its motion.
5. By adjusting the circumvention maneuver, the evader can be driven towards a desired target or along a given trajectory.

One sheep + one dog + Circumvention control

The control $k(t)$ is chosen in feedback form to align the gate, the sheep and the dog.



Inspired by this paper, the guidance-by-repulsion model for $\mathbf{u}_d, \mathbf{u}_e \in \mathbf{R}^2$ can be written with control functions $\kappa^p(t)$ and $\kappa^c(t)$:

$$\begin{cases} \dot{\mathbf{u}}_d = \mathbf{v}_d, \quad \dot{\mathbf{u}}_e = \mathbf{v}_e, \quad \mathbf{u} = \mathbf{u}_d - \mathbf{u}_e, \\ m_d \dot{\mathbf{v}}_d = -\kappa^p(t)\mathbf{u} + \kappa^c(t)\mathbf{u}^\perp - \nu_d \mathbf{v}_d, \\ m_e \dot{\mathbf{v}}_e = -f_e(|\mathbf{u}|)\mathbf{u} - \nu_e \mathbf{v}_e, \\ \mathbf{u}_d(0) = \mathbf{u}_d^0,\ \mathbf{u}_e(0) = \mathbf{u}_e^0,\ \mathbf{v}_d(0) = 0,\ \mathbf{v}_e(0) = 0, \end{cases} \tag{1}$$

where $f_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the strength of repulsion, for example, $f_e(r) = r^{-2}$.

Studies on the repulsive interactions

In [R. Escobedo, A. Ibañez, E. Zuazua, 2016], they considered bang-bang type controls with open-loop and feed-back strategies.

Similar consideration have been addressed with repulsive interactions in control theory:

- Defender-intruder strategy : [Wang, Li, 2015],
- Hunting strategy of wolves :
  [Muro, Escobedo, Spector, Coppinger, 2011 and 2014],
- Sheep-gathering problem :
  Well-posedness and Maximal principle of optimal control problems [Burger, Pinnau, Roth, Totzeck, Tse, 2016]
  and its simulations [Pinnau, Totzeck, 2018].

Guidance-by-repulsion model with many individuals

Let $\mathbf{u}_{dj}, \mathbf{u}_{ei} \in \mathbb{R}^2$ are positions of drivers and evaders for $i = 1, \cdots, N$ and $j = 1, \cdots, M$. For many evaders, we assume that the drivers follow the barycenter of evaders,

$$\mathbf{u}_{ec} := \frac{1}{N} \sum_{k=1}^{N} \mathbf{u}_{ek},$$

then the dynamics can be described by

$$\begin{cases} \ddot{\mathbf{u}}_{dj} = -\kappa_j^p(t)(\mathbf{u}_{dj} - \mathbf{u}_{ec}) + \kappa_j^c(t)(\mathbf{u}_{dj} - \mathbf{u}_{ec})^{\perp} \\ \qquad\quad - \dfrac{1}{M} \displaystyle\sum_{k=1}^{M} \psi_d(|\mathbf{u}_{dk} - \mathbf{u}_{dj}|)(\mathbf{u}_{dk} - \mathbf{u}_{dj}) - \nu_{dj} \dot{\mathbf{u}}_{dj}, \\ \ddot{\mathbf{u}}_{ei} = -\dfrac{1}{M} \displaystyle\sum_{j=1}^{M} f_e(|\mathbf{u}_{dj} - \mathbf{u}_{ei}|)(\mathbf{u}_{dj} - \mathbf{u}_{ei}) \\ \qquad\quad - \dfrac{1}{N} \displaystyle\sum_{k=1}^{N} \psi_e(|\mathbf{u}_{ek} - \mathbf{u}_{ei}|)(\mathbf{u}_{ek} - \mathbf{u}_{ei}) - \nu_{ei} \dot{\mathbf{u}}_{ei}, \\ \mathbf{u}_{dj}(0) = \mathbf{u}_{dj}^0, \ \mathbf{u}_{ei}(0) = \mathbf{u}_{ei}^0, \ \dot{\mathbf{u}}_{dj}(0) = \mathbf{v}_{dj}^0, \ \dot{\mathbf{u}}_{ei}(0) = \mathbf{v}_{ei}^0. \end{cases}$$

A simulation

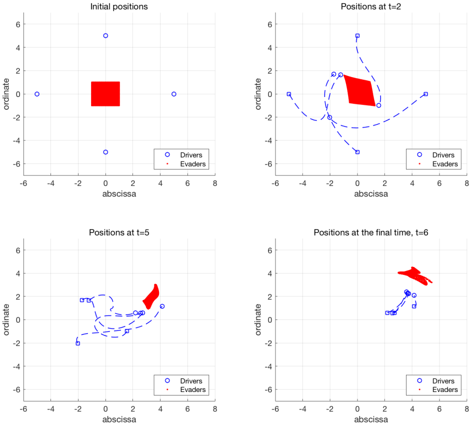


Figure: Trajectories of 4 drivers and 1024 evaders towards the point $(4, 4)$

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One driver and one evader: symmetric dissipation

To analyze the relative position, we assume the symmetric dissipation

$$\nu_e/m_e = \nu_d/m_d =: \nu > 0,$$

and we first consider constant controls

$$\kappa^p(t) \equiv 1, \quad \kappa^c(t) \equiv \kappa \in \mathbb{R}.$$

Then, $\mathbf{u} := \mathbf{u}_d - \mathbf{u}_e$ follows

$$\ddot{\mathbf{u}} + f(|\mathbf{u}|)\mathbf{u} + \nu\dot{\mathbf{u}} = \kappa\mathbf{u}^\perp.$$

where the interaction force $f(r) = 1 - f_e(r)$ deduces the potential:

$$P(r) := \int_{r_c}^{r} sf(s)ds, \quad r_c \geq 0.$$

$\mathbf{u}$ follows the damped oscillator under a central potential $P(|\mathbf{u}|)$ with control κ. Hence, we assume

$$P \geq 0, \quad P(0) = \infty \quad \text{and} \quad P \sim \frac{\gamma_m}{2}|\mathbf{u}|^2 \text{ as } r \to \infty.$$

Asymptotic motion

For the following cases, periodic (stationary) solutions of $\mathbf{u}$ arise:

- Pursuit mode: $\kappa^p(t) \equiv 1$ and $\kappa^c(t) \equiv 0$:

$$\mathbf{u}(t) = \mathbf{u}_* \in \mathbb{R}^2 \quad \text{and} \quad \mathbf{v}(t) = (0,0) \quad \text{with} \quad |\mathbf{u}_*| = r_c,$$

where the driver and evader behave uniform linear motions,

$$\mathbf{u}_\ell(t) = -\frac{f_d(\mathbf{u}_*)\mathbf{u}_*}{\nu} t + \mathbf{u}_\ell(0), \quad \ell = d, e.$$

- Circumvention mode, $\kappa^p(t) \equiv 1$ and $\kappa^c(t) \equiv \kappa \neq 0$:

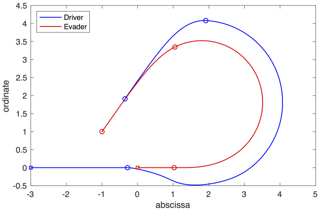
$$\mathbf{u}(t) = r_p \left(\cos\left(\frac{\kappa}{\nu} t\right), \sin\left(\frac{\kappa}{\nu} t\right) \right),$$

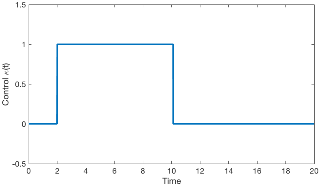
where the driver and evader have rotational motions on circles centered at the same point,

$$\mathbf{u}_\ell(t) = r_\ell \left(\cos\left(\frac{\kappa}{\nu} t + \phi_\ell\right), \sin\left(\frac{\kappa}{\nu} t + \phi_\ell\right) \right) + \mathbf{u}^*, \quad \mathbf{u}^* \in \mathbb{R}^2,\ \ell = d, e.$$

Off-Bang-Off control of the evader

Combining these two modes, we can construct an Off-Bang-Off control: choose the direction by rotations in the circumvention mode, and drive the evaders to the target in the pursuit mode.





Theorem [K.-Zuazua (preprint)]

Let $f(r)$ be as before. If $\mathbf{u}(0) \neq (0,0)$, then for any target point $\mathbf{u}_f \in \mathbb{R}^2$, there exist t_1, t_2, t_f and κ such that $\kappa^p(t) \equiv 1$ and

$$\kappa^c(t) = \begin{cases} \kappa & \text{if} \quad t \in [t_1, t_2], \\ 0 & \text{if} \quad t \in [0, t_1) \cup (t_2, t_f] \end{cases} \qquad \text{satisfy} \quad \mathbf{u}_e(t_f) = \mathbf{u}_f.$$

To control the final position of the evader, we need the following lemmas.

Lemma : Well-posedness of the model with unbounded forces

Suppose that $\kappa^p(t)$ and $\kappa^c(t)$ are bounded and $\limsup_{t\to\infty} |\kappa^c(t)| < \nu\sqrt{\gamma_m}$.

Then, the relative position $\mathbf{u}(t)$ does not hit $(0,0)$ or blow-up in a finite time. Moreover, if controls are constant, then $\mathbf{u}(t)$ is bounded.

Lemma : Global stability to reference states

The positions $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ converge exponentially
if $\kappa^p(t) \equiv 1$, $\kappa^c(t) \equiv \kappa$ and $\kappa < \nu\sqrt{\gamma_m}$:

- If $\kappa = 0$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to linear motions.
- If $|\kappa| > 0$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to rotational motions.

By combining these asymptotic steady states, we may prove the controllability of the evader's position to any desired point.

Since we can apply the Off-Bang-Off controls to any nonsingular initial data, we may use it to pass multiple target points:

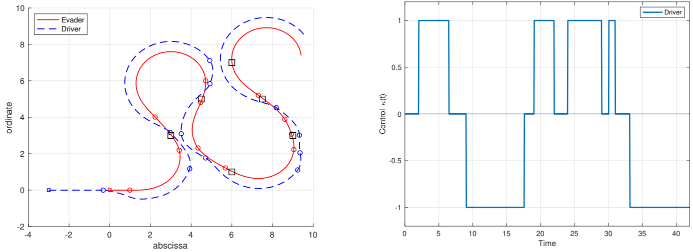


Figure: A trajectory of the evader which passes near points $(3,3)$, $(4.5,5)$, $(6,1)$, $(9,3)$, $(7.5,5)$ and $(6,7)$ denoted by black boxes.

This can be done by turning on and off $\kappa^c(t)$ using two control modes, where the dynamics converges to the corresponding steady state ('rotational motion' and 'linear motion') in a short time.

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Long-time behavior of a linear system

We want to see the asymptotic stability along time from the energy method. If the nonlinearity $f(r)$ is a constant, it became a linear model,

$$\ddot{\mathbf{u}} + \mathbf{u} + \nu\dot{\mathbf{u}} = \kappa\mathbf{u}^{\perp}, \quad \mathbf{u} \in \mathbb{R}^2,$$

which is the damped harmonic oscillator with an additional perpendicular (circumvention) interaction. We want to know when $\mathbf{u}$ decays to $(0,0)$.

The standard energy

$$E(t) := \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2), \quad \mathbf{v} = \dot{\mathbf{u}},$$

is no more non-increasing from the perpendicular term $\kappa\mathbf{u}^{\perp}$.

$$\dot{E}(t) = -\nu|\mathbf{v}|^2 + \kappa\mathbf{u}^{\perp} \cdot \mathbf{v}.$$

However, we may use hypocoercivity theory[1] to get a proper Lyapunov function. In terms of $\mathbf{x} = (\mathbf{u}, \mathbf{v})$, the equation is represented by a matrix form:

$$\dot{\mathbf{x}} + A\mathbf{x} + B\mathbf{x} = K\mathbf{x},$$

where the matrices are defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have

$$\dot{E}(t) = \mathbf{x} \cdot \dot{\mathbf{x}} = \mathbf{x} \cdot (-A - B + K)\mathbf{x} = -\mathbf{x} \cdot B\mathbf{x} + \mathbf{x} \cdot K\mathbf{x},$$

Then, so that we may add the following components to fix the energy:

$$|B\mathbf{x}|^2 = \nu|\mathbf{v}|^2, \quad B\mathbf{x}\cdot BA\mathbf{x} = \nu\mathbf{u}\cdot\mathbf{v}, \quad \mathbf{x}\cdot K\mathbf{x} = \kappa\mathbf{u}^\perp\cdot\mathbf{v} \quad \text{and} \quad K\mathbf{x}\cdot K\mathbf{x} = \kappa^2|\mathbf{u}|^2.$$

[1] [C. Villani, 2009, MEM AMS] and [K. Beauchard, E. Zuazua, 2011, ARMA]

In short, a perturbed energy $F_+(t)$,

$$F_+(t) = E(t) + \frac{\nu}{2}\left(\frac{\nu}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\dot{\mathbf{u}}\right),$$

does not increase along time,

$$\begin{aligned}\frac{d}{dt}F_+(t) &= -\frac{\nu}{2}|\mathbf{v}|^2 - \frac{\nu}{2}|\mathbf{u}|^2 + \kappa(\mathbf{u}^\perp\cdot\mathbf{v}) \\ &\le -\frac{1}{2}(\nu-\kappa)(|\mathbf{u}|^2+|\mathbf{v}|^2) = -(\nu-\kappa)E(t).\end{aligned}$$

Decaying property for small κ

$\mathbf{u}(t)$ decays exponentially if $|\kappa| < \nu$.

For the critical case, $|\kappa| = \nu$, we have another function:

$$F_\kappa(t) = E(t) - \frac{\kappa}{\nu}\mathbf{u}^\perp\cdot\mathbf{v} \quad \text{and} \quad \dot{F}_\kappa(t) = -\nu\left|\mathbf{v} - \frac{\kappa}{\nu}\mathbf{u}^\perp\right|^2 \le 0,$$

Periodic motion for critical κ

When $\kappa = \pm\nu$, $\mathbf{u}(t) = a(\cos \pm t, \sin \pm t)$, is a periodic solution.

The observed dynamics

From the relative position $\mathbf{u}$, we can recover partial information for the positions $\mathbf{u}_d$ and $\mathbf{u}_e$.

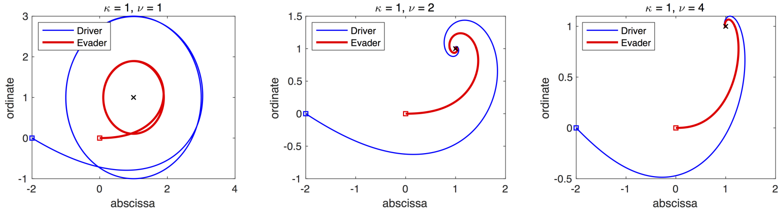

Figure: The trajectory of the driver and evader with $\kappa = 1$ and various ν: $\nu = 1$ (left), 2 (middle), and 3 (right).

This analysis can be used for our nonlinear guidance-repulsion model in order to see the long-time behavior.

Long-time behavior of the nonlinear model

Now, we get back to the guidance-by-repulsion model.

The equation of the relative position $\mathbf{u}$ with constant control $\kappa^p(t) \equiv 1$ and $\kappa^c(t) \equiv \kappa$,

$$\ddot{\mathbf{u}} + f(|\mathbf{u}|)\mathbf{u} + \nu\dot{\mathbf{u}} = \kappa\mathbf{u}^\perp, \quad \mathbf{u} \in \mathbf{R}^2,$$

is the damped oscillator with nonlinear potential and an external force.

Here, the standard energy,

$$E(t) := \frac{1}{2}|\mathbf{v}|^2 + P(|\mathbf{u}|),$$

may increase due to the perpendicular term $\kappa\mathbf{u}^\perp$.

$$\begin{aligned}
\dot{E}(t) &= \mathbf{v} \cdot \dot{\mathbf{v}} + f(|\mathbf{u}|)\mathbf{u} \cdot \dot{\mathbf{u}} \\
&= \mathbf{v} \cdot (-f(|\mathbf{u}|)\mathbf{u} - \nu\mathbf{v} + \kappa\mathbf{u}^\perp) + f(|\mathbf{u}|)\mathbf{u} \cdot \mathbf{v} \\
&= -\nu|\mathbf{v}|^2 + \kappa\mathbf{u}^\perp \cdot \mathbf{v}.
\end{aligned}$$

We use the same function as for the linear model:

$$L_{\pm}(t) = E(t) \pm \frac{\nu}{2}(\frac{\nu}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v}).$$

Then, for example, the time derivative of $L_{-}(t)$ is

$$\dot{L}_{-}(t) \leq -\frac{\nu}{2}|\mathbf{v}|^2 + \frac{\nu}{2}\left(f(|\mathbf{u}|) + \frac{\kappa^2}{\nu^2}\right)|\mathbf{u}|^2,$$

which is nonpositive if $|\mathbf{u}|$ is close to 0.

On the other hand, the boundedness of **u** can also be derived from

$$L_{\kappa}(t) = E(t) - \frac{\kappa}{\nu}\mathbf{u}^{\perp}\cdot\mathbf{v} \quad \text{and} \quad \dot{L}_{\kappa}(t) = -\nu\left|\mathbf{v} - \frac{\kappa}{\nu}\mathbf{u}^{\perp}\right|^2 \leq 0,$$

which is always nonpositive.

To control the final position of the evader, we need the following lemmas.

Lemma : Well-posedness of the model with unbounded forces

Suppose that $\kappa^p(t)$ and $\kappa^c(t)$ are bounded and $\limsup_{t\to\infty} |\kappa(t)| < \nu\sqrt{\gamma_m}$.

Then, the relative position $\mathbf{u}(t)$ does not hit $(0,0)$ or blow-up in a finite time. Moreover, if controls are constant, then $\mathbf{u}(t)$ is bounded.

Lemma : Global stability to reference states

The positions $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ converge to the steady states asymptotically if $\kappa^p(t) \equiv 1$, $\kappa^c(t) \equiv \kappa$ and $\kappa < \nu\sqrt{\gamma_m}$:

- If $\kappa = 0$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to linear motions.
- If $0 < |\kappa| < \nu\sqrt{\gamma_m}$, then $\mathbf{u}_d(t)$ and $\mathbf{u}_e(t)$ tend to rotational motions.

By combining these asymptotic steady states, we may prove the controllability of the evader's position to any desired point.

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Optimal control strategies

The off-bang-off controls can drive the evaders. How about optimal controls?

For the cost function, we suggest to minimize the final position error with the circumvention cost and the final time:

$$J(\kappa^p(\cdot), \kappa^c(\cdot)) = \frac{1}{N}\sum_{i=1}^{N} |\mathbf{u}_{ei}(t_f) - \mathbf{u}_f|^2 dt + \frac{\delta_1}{M}\sum_{j=1}^{M}\int_0^{t_f} |\kappa_j^c(t)|^2 dt + \delta_2 t_f,$$

where $\mathbf{u}_{ei}, i = 1, \ldots, N$ is the positions of the evaders, $\kappa_j^c(t)$ is the circumvention controls for $j = 1, \ldots, M$, $\mathbf{u}_f$ is the target point and t_f is the final time.

Optimal control minimizing running cost

This is an Off-Bang-Off control with $t_1 = 0$ and $t_2 = t_f$, the constant control:

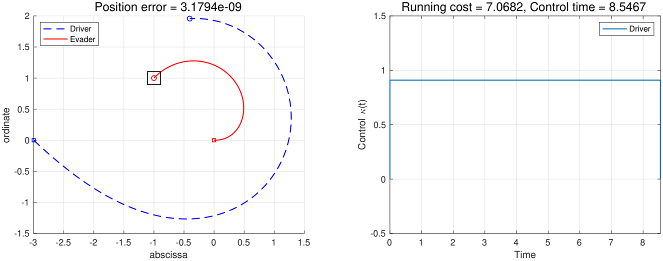

Figure: Diagrams for the constant control leading to $\mathbf{u}_e(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_d^0 = (-3, 0)$ and zero velocities.

Optimal control minimizing running cost

When we consider the circumvention cost only, then

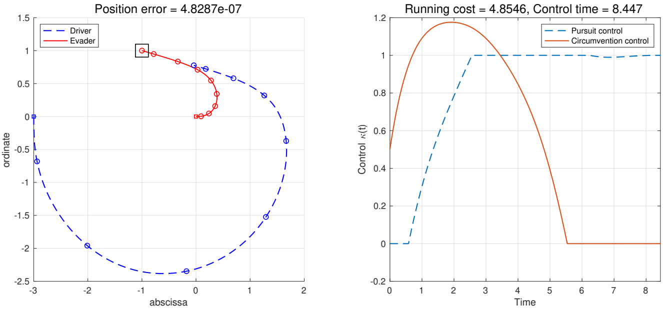

Figure: Diagrams for the control minimizing circumvention leading to $\mathbf{u}_e(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_d^0 = (-3, 0)$ and zero velocities.

Optimal control minimizing driving time

If we minimize the final time, then we need stronger circumvention, but it shares the main idea: ‘rotate and then drive’.

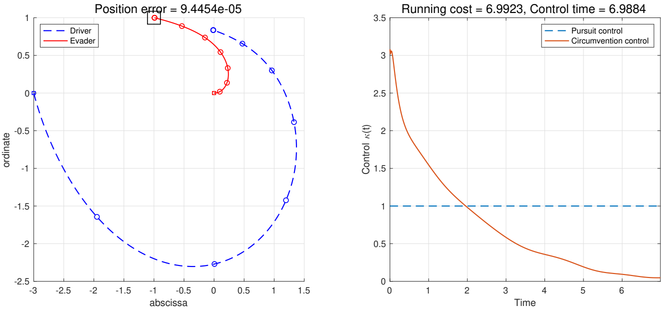

Figure: Diagrams for the control minimizing driving time leading to $\mathbf{u}_e(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_d^0 = (-3, 0)$ and zero velocities.

Feedback control mimicking the optimal control

From this idea, we may construct feedback control:

$$\kappa_j^c(t) = -\bar{\kappa}^c \frac{(\mathbf{u}_f - \mathbf{u}_{ec}) \cdot (\mathbf{u}_{dj} - \mathbf{u}_{ec})^{\perp}}{|\mathbf{u}_f - \mathbf{u}_{ec}| \cdot |\mathbf{u}_{dj} - \mathbf{u}_{ec}|}, \quad \bar{\kappa}^c = 3, \quad j = 1, 2, \cdots .$$

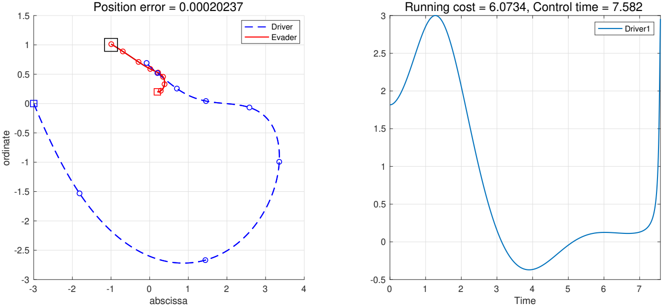

Figure: Diagrams for the feedback control leading to $\mathbf{u}_e(t_f) \simeq (-1, 1)$ when initially $\mathbf{u}_e^0 = (0, 0)$, $\mathbf{u}_d^0 = (-3, 0)$ and zero velocities.

The effect of the number of evaders

If the evaders are gathered initially, the dynamics are similar to the one evader case, as we have one fat evader.

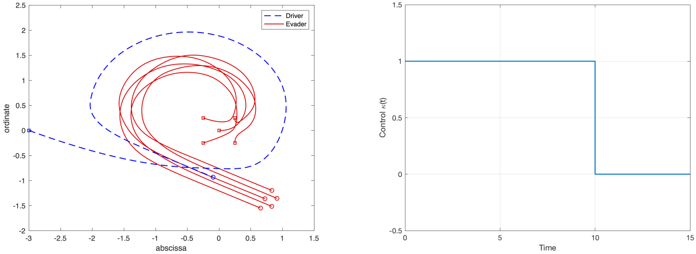

Figure: Trajectories of five evaders with a bang-off control $\kappa(t)$.

$$f_e(r) = \frac{1}{r^2}, \quad \psi_d(r) = -\frac{1}{2r^4} \quad \text{and} \quad \psi_e(r) = 10\left(\frac{(0.1)^2}{r^2} - \frac{(0.1)^4}{r^4}\right).$$

Feedback for many drivers and evaders

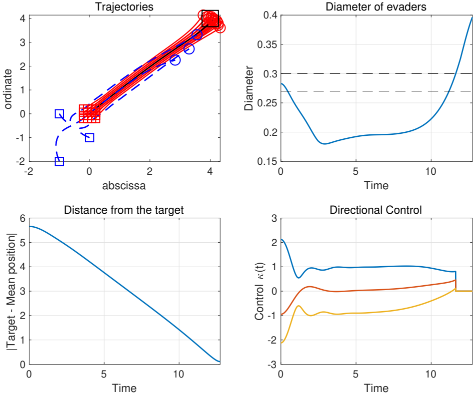

Figure: Trajectories, diameter, distance and control for the feedback control functions.

Trapping problem?

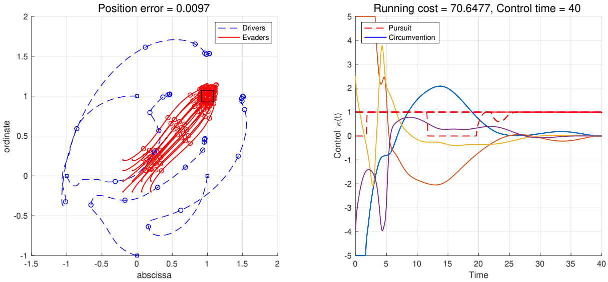

Figure: Optimal control leading to $(1, 1)$ which traps the evaders at the final time.

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Summary

- The guidance-by-repulsion problem is a bi-linear second-hand control on partial states. (Null-controllability is trivially false)
- In summary, one-driver and one-evader model with good assumptions (symmetric dissipation, constant control, potential condition) leads to the controllability of the evader's position.

THANK YOU FOR YOUR ATTENTION!

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