Methods based on shape derivative for the optimal design on annulus

Petar Kunštek
Joint work with Marko Vrdoljak
University of Zagreb, Croatia

VIII Partial differential equations, optimal design and numerics
21 August 2019

IP:3706

Part I: Construction of classical solutions for optimal design problems

Introduction

Let $\Omega \subset \mathbb{R}^d$ be open and bounded set.
Two phases each with different isotropic conductivity: α, β $(0 < \alpha < \beta)$.
q_α is the prescribed volume of the first phase α $(0 < q_\alpha < |\Omega|)$.
$\chi \in \mathrm{L}^\infty(\Omega)$ such that

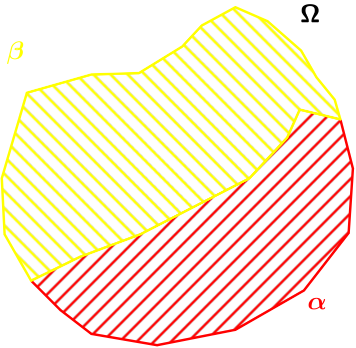
$$\begin{cases} \chi = 1, & \text{phase } \alpha \\ \chi = 0, & \text{phase } \beta \end{cases}.$$

Conductivity can be expressed as

$$\mathbf{A}(\chi) := \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I},$$

where

$$\int_\Omega \chi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = q_\alpha.$$



Introduction

State functions $u_i \in \mathrm{H}_0^1(\Omega),\ i = 1, 2, ..., m$ are given as a solution of the following boundary value problems:

$$(\mathrm{S}) \qquad \begin{cases} -\mathrm{div}(\mathbf{A}\,\nabla u_i) = f_i & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, 2, ..., m,$$

with $\mathbf{A} = \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I}$. Denote $\boldsymbol{u} = (u_1, ..., u_m)$.
Energy functional:

$$J(\chi) := \sum_{i=1}^{m} \mu_i \int\limits_{\Omega} f_i(\boldsymbol{x}) u_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

where $\mu_i > 0,\ i = 1, 2, ..., m$.

Statement of the problem

Optimal design problem:

$$
\text{(P)} \qquad \begin{cases} J(\chi) = \displaystyle\sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, \mathrm{d}\boldsymbol{x} \to \max \\ \text{s.t.} \quad \chi \in L^\infty(\Omega, \{0,1\}), \quad \displaystyle\int_\Omega \chi \, \mathrm{d}\boldsymbol{x} = q_\alpha, \\ \boldsymbol{u} \text{ solves (S) with } \mathbf{A} = \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I}. \end{cases}
$$

If solution χ exists for (P) we call it *classical solution*.

Important: For general optimal design problems the classical solutions usually do not exist.

Results from general theory

$$
\text{(I)} \qquad \begin{cases} I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, \mathrm{d}\boldsymbol{x} \to \max \\ s.t. \quad \theta \in L^\infty(\Omega, [0,1]), \ \int_\Omega \theta = q_\alpha, \text{ where } u_i \text{ satisfies} \\ -\operatorname{div}(\lambda_\theta^- \nabla u_i) = f_i, \quad u_i \in \mathrm{H}_0^1(\Omega), \ i = 1, 2, \dots, m \end{cases}
$$

where $\lambda_\theta^-(x) = \left(\frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta}\right)^{-1}$.
For spherically symmetric problem such that:

- $\Omega = R(\Omega)$ for any rotation R
- f_i are radial functions

it can be proved that there exists radial solution θ_R^* of (I).

In particular, it can be shown that if θ_R^* is classical it is also a solution of problem (P). Also, state functions u_i and fluxes $\sigma_i = \mathbf{a}\nabla u_i$ are radial functions and σ_i are unique.

Define

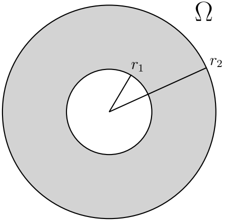
$$\Psi := \sum_{i=1}^{m} \mu_i |\boldsymbol{\sigma}_i{}^*|^2.$$

Lemma

The necessary and sufficient condition of optimality for solution θ^* of optimal design problem (I) *simplifies to the existence of a Lagrange multiplier $c \geq 0$ such that*

$$\begin{aligned} \Psi > c \quad &\Rightarrow \quad \theta^* = 1\,, \\ \Psi < c \quad &\Rightarrow \quad \theta^* = 0\,. \end{aligned} \tag{1}$$

Single state optimal design problem



Single state equation:

$$(2)\qquad \begin{cases} -\operatorname{div}(\lambda_\theta^-(x)\nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda_\theta^-(x) = \left(\frac{\theta(x)}{\alpha} + \frac{1-\theta(x)}{\beta}\right)^{-1}$.

Optimization problem:

$$(3)\qquad \begin{cases} I(\theta) = \int_\Omega u\,\mathrm{d}\boldsymbol{x} \to \max \\ s.t. \quad \theta \in \mathrm{L}^\infty(\Omega,[0,1]),\ \int_\Omega \theta = q_\alpha,\ \text{where } u \text{ satisfies (2)} \end{cases}$$

Single state optimal design problem

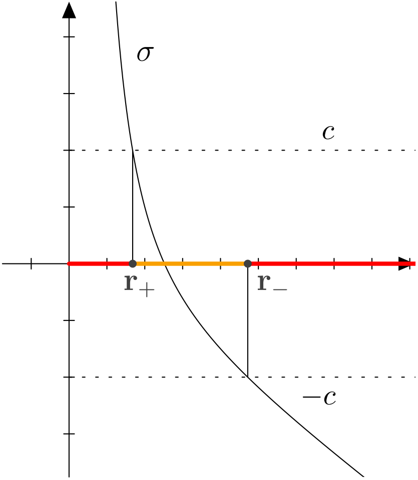
One can rewrite (2) in polar coordinates :

$$-\frac{1}{r^{d-1}}(r^{d-1}\underbrace{\lambda_\theta^- u'(r)}_{\sigma})' = 1 \text{ in } \langle r_1, r_2\rangle\,, \quad u(r_1) = u(r_2) = 0.$$

Observe that σ satisfies

$$\sigma = -\frac{r}{d} + \frac{\gamma}{r^{d-1}}, \quad \gamma > 0$$

$\sigma(r) : \langle 0, \infty\rangle \to \mathbb{R}$ is a strictly decreasing function.



The necessary and sufficient condition of optimality for θ^* states

$$
\begin{aligned}
|\sigma^*| > c \quad &\Rightarrow \quad \theta^* = 1\,, \\
|\sigma^*| < c \quad &\Rightarrow \quad \theta^* = 0\,.
\end{aligned}
$$

There are only three possible candidates for optimal design:

1) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$

2) $\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$

3) $\theta^*(r) = \begin{cases} 0, & r \in [r_1, r_-\rangle \\ 1, & r \in [r_-, r_2\rangle \end{cases}$

Simplification to a non-linear system

From condition of optimality a non-linear system (with unknowns γ, c, r_+, r_-) is created:

$$
\text{(NS)} \qquad \begin{cases} S_d \displaystyle\int_{r_1}^{r_2} \theta(\rho)\rho^{d-1}\,\mathrm{d}\rho = q_\alpha \\ u(r_2) = 0 \iff \gamma \displaystyle\int_{r_1}^{r_2} \left(\frac{1}{a(\rho)\rho^{d-1}}\right) \mathrm{d}\rho = \int_{r_1}^{r_2} \frac{\rho}{a(\rho)}\,\mathrm{d}\rho \\ \sigma(r_+) = c, \quad \sigma(r_-) = -c, \qquad \text{where } c > 0 \end{cases}
$$

where

$$
\sigma(r) = \frac{\gamma}{r^{d-1}} - \frac{r}{d}, \quad \& \quad a(r) = \left(\frac{\theta(r)}{\alpha} + \frac{1-\theta(r)}{\beta}\right)^{-1}.
$$

Important: For solving (NS) optimal design is assumed.

(Optimal design for annulus $d = 2, 3,\ f = 1$)

With previous assumptions problem (I) admits optimal solution with two possible designs:

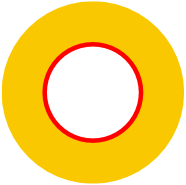
1) $$\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_-\rangle \\ 1, & r \in [r_-, r_2] \end{cases}$$ **alpha-beta-alpha**

2) $$\theta^*(r) = \begin{cases} 1, & r \in [r_1, r_+\rangle \\ 0, & r \in [r_+, r_2\rangle \end{cases}$$ **alpha-beta**

If q_α is small design 2) is optimal.

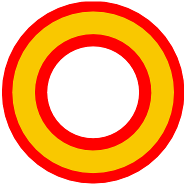
alpha-beta

($q_\alpha <$ critical value)



alpha-beta-alpha

($q_\alpha >$ critical value)



Part II: Numerical methods based on shape derivative

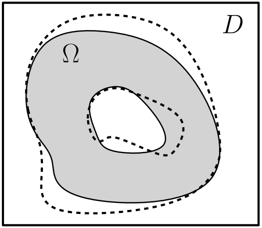
description of methods, numerical solutions in 2D & 3D.

Shape derivative

Perturbation of the set Ω is given with

$$\Omega_t = (\mathrm{Id} + t\psi)\Omega$$

where $\psi \in W^{k,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.



Definition (Shape derivative)

Let $J = J(\Omega)$ be a shape functional. J is said to be shape differentiable at Ω in direction ψ if

$$J'(\Omega, \psi) := \lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

exists and the mapping $\psi \mapsto J'(\Omega, \psi)$ is linear and continuous. $J'(\Omega, \psi)$ is called the **shape derivative**.

Single state problem

For single state optimal design problem (with transmission condition):

$$
(4) \qquad \begin{cases} J(\chi) = \int_\Omega f u \, \mathrm{d}\boldsymbol{x} \to \max \\ \text{s.t.} \quad \chi \in L^\infty(\Omega, \{0,1\}), \quad \int_\Omega \chi \, \mathrm{d}\boldsymbol{x} = q_\alpha, \\ u \text{ solves (S) with } \mathbf{A} = \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I} \end{cases}
$$

shape derivative is given with:

$$
\begin{aligned} J'(\Omega, \psi) = & \int_\Omega \mathbf{A}(-\operatorname{div}(\psi) + \nabla\psi + \nabla\psi^\tau)\nabla u \cdot \nabla u \, \mathrm{d}\boldsymbol{x} \\ & + \int_\Omega 2\operatorname{div}(f\psi) u \, \mathrm{d}\boldsymbol{x} \end{aligned}
$$

where u is solution of BVP (S) on domain Ω with $\mathbf{A} = \chi\alpha\,\mathbf{I} + (1-\chi)\beta\,\mathbf{I}$.

Gradient method based on shape derivative

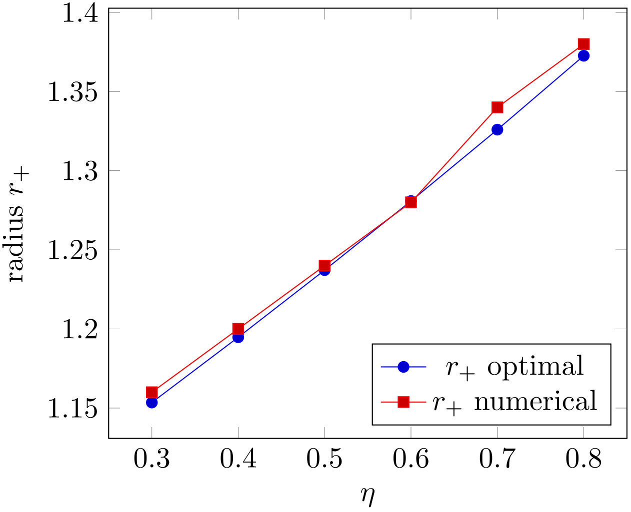
Heuristics: do several iterations of the method, check results and adapt parameters.

Algorithm 1: iteration of the method

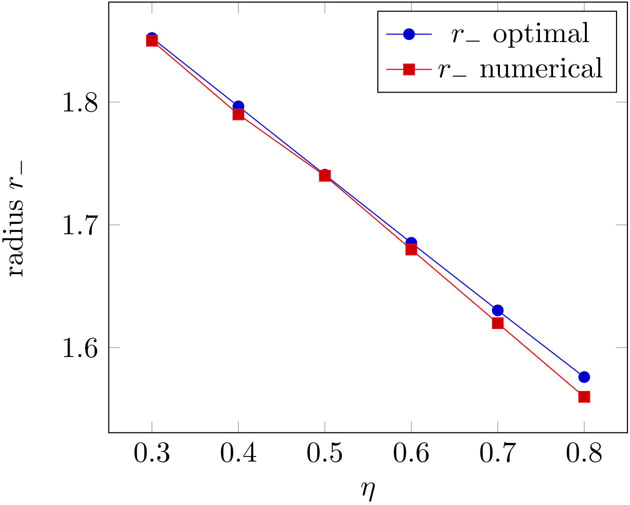
1. Input : interface is given implicitly (LSF) or explicity as discretized set of points - triangulation mesh $\mathcal{T}_k$
2. Create function space `Vh` na $\mathcal{T}_k$ (P1,P2,...)
3. Determine ascent vector $\psi \in$ `Vh` from shape derivative (consists of solving several PDEs)
4. Output: update interface (depends highly on considered reprezentation of interface)

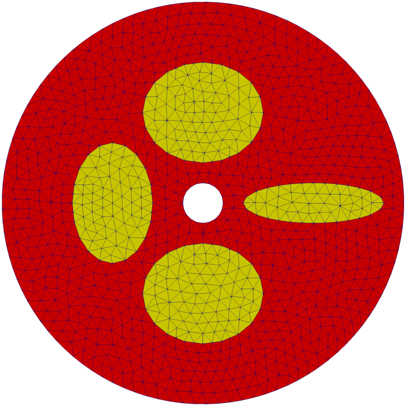
- above implemented methods are fairly stable with minimal user intervention
- in 2D it quickly approximates the optimal shape and script is under 100 lines of code

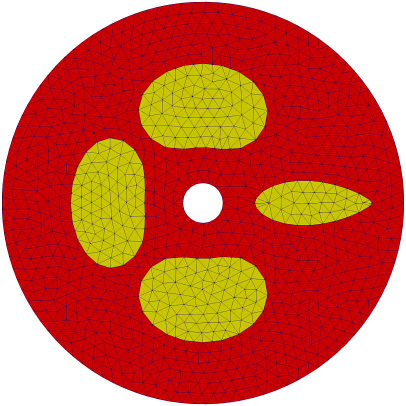
Numerical results

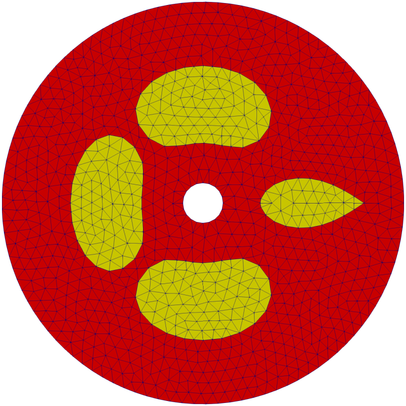


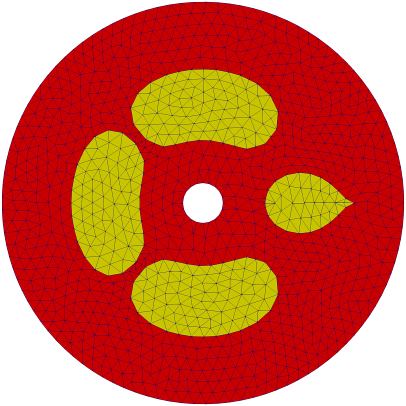
Numerical results

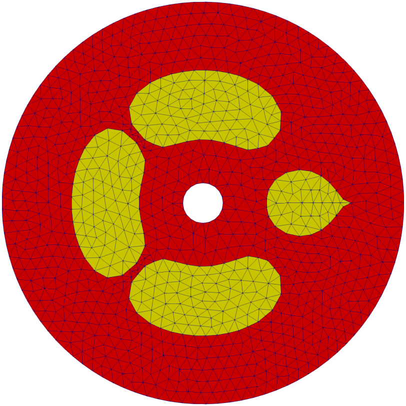


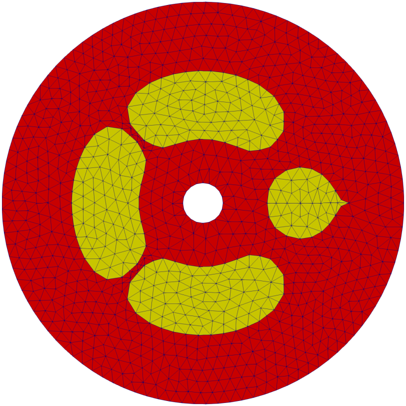


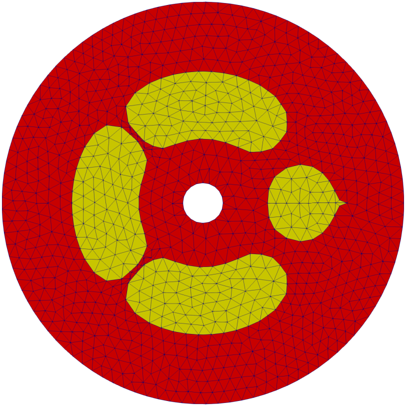


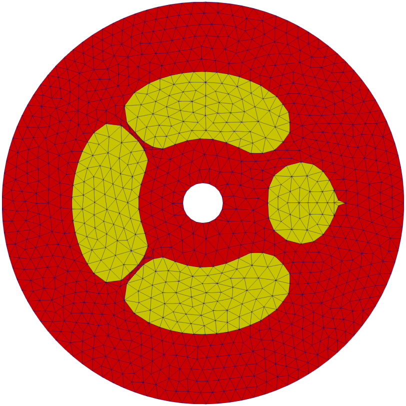


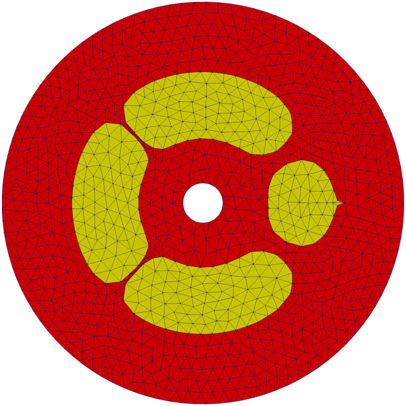


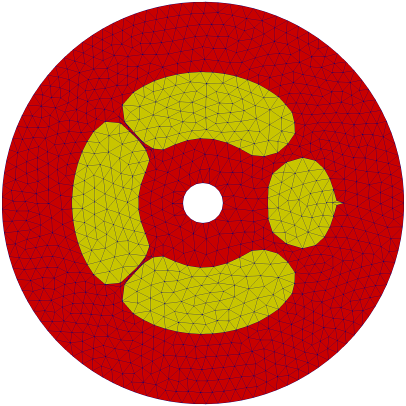


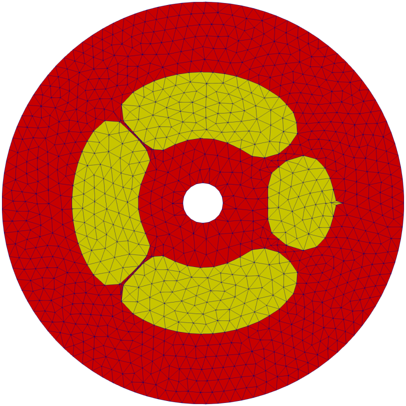


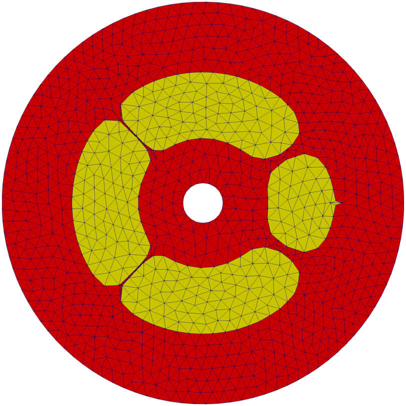


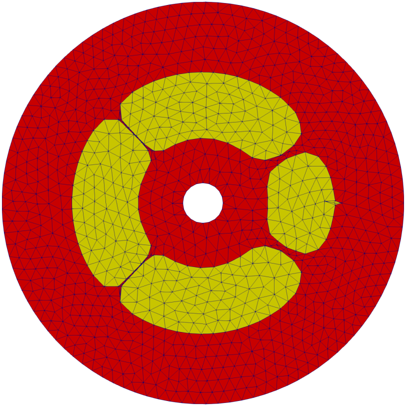


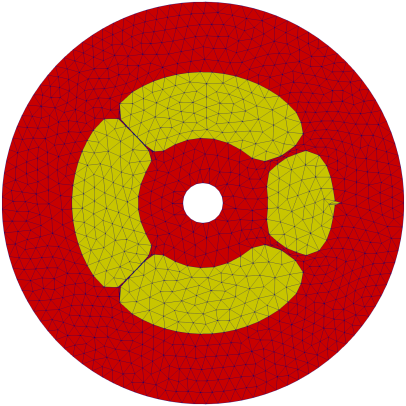


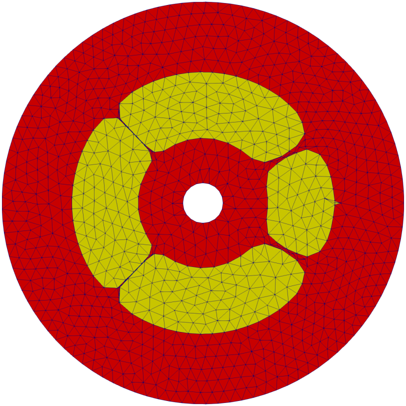


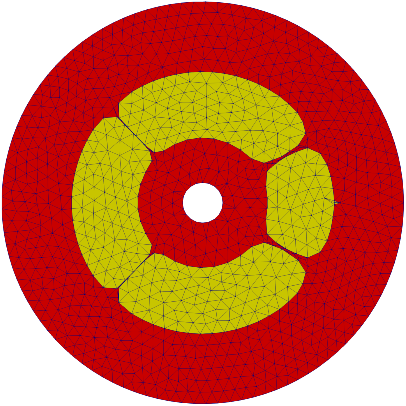


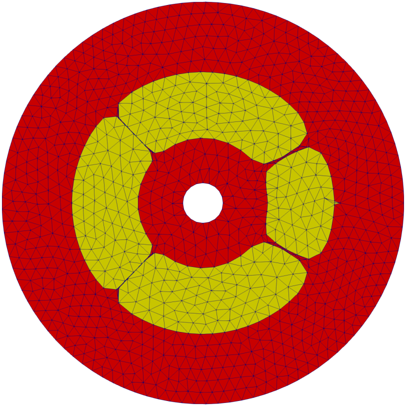


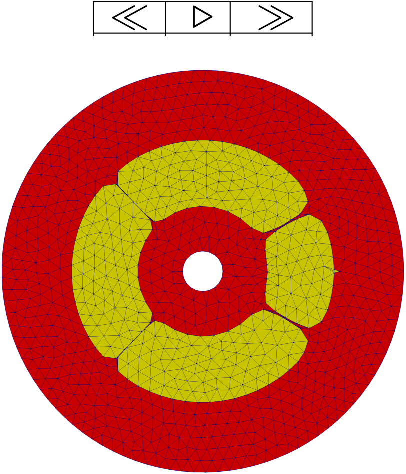


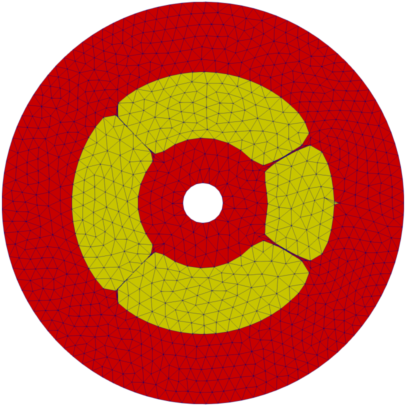


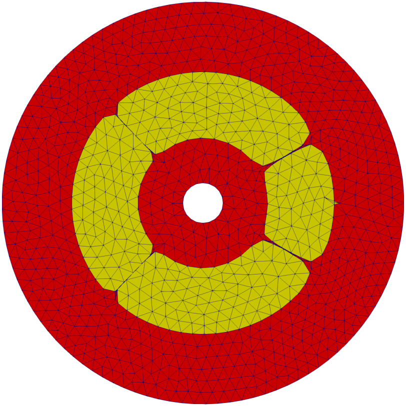


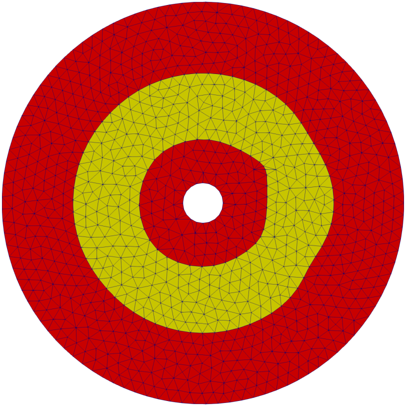


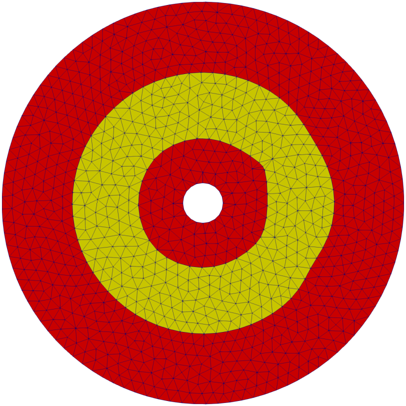


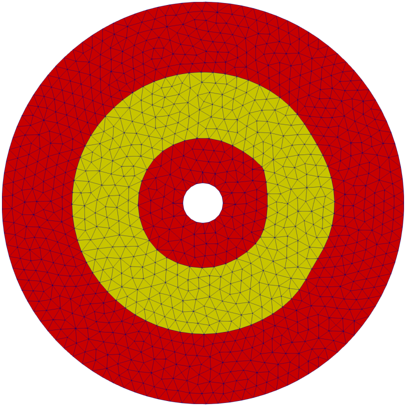


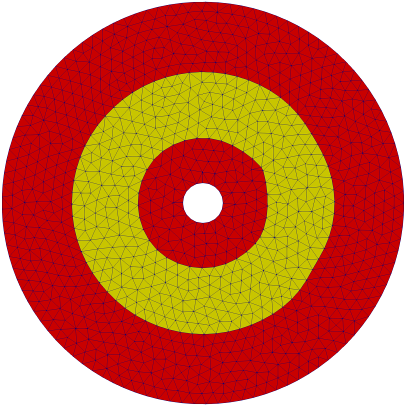


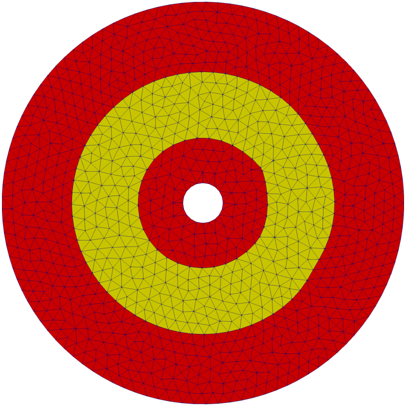


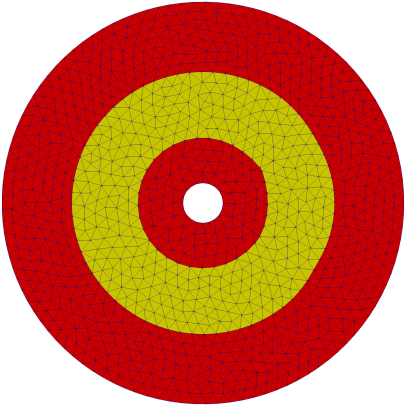


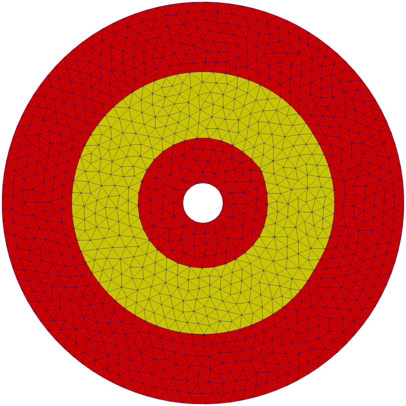


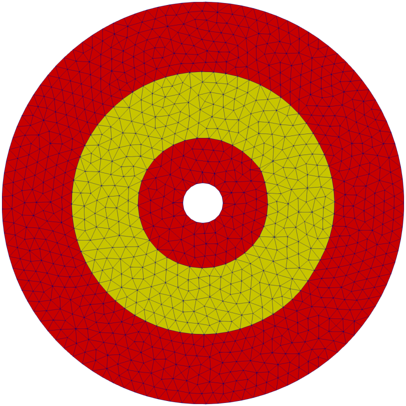


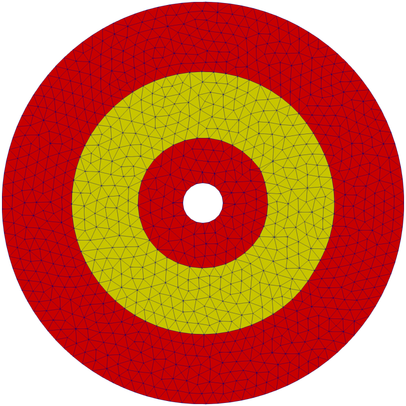


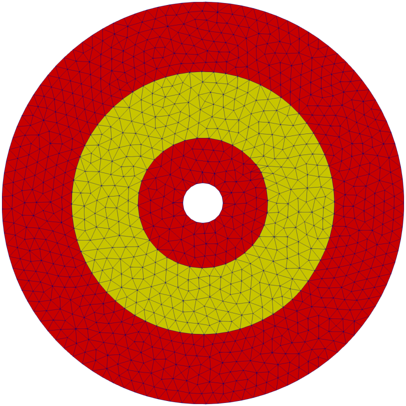


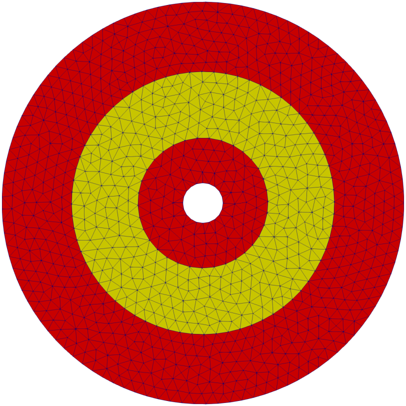




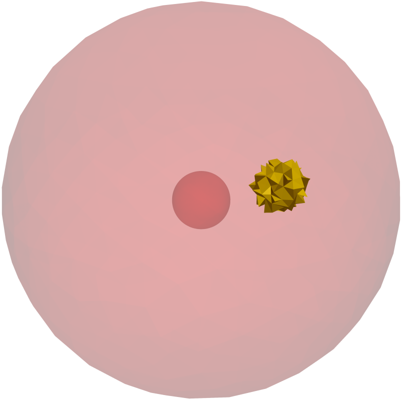




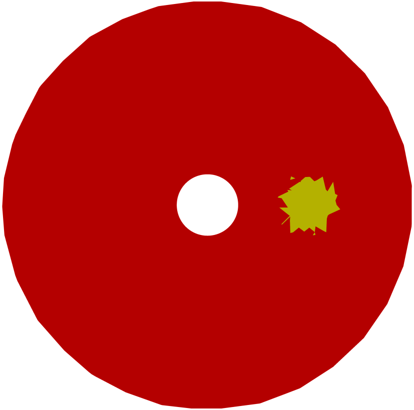




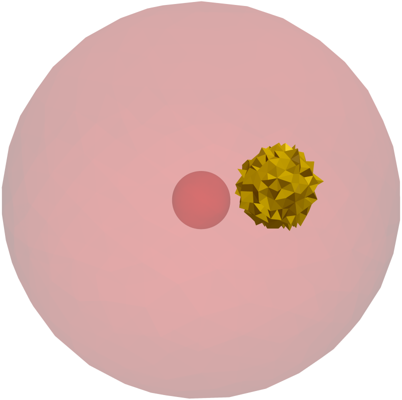
(a) 3D representation of material β (yellow)



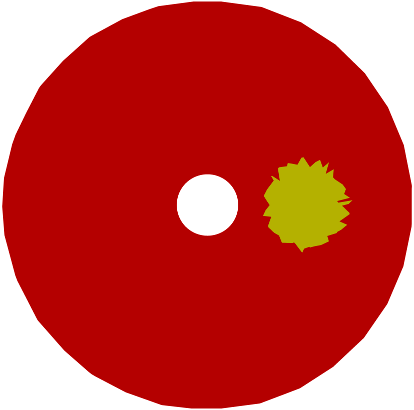
(b) slice of volume representation at $z = 0$

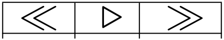


(a) 3D representation of material β (yellow)

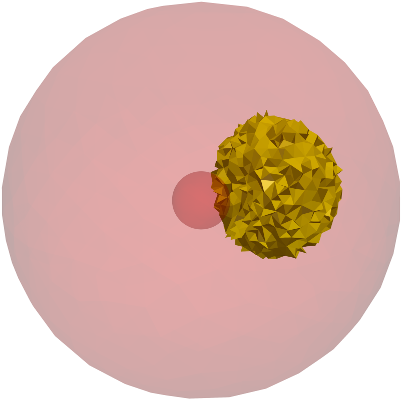


(b) slice of volume representation at $z = 0$

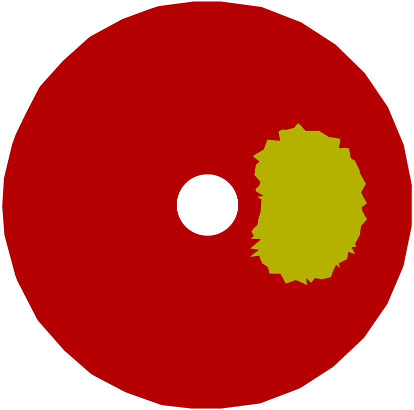




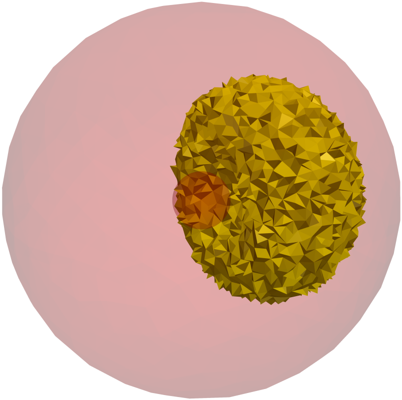
(a) 3D representation of material β (yellow)



(b) slice of volume representation at $z = 0$

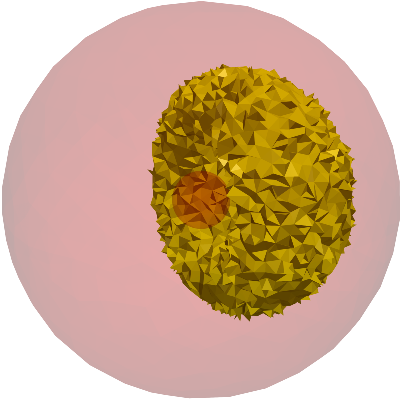


(a) 3D representation of material β (yellow)



(b) slice of volume representation at $z = 0$

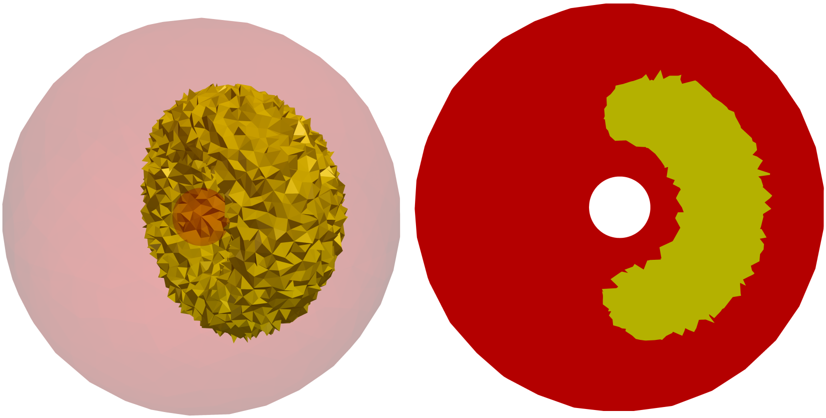
(a) 3D representation of material β (yellow)



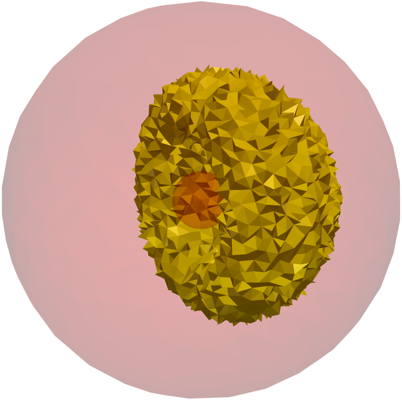
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$

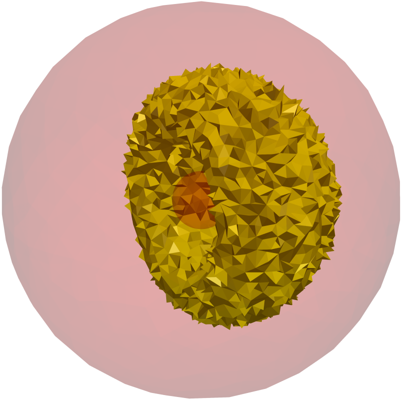


(a) 3D representation of material β (yellow)



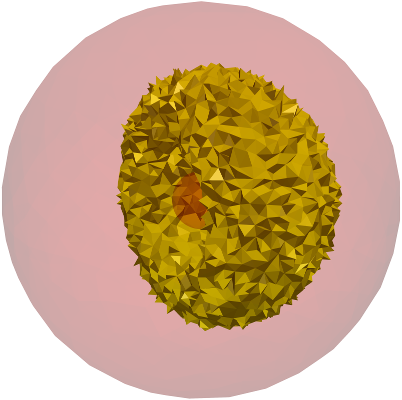
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



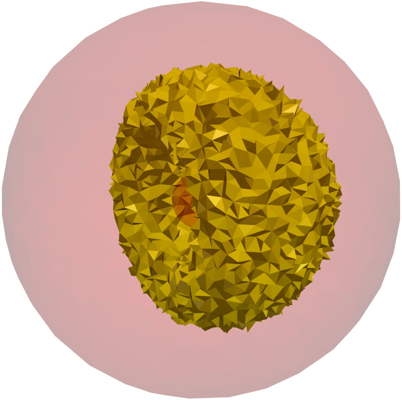
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



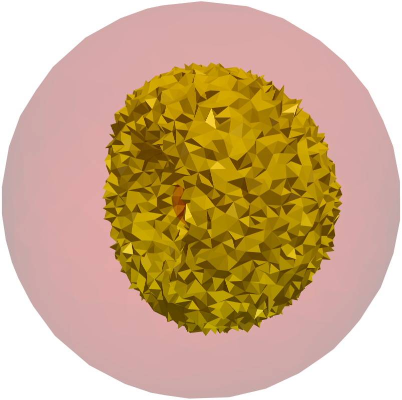
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



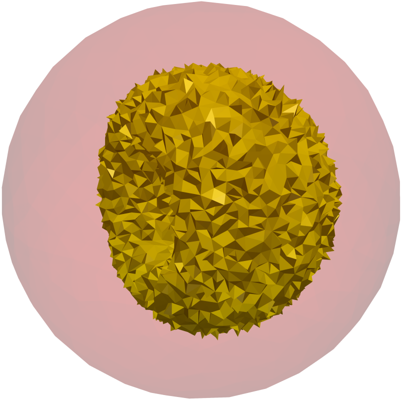
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



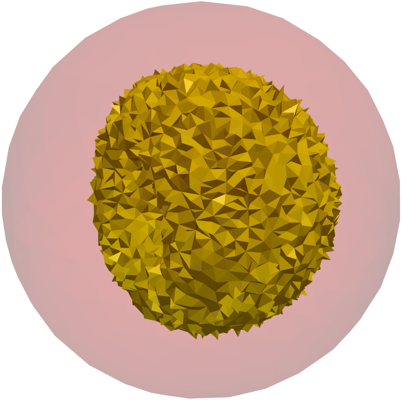
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



(b) slice of volume representation at $z = 0$

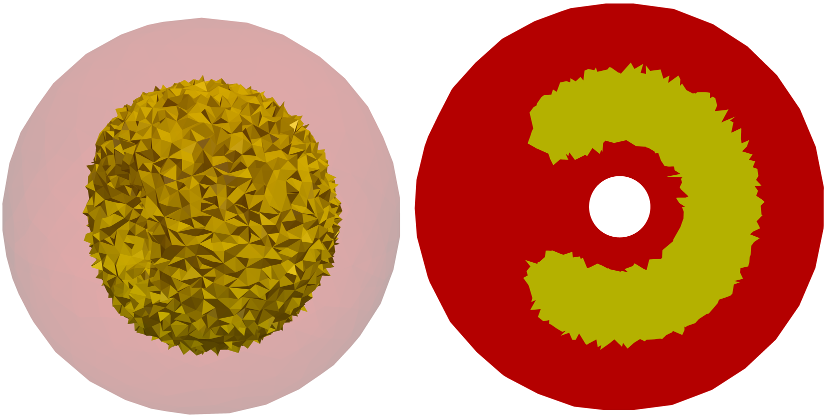
(a) 3D representation of material β (yellow)



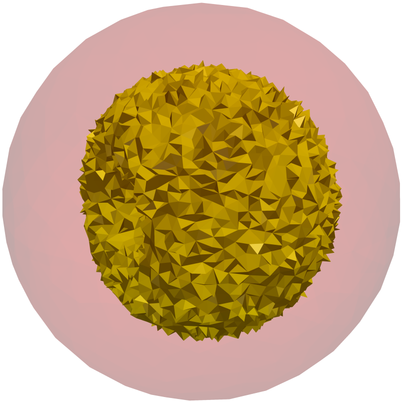
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$

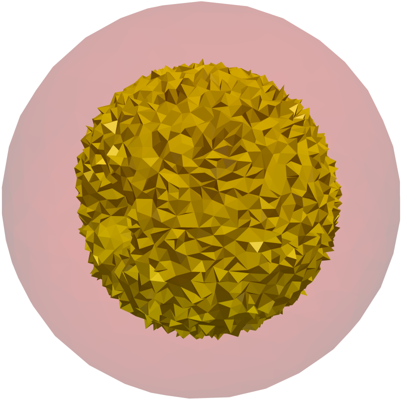


(a) 3D representation of material β (yellow)



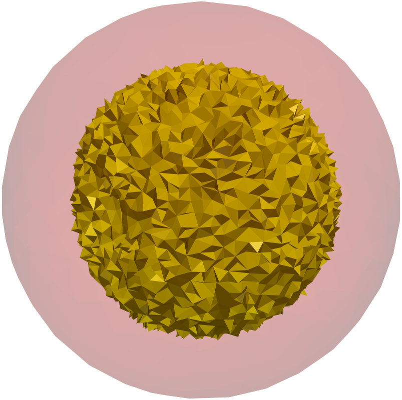
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



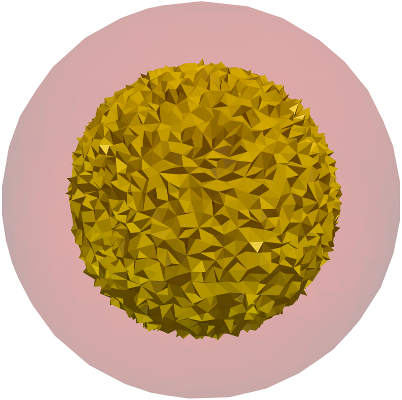
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



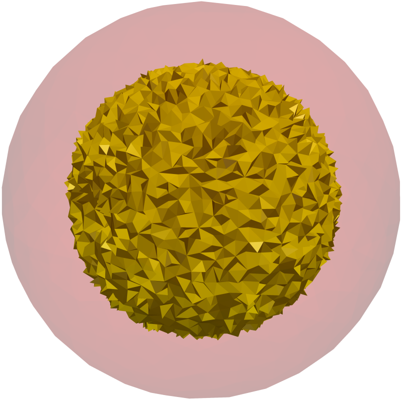
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



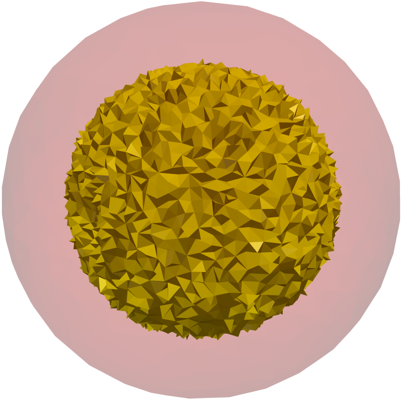
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



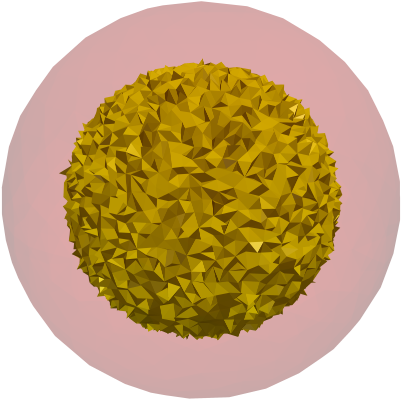
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



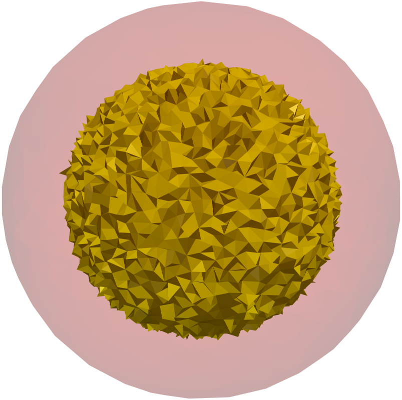
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



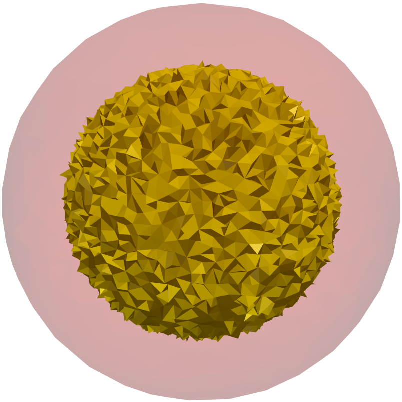
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



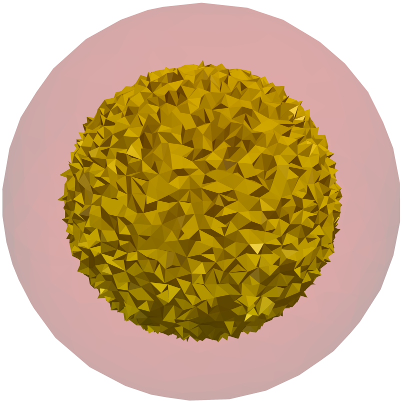
(b) slice of volume representation at $z = 0$

(a) 3D representation of material β (yellow)



(b) slice of volume representation at $z = 0$

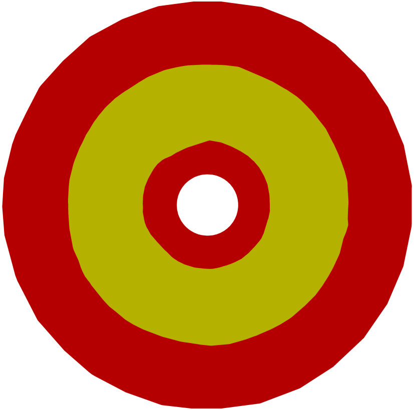
(a) 3D representation of material β (yellow)



(b) slice of volume representation at $z = 0$

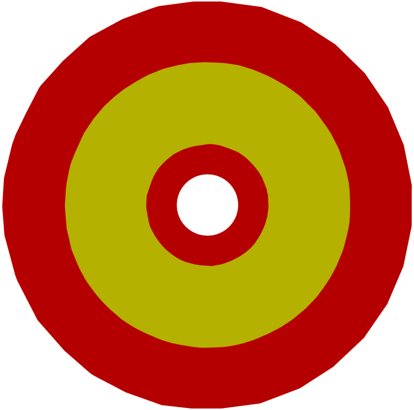
(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$



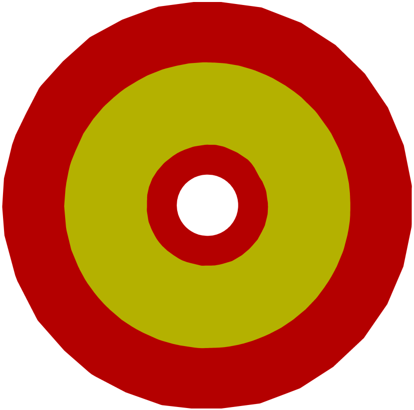
(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$



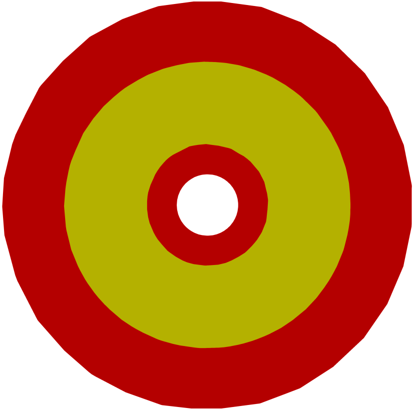
(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$



(a) 3D representation of material β (yellow)

(b) slice of volume representation at $z = 0$



Part III: Second order shape derivative

Work in progress

We are interested in the second order expansions of J:

$$J((\mathrm{Id}+\psi)\Omega) = J(\Omega) + J'(\Omega;\psi) + \frac{1}{2}J''(\Omega;\psi,\psi) + \mathrm{o}(\|\psi\|_k^2)$$

Important: This is not a variations of first order shape derivative

$$J''(\Omega;\psi_1,\psi_2) \neq (J'(\Omega;\psi_1))'(\Omega;\psi_2) = \lim_{t\to 0}\frac{1}{t}(J'(\mathrm{Id}+t\psi_2)\Omega;\psi_1) - J'(\Omega;\psi_1).$$

but the following identity holds

$$J''(\Omega;\psi_1,\psi_2) = (J'(\Omega;\psi_1))'(\Omega;\psi_2) - J'(\Omega;\nabla\psi_1\psi_2).$$

By standard method of using local derivative $u'(\psi)$:

$$
\begin{aligned}
&J''(\Omega;\psi_1,\psi_2) = \\
&\quad \alpha \int\limits_{\Gamma} (\psi_1 \cdot \boldsymbol{n}_\alpha)(\psi_2 \cdot \boldsymbol{n}_\alpha) \left\{ \mathbf{H} \left[2\left|\frac{\partial u_\alpha}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\alpha|^2 \right] + \frac{\partial}{\partial \boldsymbol{n}_\alpha} \left[2\left|\frac{\partial u_\alpha}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\alpha|^2 \right] \right\} \mathrm{d}S \\
&\quad - \beta \int\limits_{\Gamma} (\psi_1 \cdot \boldsymbol{n}_\alpha)(\psi_2 \cdot \boldsymbol{n}_\alpha) \left\{ \mathbf{H} \left[2\left|\frac{\partial u_\beta}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\beta|^2 \right] + \frac{\partial}{\partial \boldsymbol{n}_\alpha} \left[2\left|\frac{\partial u_\beta}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\beta|^2 \right] \right\} \mathrm{d}S \\
&\quad - \frac{2(\beta+\alpha)\alpha}{\beta-\alpha} \int\limits_{\Omega_\alpha} \nabla u'_\alpha(\psi_1) \cdot \nabla u'_\alpha(\psi_2)\, \mathrm{d}x + \frac{2(\beta+\alpha)\beta}{\beta-\alpha} \int\limits_{\Omega_\beta} \nabla u'_\beta(\psi_1) \cdot \nabla u'_\beta(\psi_2)\, \mathrm{d}x \\
&\quad + \frac{2\alpha\beta}{\beta-\alpha} \int\limits_{\Gamma} u'_\beta(\psi_1) \frac{\partial u'_\alpha(\psi_2)}{\partial \boldsymbol{n}_\alpha} + u'_\alpha(\psi_2) \frac{\partial u'_\beta(\psi_1)}{\partial \boldsymbol{n}_\alpha} + u'_\alpha(\psi_1) \frac{\partial u'_\beta(\psi_2)}{\partial \boldsymbol{n}_\alpha} + u'_\beta(\psi_2) \frac{\partial u'_\alpha(\psi_1)}{\partial \boldsymbol{n}_\alpha}\, \mathrm{d}S \\
&\quad + \alpha \int\limits_{\Gamma} Z(\psi_1,\psi_2) \left[2\left|\frac{\partial u_\alpha}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\alpha|^2 \right] \mathrm{d}S - \beta \int\limits_{\Gamma} Z(\psi_1,\psi_2) \left[2\left|\frac{\partial u_\beta}{\partial \boldsymbol{n}_\alpha}\right|^2 - |\nabla u_\beta|^2 \right] \mathrm{d}S
\end{aligned}
$$

where

$$Z(\psi_1,\psi_2) = \nabla \boldsymbol{n}_\alpha^{\mathcal{T}}(\psi_1)_\Gamma \cdot (\psi_2)_\Gamma - \nabla_\Gamma(\psi_1 \cdot \boldsymbol{n}_\alpha) \cdot (\psi_2)_\Gamma - \nabla_\Gamma(\psi_2 \cdot \boldsymbol{n}_\alpha) \cdot (\psi_1)_\Gamma$$

and u is a solution of (S). Local derivative $u'(\psi) \in \mathrm{H}^1(\Omega_\alpha \cup \Omega_\beta)$ is a solution of the following transmission problem with discontinuous jumps on the interface:

(5)

$$\begin{cases} \Delta u'(\psi) = 0 & \text{in } \Omega_\alpha \cup \Omega_\beta, \\ u'_\alpha(\psi) - u'_\beta(\psi) = \frac{\alpha-\beta}{\beta}(\nabla u_\alpha \cdot \boldsymbol{n}_\alpha)(\psi \cdot \boldsymbol{n}_\alpha) & \text{on } \Gamma, \\ \alpha \nabla u'_\alpha(\psi) \cdot \boldsymbol{n}_\alpha - \beta \nabla u'_\beta(\psi) \cdot \boldsymbol{n}_\alpha = (\alpha - \beta)\,\mathrm{div}_\Gamma(\nabla_\Gamma u(\psi \cdot \boldsymbol{n}_\alpha)) & \text{on } \Gamma, \\ u'(\psi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Volume representation:

(6)

$$
\begin{aligned}
J''(\Omega;\psi_1,\psi_2) &= \int_\Omega \mathbf{a}\left[-\operatorname{div}\psi_1\operatorname{div}\psi_2\,\mathbf{I}+\nabla\psi_1:\nabla\psi_2^{\mathcal{T}}\,\mathbf{I}-\nabla\psi_1\nabla\psi_2^{\mathcal{T}}-\nabla\psi_2\nabla\psi_1^{\mathcal{T}}\right]\nabla u\cdot\nabla u\,\mathrm{d}x\\
&+\int_\Omega \mathbf{a}\left[-\nabla\psi_1\nabla\psi_2-\nabla\psi_2\nabla\psi_1-\nabla\psi_1^{\mathcal{T}}\nabla\psi_2^{\mathcal{T}}-\nabla\psi_2^{\mathcal{T}}\nabla\psi_1^{\mathcal{T}}\right]\nabla u\cdot\nabla u\,\mathrm{d}x\\
&+\int_\Omega \mathbf{a}\left[\operatorname{div}\psi_1(\nabla\psi_2+\nabla\psi_2^{\mathcal{T}})+\operatorname{div}\psi_2(\nabla\psi_1+\nabla\psi_1^{\mathcal{T}})\right]\nabla u\cdot\nabla u\,\mathrm{d}x\\
&+2\int_\Omega \left[f\operatorname{div}\psi_1\operatorname{div}\psi_2+\psi_1\cdot\nabla f\operatorname{div}\psi_2+\psi_2\cdot\nabla f\operatorname{div}\psi_1+Hf\psi_2\cdot\psi_1\right]u\,\mathrm{d}x\\
&-2\int_\Omega \nabla\psi_1:\nabla\psi_2^{\mathcal{T}} f u\,\mathrm{d}x+\frac{1}{2}\int_\Omega \mathbf{a}\nabla v(\psi_1)\cdot\nabla v(\psi_2)\,\mathrm{d}x
\end{aligned}
$$

where u is a solution of (S) and $v(\psi)\in \mathrm{H}_0^1(\Omega)$ satisfies following equality for any $\varphi\in\mathrm{H}_0^1(\Omega)$:

$$
(7)\quad \int_\Omega \mathbf{a}\nabla v(\psi)\cdot\nabla\varphi\,\mathrm{d}x = 2\int_\Omega \operatorname{div}(f\psi)\varphi\,\mathrm{d}x + 2\int_\Omega \mathbf{a}\left[-\operatorname{div}(\psi)\,\mathbf{I}+\nabla\psi+\nabla\psi^{\mathcal{T}}\right]\nabla u\cdot\nabla\varphi\,\mathrm{d}x.
$$

References

- Allaire, G., Pantz, O., *Structural optimization with* `FreeFem++` Structural and Multidisciplinary Optimization 32.3: 173-181. (2006)
- Dapogny, C., Frey, P., *Computation of the signed distance function to a discrete contour on adapted triangulation*, Calcolo, Volume 49, Issue 3, pp.193-219 (2012).
- Hecht, F. *New development in FreeFem++* J. Numer. Math. 20, no. 3-4, 251–265. 65Y15 (2012)
- Murat, F., Tartar L. *Calcul des Variations et Homogénéisation, Les Méthodes de l'Homogénéisation Théorie et Applications en Physique*, Coll. Dir. Etudes et Recherches EDF, 57, Eyrolles,Paris, pp.319-369 (1985)
- Henrot, A., Pierre M. *Shape Variation and Optimization* (2018)
- Laurain A., Sturm, K., *Domain expression of the shape derivative and application to electrical impedance tomography* Technical report 1863, Weierstrass - Institute for applied analysis and stochastics (2013)
- Vrdoljak, M. *Classical Optimal Design in Two-Phase Conductivity Problems*, SIAM Journal on Control and Optimization: 2020-2035 (2016)