Non local models based on hyperelasticity.

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Classical non linear elasticity

We say that a solid $\Omega \subset \mathbb{R}^3$ is elastic when it deforms under the action of an external load and recovers its original shape when the load stops acting.
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Let $u(x, t)$ be the position occupied by the material point $x \in \Omega$ at time $t$. $u : \Omega \times (0, T) \rightarrow \mathbb{R}^3$. 
Classical non linear elasticity

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Let $u(x, t)$ be the position occupied by the material point $x \in \Omega$ at time $t$. $u : \Omega \times (0, T) \rightarrow \mathbb{R}^3$.

The deformation gradient is the differential of $u$ with respect to $x$, $F = Du$. In components, $F_{i\alpha} = u_{i,\alpha} = \frac{\partial u_i}{\partial x_\alpha}$. 
Cauchy’s equation of motion

After imposing second Newton’s law, it is obtained:

\[(\rho_R \dot{v}) = \text{Div } T_R + \rho_R b\]
Cauchy’s equation of motion

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Density \cdot acceleration.
Cauchy’s equation of motion

After imposing second Newton’s law, it is obtained:

\[
(\rho_R \ddot{v}) = \nabla \cdot T_R + \rho_R b
\]

Piola-Kirchhoff stress tensor.
Cauchy’s equation of motion

After imposing second Newton’s law, it is obtained:

\[(\rho_R \dot{\gamma}) = \text{Div } T_R + \rho_R b\]

External forces.
Cauchy’s equation of motion

After imposing second Newton’s law, it is obtained:

\[ 0 = \text{Div}T_R + \rho_R b \quad \star \]
Cauchy's equation of motion

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Hyperelastic materials: There exists an energy function \( W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) such that \( T_R = D_F W(F) \).
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- Hyperelastic materials: There exists an energy function \( W : \mathbb{R}^{n \times n} \to \mathbb{R} \) such that \( T_R = D_F W(F) \).

- Equilibrium equations (\( * \)) as Euler-Lagrange equations.
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- Equilibrium equations (\( \star \)) as Euler-Lagrange equations.
- Deformation can be searched as a minimizer of the functional

\[ I(u) = \int_{\Omega} [W(Du) - Bu] \, dx. \]
Direct method of Calculus of Variations

The direct method of Calculus of Variations is a way of determining the existence of solution (a minimizer) of a variational problem provided the following ingredients:

1. Coercivity: \( \lim_{||u|| \to \infty} I(u) = +\infty \).

2. Weak lower semi-continuity: For every \( u_j \rightharpoonup u \) (weakly), we have the following inequality \( I(u) \leq \liminf I(u_j) \).
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I(u) \leq \lim \inf I(u_j).
\]
Polyconvexity

- Scalar case ($n = 1$ or $m = 1$): The s.w.l.s.c. of $I$ is obtained through the convexity of $W(x, u, \cdot)$. 

Definition: Polyconvexity

$W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be polyconvex iff there exists a convex function $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $W(A) = h(A, \text{cof } A, \text{det } A)$.
Polyconvexity

- Scalar case \((n = 1 \text{ or } m = 1)\): The s.w.l.s.c. of \(I\) is obtained through the convexity of \(W(x, u, \cdot)\).
- Vectorial case \((n, m > 1)\): Convexity may be a condition too strong but there are other weaker notions that provide the s.w.l.s.c.
Polyconvexity

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**Definition: Polyconvexity**

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W : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ is said to be polyconvex iff there exists a convex function } h : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R} \text{ such that }
\]

\[
W(A) = h(A, \text{cof} A, \det A).
\]
Polyconvexity of $W$
Polyconvexity of $W$ + Weak continuity of $\det Du$

$\Downarrow$

Weak lower semi-continuity of the functional $I = \int W(x, u, Du)$
Piola Identity
\[ \text{div}(\text{cof} Du) = 0 \]

Polyconvexity of \( W \)

Weak continuity of \( \det Du \)

Weak lower semi-continuity of the functional
\[ I = \int W(x, u, Du) \]
Material symmetry: Isotropy

Isotropic material

$W$ is isotropic if $SO(3) \subset S$, i.e.,

$$W(F) = W(FR) \quad \forall R \in SO(3).$$
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$$W(F) = W(FR) \quad \forall R \in SO(3).$$

In this case, the Rivlin-Ericksen theorem establishes that there exists $\tilde{h} : (0, +\infty)^3 \to \mathbb{R}$ such that

$$W(A) = \tilde{h}(|A|^2, |\text{cof}A|^2, (\det A)^2).$$

It suits very well with the polyconvexity assumption!!
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W(A) = \tilde{h}(|A|^2, |\text{cof}A|^2, (\det A)^2).
\]

It suits very well with the polyconvexity assumption!!
Existence theorem

Theorem (John Ball)

- There exists a convex function \( \hat{W} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times (0, \infty) \to (0, \infty) \) such that
  \[
  W(F) = \hat{W}(F, \text{cof} F, \det F) \quad \forall F \in \mathbb{R}^{3 \times 3}_+.
  \]
- \( W(F) \to \infty \) when \( \det F \to 0 \).
- \( W(F) \geq c_1(|F|^p + |\text{cof} F|^q + (\det F)^r) - c_2 \).

Then there exists a minimizer of the functional
  \[
  I[u] := \int_{\Omega} W(x, u, Du)dx.
  \]
Isotropic models

- **Money Rivlin materials**

\[ W(F) = \alpha |F|^2 + \beta |\text{cof}F|^2 + J(\det F), \]

with \( \lim_{t \to \infty} J(t) = +\infty \). \( \alpha, \beta > 0 \).

- **Neo-Hookean materials** in the case \( \beta = 0 \).

  **Curious note:** Pixar’s characters simulation: “Stable Neo-Hookean Flesh Simulation”.

- **Odgen materials**
Previously on solid mechanics and variational calculus...

Fractional model of hyperelasticity

- Motivation
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- A little bit of calculus in $H^{s,p}$
- Fractional Piola Identity
- Weak continuity of the determinant
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Why a fractional model of hyperelasticity?
Problem

It is no valid any more when the functions stop being continuous and some singularities arise, like fractures.
Motivation

- Lavrentiev phenomenon: minimizers may change as the functional space changes.
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- When a solid is subjected to great loads, singularities may appear such as fracture and cavitation (the sudden formation of voids in the material).
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- Lavrentiev phenomenon: minimizers may change as the functional space changes.

- When a solid is subjected to great loads, singularities may appear such as fracture and cavitation (the sudden formation of voids in the material).

- $W^{1,p}$ with $p > n$ forces functions to be continuous.
Previous approach

\[ I = \int \int w(x, x', u(x), u(x')) \, dx \, dx' \]

- Existence of solution and \( \Gamma \)-convergence were studied (Bellido, Mora-Corral 2014; Bellido, Mora-Corral, Pedregal 2015).
- Not suitable in hyperelasticity.
Previous approach

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- Not suitable in hyperelasticity.

\[ I = \int W(\int \cdots) \]
Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s} \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy.$$
Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy.$$ 

$$D^s u = \left( \frac{c_{n,s}}{|x|^{n-(1-s)}} \right) \ast Du$$
Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s} \text{ p. v.}_x \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy.$$ 

- $D^s u = \left( \frac{c_{n,s}}{|x|^{n-(1-s)}} \right) \ast Du$

- $\widehat{D^s u}(\xi) = \frac{1}{|2\pi \xi|^{1-s}} \widehat{Du} = \frac{2\pi i \xi}{|2\pi \xi|^{1-s}} \hat{u}(\xi)$
Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s} \text{ p. v.}_x \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy.$$ 

- $D^s u = \left( \frac{c_{n,s}}{|x|^{n-(1-s)}} \right) * Du$
- $(\Delta)^s u = - \sum_{j=1}^{N} \frac{\partial^s}{\partial x_j^s} \frac{\partial^s}{\partial x_j^s} u$
- $\hat{D}^s u(\xi) = \frac{1}{|2\pi \xi|^{1-s}} \hat{D}u$

$$= \frac{2\pi i \xi}{|2\pi \xi|^{1-s}} \hat{u}(\xi)$$
Functional space $H^{s,p}$

$$D^s u(x) := c_{n,s} \mathcal{P} \mathcal{V}_x \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy.$$ 

- $D^s u = \left( \frac{c_{n,s}}{|x|^{n-(1-s)}} \right) * Du$

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- $\hat{D^s u}(\xi) = \frac{1}{|2\pi \xi|^{1-s}} \hat{Du}$

- $= \frac{2\pi i \xi}{|2\pi \xi|^{1-s}} \hat{u}(\xi)$

- $D^s u \to Du$
Our goal was to prove the existence of minimizer of the functional

$$I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x)) \, dx,$$

where we have substituted the gradient by the so-called Riesz \( s \)-fractional gradient

$$D^s u(x) := c_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy.$$
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And so, the new space we are going to search the minimizers in is

$$H^{s,p}_g(\Omega) := \{ u \in H^{s,p}(\mathbb{R}^n) : u = g \text{ in } \Omega^c \},$$

where

$$H^{s,p}(\mathbb{R}^n) := \{ u \in L^p(\mathbb{R}^n) : D^s u \in L^p(\mathbb{R}^n; \mathbb{R}^n) \}.$$
Our goal was to prove the existence of minimizer of the functional

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where we have substituted the gradient by the so-called Riesz $s$-fractional gradient

$$D^s u(x) := c_{n,s} \text{ p.v. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \otimes \frac{x - y}{|x - y|} \, dy.$$

And so, the new space we are going to search the minimizers in is

$$H^{s,p}_g(\Omega, \mathbb{R}^m) := \{ u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) : u = g \text{ in } \Omega^c \},$$

where

$$H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) := \{ u \in L^p(\mathbb{R}^n, \mathbb{R}^m) : D^s u \in L^p(\mathbb{R}^n; \mathbb{R}^{n \times m}) \},$$
Functional space $H^{s,p}$

**Proposition**

a) $C^\infty_0(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$.

b) $H^{s,p}(\mathbb{R}^n)$ is reflexive.

c) If $s < t < 1$ and $1 < q \leq p \leq \frac{nq}{n-(t-s)q}$, then $H^{t,q}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$.

d) If $0 < \mu \leq s - \frac{n}{p}$, then $H^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n)$.

e) If $p = 2$, then $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ with equivalence of norms.

f) If $0 < s_1 < s < s_2 < 1$ then $H^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s_1,p}(\mathbb{R}^n)$. 
# Functional space $H^{s,p}$

## Proposition

1. $C_c^\infty(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$.
2. $H^{s,p}(\mathbb{R}^n)$ is reflexive.
3. If $s < t < 1$ and $1 < q \leq p \leq \frac{nq}{n-(t-s)q}$, then $H^{t,q}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$.
4. If $0 < \mu \leq s - \frac{n}{p}$, then $H^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n)$.
5. If $p = 2$, then $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ with equivalence of norms.
6. If $0 < s_1 < s < s_2 < 1$ then $H^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s_1,p}(\mathbb{R}^n)$.
Functional space $H^{s,p}$

**Theorem (Shieh-Spector 2015)**

Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists $C = C(|\Omega|, n, p, s) > 0$ such that

$$
||u||_{L^q(\Omega)} \leq C||D^s u||_{L^p(\mathbb{R}^n)}
$$

for all $u \in H^{s,p}(\mathbb{R}^n)$, and any $q$ satisfying

$$
\begin{cases}
q \in [1, p^*] & \text{if } sp < n, \\
q \in [1, \infty) & \text{if } sp = n, \\
q \in [1, \infty] & \text{if } sp > n.
\end{cases}
$$

$p^* = \frac{np}{n-sp}$. 
Functional space \( H^{s, p} \)

**Theorem (Shieh–Spector 2017)**

Set \( 0 < s < 1 \) and \( 1 < p < \infty \). Let \( \Omega \subset \mathbb{R}^n \) be open and bounded and \( g \in H^{s, p}(\mathbb{R}^n) \). Then for any sequence \( \{u_j\}_{j \in \mathbb{N}} \subset H^{s, p}_g(\Omega) \) such that

\[
u_j \rightharpoonup u \quad \text{in} \quad H^{s, p}(\mathbb{R}^n), \]

for some \( u \in H^{s, p}(\mathbb{R}^n) \), one has \( u \in H^{s, p}_g(\Omega) \) and

\[
u_j \rightarrow u \quad \text{in} \quad L^q(\mathbb{R}^n), \]

for every \( q \) satisfying

\[
\begin{cases}
q \in [1, p^*) & \text{if } sp < n, \\
q \in [1, \infty) & \text{if } sp = n, \\
q \in [1, \infty) & \text{if } sp > n.
\end{cases}
\]
Singularity: fracture and cavitation

$H^{s,p}$ may include functions with singularities forbidden in Sobolev spaces of interest in a pure mathematical point of view as well as an applied one.
Singularities: fracture and cavitation

$H^{s,p}$ may include functions with singularities forbidden in Sobolev spaces of interest in a pure mathematical point of view as well as an applied one.

- **Fracture:** Let $Q = (0, 1)^n$ and $\varphi_2, \ldots, \varphi_n \in C_\infty^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \ldots, \varphi_n)$. Then

  \[ u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{1}{s}, \quad \text{and} \quad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{1}{s}. \]
Singularities: fracture and cavitation

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- **Fracture:** Let $Q = (0, 1)^n$ and $\varphi_2, \ldots, \varphi_n \in C_\infty^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \ldots, \varphi_n)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{1}{s}, \quad \text{and} \quad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{1}{s}.$$

- **Cavitation:** Let $\varphi \in C_\infty^\infty([0, \infty))$ be such that $\varphi(0) > 0$, and $u(x) = \frac{x}{|x|} \varphi(|x|)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{n}{s}, \quad \text{and} \quad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{n}{s}.$$

**Cavitation:**

The sudden formation of voids in a material.
Previously on solid mechanics and variational calculus...

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Theorem

Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:

(a) $W(x, y, \cdot)$ is polyconvex.

(b) There exists $h : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} \frac{h(t)}{t} = \infty$ and

$$
W(x, y, F) \geq a(x) + c |F|^p + c |\text{cof} F|^q + h(|\det F|), \quad \text{if } sp < n,
$$

$$
W(x, y, F) \geq a(x) + c |F|^p, \quad \text{if } sp \geq n,
$$

for $a \in L^1$ and some $q > \frac{p^*}{p^* - 1}$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $u_0 \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$.

Then there exists a minimizer of $I$ in $H^{s,p}_{u_0}(\Omega, \mathbb{R}^n)$. 
How do we do it?
Previously on solid mechanics and variational calculus...

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Fractional divergence

The definition of the $s$-fractional divergence is given in order to fulfil an integration by parts/Divergence Theorem.

$$\text{div}^s \varphi(x) = -c_{n,s} \text{ p. v.} \int_{\mathbb{R}^n} \frac{\varphi(x) + \varphi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy.$$
Fractional divergence

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**Theorem: Integrations by parts**

Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $D^s u \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, then for all $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

\[ \int D^s u(x) \cdot \phi(x) \, dx = - \int u(x) \text{div}^s \phi(x) \, dx. \]
Polyconvexity of $W$ +
Polyconvexity of $W$ + Weak continuity of $\text{det } D^s u$

$\Rightarrow$

Weak lower semi-continuity of the functional $I = \int W(x, u, D^s u)$
Piola Identity
\[
\text{div}^s(\text{cof}D^s u) = 0
\]

Polyconvexity of \( W \)

Weak continuity of \( \text{det} D^s u \)

\[
\downarrow \quad \downarrow
\]

Weak lower semi-continuity of the functional
\[
I = \int W(x, u, D^s u)
\]
Piola Identity
\[ \text{div}^s(\text{cof} D^s u) = 0 \]

Polyconvexity of \( W \) + Weak continuity of \( \det D^s u \)

Weak lower semi-continuity of the functional \( I = \int W(x, u, D^s u) \)
Piola Identity
\[ \text{div}^S(\text{cof}D^S u) = 0 \]

⇒ Polyconvexity of \( W \)

⇒ Weak continuity of \( \det D^S u \)

⇒ Weak lower semi-continuity of the functional
\[ I = \int W(x, u, D^S u) \]
s-fractional divergence of the product

**Lemma**

Let $g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in C^1_c(\mathbb{R}^n)$. Then $\varphi g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$,

$$\text{div}^s(\varphi g)(x) = \varphi(x)\text{div}^s g(x) + K_{\varphi(x)}(g^T)(x),$$

where the operator $K_{\varphi} : L^q(\mathbb{R}^n, \mathbb{R}^{k \times n}) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^k)$ defined as

$$K_{\varphi}(U)(x) = c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} U(y) \frac{x - y}{|x - y|} \, dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

is linear and bounded for all $p \in [1, q]$. 
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Piola Identity

**Fractional Piola Identity**

Let $u \in C_c^\infty(\mathbb{R}^n)$, $s \in (0, 1)$. Then

$$\text{Div}^s(\text{cof}(D^s u)) = 0.$$
Piola Identity

**Fractional Piola Identity**

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In the case of the first row:

$$-c_{n,s} \rho v_x \int_{\mathbb{R}^n} \frac{(\text{cof}(D^s u))_1(x')}{|x - x'|^{n+s}} \cdot \frac{x - x'}{x - x'} dx' = 0$$

Special attention had to be paid to the limits in the singularities!
Previously on solid mechanics and variational calculus...

Fractional model of hyperelasticity
- Motivation
- Existence
- A little bit of calculus in $H^{s,p}$
- Fractional Piola Identity
- **Weak continuity of the determinant**
- $\Gamma$-convergence
- Bounded domains
Integration by parts of the determinant

Classical result

\[
\int \det(Du)(x) \varphi(x) \, dx = -\frac{1}{n} \int u(x) \cdot \nabla \varphi \text{cof}(Du)(x) \, dx
\]

\[
\uparrow
\]

\[
\det(Du) = \text{div}(u_k(\text{cof}Du)_k) \quad \forall k = 1, \ldots, n
\]
Integration by parts of the determinant

**Classical result**

\[
\int \det(Du)(x) \varphi(x) \, dx = -\frac{1}{n} \int u(x) \cdot \nabla \varphi \text{cof}(Du)(x) \, dx
\]

**Lemma**

For every \( \varphi \in C^\infty_c(\mathbb{R}^n) \) we have that \( u \cdot K_\varphi(\text{cof}D^s u) \in L^1(\mathbb{R}^n) \) and

\[
\int \det(D^s u)(x) \varphi(x) \, dx = -\frac{1}{n} \int u(x) \cdot K_\varphi(\text{cof}(D^s u))(x) \, dx.
\]
Weak continuity of the determinant

Weak continuity of the minors

Let \( u_j \rightharpoonup u \) in \( H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \) and \( M : \mathbb{R}^{n \times n} \to \mathbb{R} \) a minor of order \( r \), then

\[
M(D^s u_j) \rightharpoonup M(D^s u)
\]

in \( L^p_r(\mathbb{R}^n) \).
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Convergence of the fractional gradients

For $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ we have that

$$D^s u \to Du$$

as $s$ goes to $1^-$, strongly in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times m})$. 
Recovering the classical model

**Γ-convergence in fractional hyperelasticity**

Let \( W : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \to \mathbb{R} \) such that \( W(x, u, \cdot) \) is quasiconvex for a.e. \( \mathbb{R}^n \) and all \( u \in \mathbb{R}^m \). Let

\[
I_s(u) = \int_{\mathbb{R}^n} W(x, u, D^s u) \, dx
\]

be defined in \( H^{s,p}_g(\Omega; \mathbb{R}^m) \), and let

\[
I(u) = \int_{\mathbb{R}^n} W(x, u, Du) \, dx
\]

be defined on \( W^{1,p}_g(\Omega; \mathbb{R}^m) \). Then

\( I_s \) Γ-converges to \( I \).
Γ-convergence

Fractional Mean Value Theorem ($p = 1$ Comi-Stefani 2019)

Let $u \in H^{s,p}(\mathbb{R}^n)$. Then, for every $s_0 > 0$ there exists a constant $C > 0$ such that for every $s$, $s_0 \leq s < 1$ and for every $h \in \mathbb{R}^n$

$$
\int_{\mathbb{R}^n} |u(x + h) - u(x)|^p \, dx \leq C |h|^{sp} \|D^s u\|_{L^p(\mathbb{R}^n)}^p.
$$

Proposition

Let $\{u_s\}_{s \in (0,1)}$ where each $u_s \in H^{s,p}_g(\Omega)$. If

$$
\|D^s u_s\|_{L^p(\mathbb{R}^n)} \leq C,
$$

then there exists $u \in W^{1,p}(\mathbb{R}^n)$ such that $D^s u_s \rightharpoonup Du$ in $L^p(\mathbb{R}^n)$. 
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New model on bounded domains

Motivated by applications, we would like to obtain similar results but over bounded domains and including different kinds of boundary conditions.

\[ I[u] = \int_{\Omega} W(G_{\rho} u(x)) \, dx \]

where \( G_{\rho} u \) is a non-local gradient,

\[ G_{\rho}(u) = \int_{\Omega} \frac{u(x) - u(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho(|x - y|) \, dy, \]

and \( W(F) = \hat{W}(F, \text{cof} F, \text{det} F) \),

with \( \hat{W} \) convex.
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\[ \rho(|x - y|) = \frac{1}{|x - y|^{n(1-s)}} \chi_{\mathbb{B}(0,1-s)} \]
Thank you very much for your attention.
References
