CONTINUOUS OPTIMIZATION

Department MATHEMATIK

FAU FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
FACULTY OF SCIENCES

Recent results on multi-dimensional nonlocal balance laws

VIII Partial differential equations, optimal design and numerics
Benasque, 18.08.-30.08.2019

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20.08.2019

Applications

- Traffic flow
- Supply chains
- Sedimentation models

and especially in the case where multiple spatial variables are considered

- Crowd dynamics: Look around behavior of individuals
- Particle size evolution: multi-dimensional nonlocal population balance equations (PBE) describe the dynamics of the particle "shape" distribution
  - ... if the growth kinetics of particles depend on information of the whole population (e.g. total surface or total mass)...
  - ... and if the particles are described by multiple parameters (e.g. length and width of needle shaped particles or polygonal approximation of the shape of crystals)

Mathematical formulation

We call $q:\overline{\Omega_T}\to\mathbb{R}$ with $\Omega_T:=(0,T)\times\mathbb{R}^n$ a (weak) solution iff it satisfies the following initial value problem (in a weak sense)

$$
\begin{aligned}
q_t(t,\boldsymbol{x})+\operatorname{div}_{\boldsymbol{x}}\Big(\lambda\Big[W[q,\gamma,\mathscr{A}]\Big](t,\boldsymbol{x})q(t,\boldsymbol{x})\Big)&=0 && (t,\boldsymbol{x})\in\Omega_T\\
q(0,\boldsymbol{x})&=q_0(\boldsymbol{x}) && \boldsymbol{x}\in\Omega\\
W[q,\gamma,\mathscr{A}](t,\boldsymbol{x})&:=\iint_{\mathscr{A}(t)}\gamma(t,\boldsymbol{x},\boldsymbol{y})q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} && (t,\boldsymbol{x})\in\Omega_T\\
\lambda\left[W[q,\gamma,\mathscr{A}]\right](t,\boldsymbol{x})&:=\lambda\left(W[q,\gamma,\mathscr{A}](t,\boldsymbol{x}),t,\boldsymbol{x}\right) && (t,\boldsymbol{x})\in\Omega_T
\end{aligned}
$$

with

- $q_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $\gamma\in C([0,T];C^1_{\mathrm{b}}(\mathbb{R}^n\times\mathbb{R}^n))$
- $\lambda\in C^1(\mathbb{R};C([0,T];C^1(\mathbb{R}^n;\mathbb{R}^n)))$ satisfying for given $W\in C([0,T];C^1_{\mathrm{b}}(\mathbb{R}^n))$ the following estimates:

$$
\begin{aligned}
\exists A\in L^\infty_{\mathrm{loc}}(\mathbb{R}_{>-1}):&\quad \|\mathrm{D}_3\lambda[W]\|_{C([0,T];C(\mathbb{R}^n;\mathbb{R}^{n\times n}))}\leq A\left(\|W\|_{C([0,T];C(\mathbb{R}^n))}\right)\\
\exists B\in L^\infty_{\mathrm{loc}}(\mathbb{R}_{>-1}):&\quad \|\partial_1\lambda[W]\|_{C([0,T];C(\mathbb{R}^n;\mathbb{R}^n))}\leq B\left(\|W\|_{C([0,T];C(\mathbb{R}^n))}\right).
\end{aligned}
$$

Multi-dimensional nonlocal balance laws in the literature

- In [A. Aggarwal, R.M. Colombo, and P. Goatin. *Nonlocal systems of conservation laws in several space dimensions*. SIAM Journal on Numerical Analysis, 2015] convergence of a subsequence of solutions to a modified Lax-Friedrichs scheme resulting in a weak Entropy solution, also known as Kružkov solution, is shown
- In [R.M. Colombo and M. Lécureux-Mercier. *Nonlocal crowd dynamics models for several populations*. Acta Mathematica Scientia, 2012] this PDE is considered

$$\begin{aligned} \partial_t q(t,\boldsymbol{x}) + \operatorname{div}_{\boldsymbol{x}}\big(q(t,\boldsymbol{x})v\big((q*\eta)(t,\boldsymbol{x})\big)\vec{\boldsymbol{v}}(\boldsymbol{x})\big) &= 0 && (t,\boldsymbol{x}) \in \Omega_T \\ q(0,\boldsymbol{x}) &= q_0(\boldsymbol{x}) && \boldsymbol{x} \in \Omega \end{aligned}$$

together with the setting

$$v \in C^2_{\mathrm{b}}(\mathbb{R}),\ \vec{\boldsymbol{v}} \in C^2(\mathbb{R}^n;\mathbb{S}^{n-1}) \cap W^{2,1}(\mathbb{R}^n;\mathbb{S}^{n-1}),\ \eta \in C^2_{\mathrm{b}}(\mathbb{R}^n;[0,1]) \text{ with } \|\eta\|_{L^1(\mathbb{R})}$$

The setting in the previous slide covers this already for

$$v \in C^1_{\mathrm{b}}(\mathbb{R}) \qquad \vec{\boldsymbol{v}} \in C^1_{\mathrm{b}}(\mathbb{R}^n;\mathbb{R}^n) \quad \eta \in C^1_{\mathrm{b}}(\mathbb{R}^n).$$

$\Rightarrow$ Generalization by the here presented framework; also no further Entropy conditions are required

Feasible integration areae in the nonlocal term

- For $n \in \mathbb{N}_{\geq 1}, K \in \mathbb{R}_{\geq 0}$ we define the sets

$$\begin{aligned}
\mathscr{M}^n &:= \{\mathscr{A} \in \mathscr{L}(\mathbb{R}^n) : \ (\mathscr{A} \in \mathscr{L}_\mathrm{b}(\mathbb{R}^n) \vee (\mathbb{R}^n \setminus \mathscr{A}) \in \mathscr{L}_\mathrm{b}(\mathbb{R}^n)) \\
&\qquad \wedge \ \partial\mathscr{A} \ (n-1)\text{-rectifiable}\} \\
\mathscr{M}^n_K &:= \{\mathscr{A} \in \mathscr{M}^n : \mathscr{H}^{n-1}(\partial\mathscr{A}) \leq K\} \\
C([0,T];\mathscr{M}^n_K) &:= \Big\{\mathscr{F} : [0,T] \to \mathscr{M}^n_K : \lim_{[0,T]\ni t \to s} \mathrm{d}^n_\Delta(\mathscr{F}(t), \mathscr{F}(s)) = 0, \\
&\qquad \forall s \in [0,T]\Big\}.
\end{aligned}$$

- $C([0,T];\mathscr{M}^n_K)$ is called the set of **feasible integration areae** (of the nonlocal term) for given $K \in \mathbb{R}_{\geq 0}$. For $\mathscr{A}, \mathscr{B} \in \mathscr{L}(\mathbb{R}^n)$ the term $\mathrm{d}^n_\Delta(\mathscr{A}, \mathscr{B})$ denotes the n-dimensional Lebesgue-measure of the symmetric difference of both sets.
- Roughly speaking: The continuity condition will ensure the continuity of solutions of the nonlocal balance laws w.r.t. time and the rectifiability condition a contraction property in a later required fixed-point equation

Characteristics

Let $T \in \mathbb{R}_{>0}$ and a nonlocal term $w \in C([0,T_1]; C^1_{\mathrm{b}}(\mathbb{R}^n))$ be given. We call ξ_w for $(t,\boldsymbol{x}) \in (0,T) \times \mathbb{R}^n$ satisfying

$$\xi_w[t,\boldsymbol{x}](\tau) = \boldsymbol{x} + \int_t^\tau \lambda[w]\big(s, \xi_w[t,\boldsymbol{x}](s)\big)\,\mathrm{d}s, \quad \tau \in [0,T]$$

the **characteristics** corresponding to the velocity function $\lambda[w]$.

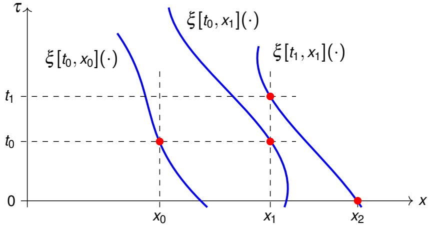


Figure 1: Characteristics for different initial values $x_0, x_1, x_2 \in \mathbb{R}$ and times $t_0, t_1 \in (0,T]$ with $\xi[0,x_2](t_1) = \xi[t_1,x_1](0)$

Main theorem

Theorem (Existence and uniqueness of a weak solution on any finite time horizon)

For any $T \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}_{\geq 1}$ and the assumptions

- $q_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$
- $\gamma \in C([0,T]; C_b^1(\mathbb{R}^n \times \mathbb{R}^n))$
- $\lambda \in C^1(\mathbb{R}; C([0,T]; C^1(\mathbb{R}^n;\mathbb{R}^n)))$ *satisfying for given* $W \in C([0,T]; C_b^1(\mathbb{R}^n))$ *the following estimates:*

$$\begin{aligned}
\exists A \in L^\infty_{\text{loc}}(\mathbb{R}_{>-1}): \quad & \|\mathrm{D}_3\lambda[W]\|_{C([0,T];C(\mathbb{R}^n;\mathbb{R}^{n\times n}))} \leq A\left(\|W\|_{C([0,T];C(\mathbb{R}^n))}\right)\\
\exists B \in L^\infty_{\text{loc}}(\mathbb{R}_{>-1}): \quad & \|\partial_1\lambda[W]\|_{C([0,T];C(\mathbb{R}^n;\mathbb{R}^n))} \leq B\left(\|W\|_{C([0,T];C(\mathbb{R}^n))}\right)
\end{aligned}$$

and for $\mathscr{A} \in C([0,T]; \mathscr{M}_K^n)$ with $K \in \mathbb{R}_{\geq 0}$ the nonlocal balance law admits a unique weak solution $q \in C([0,T]; L^1(\mathbb{R}^n))$.

Sketch of proof

- Assumption: the nonlocal term $W[q,\gamma,\mathscr{A}](t,x)$ is given by $w(t,x)$, a function explicitly depending only on t,x
- Obtaining a **linear** balance law by plugging it in the nonlocal balance law $\Rightarrow$ unique solution of linear balance law
- Banach's Fixed Point Theorem: Unique solution of the fixed-point equation $\mathbf{F}[w]=w$ for small times with

$$\mathbf{F}:\begin{cases} C([0,T_1];C^1_{\mathrm{b}}(\mathbb{R}^n)) & \to C([0,T_1];C^1_{\mathrm{b}}(\mathbb{R}^n)) \\ w & \mapsto \left((t,\boldsymbol{x})\mapsto \iint\limits_{\xi_w[t,\mathscr{A}(t)](0)} \gamma(t,\boldsymbol{x},\xi_w[0,\boldsymbol{y}](t))q_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\end{cases}$$

- Specific class of test functions and the fundamental lemma of calculus of variation $\to$ **unique** solution of the nonlocal balance law for small times (no Entropy condition needed!)
- Clustering argument in time $\to$ solution up to time $t=T$
- Solution representation

$$q_{w^*}(t,x):=q_0(\xi_{w^*}[t,x](0))\partial_2\xi_{w^*}[t,x](0)$$

with the unique solution of the fixed-point equation w^*.

Ingredients for contraction I/II

- In the fixed-point equation: When showing the contraction/Lipschitz continuity of the fixed-point mapping **F**, we obtain:

$$\|\mathbf{F}[w]-\mathbf{F}[\tilde{w}]\| \leq \mathrm{d}_{\Delta}(\xi_w[t,\mathscr{A}(t)](0),\xi_{\tilde{w}}[t,\mathscr{A}(t)](0)) + \text{other terms.}$$

$\Rightarrow$ Goal: $\mathrm{d}_{\Delta}(\xi_w[t,\mathscr{A}(t)](0),\xi_{\tilde{w}}[t,\mathscr{A}(t)](0)) \leq C\|\xi_w-\xi_{\tilde{w}}\| \leq \hat{C}\|w-\tilde{w}\|.$

- In the following, the two inequalities in the upper goal will be addressed (starting by the second one)

Lemma (Stability of characteristics w.r.t. nonlocal terms)

For every $t,\tau \in [0,T]$ and $w,\tilde{w} \in C(\Omega_T;\mathbb{R}^n)$ with $\|w\|_{C(\Omega_T;\mathbb{R}^n)} \leq M$ and $\|\tilde{w}\|_{C(\Omega_T;\mathbb{R}^n)}\} \leq M$ for a $M \in \mathbb{R}_{>0}$ we obtain

$$\begin{aligned}&\|\xi_w[t,\cdot](\tau)-\xi_{\tilde{w}}[t,\cdot](\tau)\|_{C(\mathbb{R}^n;\mathbb{R}^n)}\\ &\leq |t-\tau|\|w-\tilde{w}\|_{C([0,T];C(\mathbb{R}^n))}\|B\|_{L^\infty((0,M))}\cdot \mathrm{e}^{T\left(\|B\|_{L^\infty((0,M))}\|\nabla\tilde{w}\|_{C(\Omega_T;\mathbb{R}^n)}+\|A\|_{L^\infty((0,M))}\right)}.\end{aligned}$$

Motivation: symmetric difference of sets and characteristics

- W.l.o.g.: $\xi_w[t,\cdot](0) \equiv \mathrm{Id}$. Define $\mathscr{B}(t) := \xi_{\tilde{w}}[t,\mathscr{A}(t)](0)$ where $\mathscr{A} \in C([0,T];\mathscr{M}_K^n)$ and $\mathscr{A}(t)$ is a circle with radius $R \in \mathbb{R}_{>0}$
- $r := \|\xi_w[\cdot,*](0) - \tilde{\xi}_{\tilde{w}}[\cdot,*](0)\|_{C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)}$

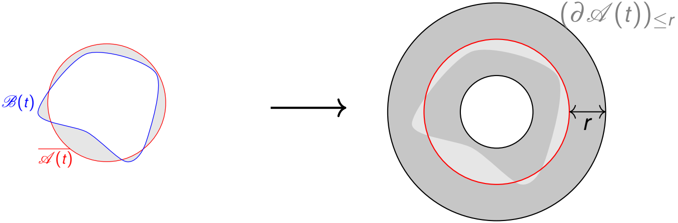


Figure 2: Estimating the Lebesgue-measure of the symmetric difference by analyzing the Lebesgue-measure of parallel sets.

In this case:

$$\begin{aligned}&\mathrm{d}_\Delta(\xi_w[t,\mathscr{A}(t)](0),\xi_{\tilde{w}}[t,\mathscr{A}(t)](0))\\ &\leq \pi((R+r)^2-(R-r)^2) = 4\pi Rr\\ &= 2\mathscr{H}^1(\partial\mathscr{A}(t))\|\xi_w[\cdot,*](0) - \tilde{\xi}_{\tilde{w}}[\cdot,*](0)\|_{C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)}.\end{aligned}$$

Ingredients for contraction II/II

Lemma (Estimate for the volume of the symmetric difference w.r.t. characteristics)

For $n \in \mathbb{N}_{\geq 1}$ and $T \in \mathbb{R}_{>0}$ let $\mathscr{A} \in C([0,T]; \mathscr{M}_K^n)$ for a $K \in \mathbb{R}_{\geq 0}$ be given. Moreover, define for ξ_w and $\xi_{\tilde{w}}$ and $w, \tilde{w} \in C([0,T]; C_b^1(\mathbb{R}^n))$ for every $t, s \in [0,T]$

$$L_{w,\tilde{w}}(t,s) := \max\left\{ \|\det\left(\mathrm{D}_2\xi_w[t,\cdot](s)\right)\|_{C(\mathbb{R}^n)}, \ \|\det\left(\mathrm{D}_2\xi_{\tilde{w}}[t,\cdot](s)\right)\|_{C(\mathbb{R}^n)} \right\}.$$

Then we obtain

$$\mathrm{d}_\Delta^n\left(\xi_w[t,\mathscr{A}(t)](s), \xi_{\tilde{w}}[t,\mathscr{A}(t)](s)\right) \leq 4KL_{w,\tilde{w}}(t,s)\|\xi_w[t,\cdot](s) - \xi_{\tilde{w}}[t,\cdot](s)\|_{C(\mathbb{R}^n)}.$$

Moreover, $L_{w,\tilde{w}}(t,s)$ can be bounded uniformly in $t, s \in [0,T]$, i.e.

$$\|L_{w,\tilde{w}}\|_{C([0,T]^2)} \leq \|\max\{\det\left(\mathrm{D}_2\xi_w\right), \det\left(\mathrm{D}_2\xi_{\tilde{w}}\right)\}\|_{C(\Omega_T \times [0,T])}.$$

Remark: Minimizing topological boundaries

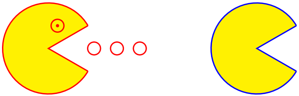


Figure 3: Topological boundary in red and minimized topological boundary in blue.

- In PBE itself the Lebesgue-measure λ^n of $\mathscr{A}$ is used but not the Hausdorff-measure of $\partial\mathscr{A}$
- **Question**: How to “minimize” the topological boundary such that the Lebesgue measure does not change but the Hausdorff measure decreases
- **Answer**: If $\mathscr{A} \in \mathscr{M}^n$ is a Borel-set, then exists a Borel set $\tilde{\mathscr{A}} \in \mathbb{R}^n$ so that $\mathrm{d}^n_\Delta(\mathscr{A}, \tilde{\mathscr{A}}) = 0$ where the topological boundary of $\tilde{\mathscr{A}}$ satisfies

$$\partial\tilde{\mathscr{A}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : 0 < \frac{\lambda^n(\mathscr{A} \cap B_r(\boldsymbol{x}))}{\lambda^n(B_r(\boldsymbol{x}))} < 1 \quad \forall r \in \mathbb{R}_{>0} \right\}.$$

- Roughly speaking: Neglection of points with Lebesgue density zero (isolated points, lines, etc.) or one (points, lines, etc. in the interior of the set)

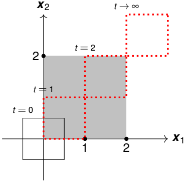
Example 1: Moving cuboid I/II

$$q_t(t,\boldsymbol{x}) + \operatorname{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) = 0$$

$$q(0,\boldsymbol{x}) := \chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\boldsymbol{x})$$

$$\lambda[W[q]](t,\boldsymbol{x}) = \begin{pmatrix} W[q](t) \\ W[q](t) \end{pmatrix}$$

$$W[q](t) = \iint\limits_{[0,2]^2} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$



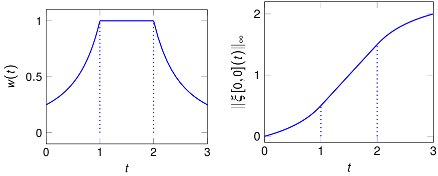
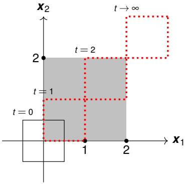


Figure 4: **Top right:** Illustration of the “moving” support of the solution q. **Bottom:** nonlocal term and the L^∞-norm of the initial characteristic.

Example 1: Moving cuboid II/II

$$
\begin{aligned}
q_t(t,\boldsymbol{x}) + \operatorname{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) &= 0\\
q(0,\boldsymbol{x}) &:= \chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\boldsymbol{x})\\
\lambda[W[q]](t,\boldsymbol{x}) &= \begin{pmatrix} W[q](t) \\ W[q](t) \end{pmatrix}\\
W[q](t) &= \iint_{[0,2]^2} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.
\end{aligned}
$$



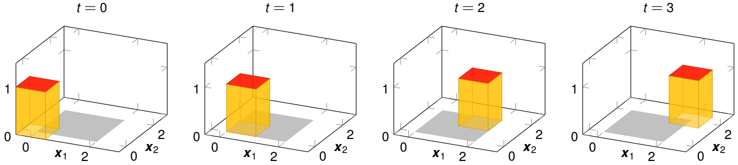


Figure 5: A time series of the solution q, gray area denotes the nonlocal area of integration, illustrated for $t \in \{0,1,2,3\}$.

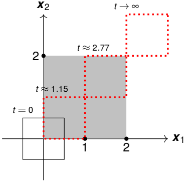
Example 2: Moving and shrinking cuboid I/II

$$q_t(t,\boldsymbol{x}) + \mathrm{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) = -\tfrac{1}{4}q(t,\boldsymbol{x})$$

$$q(0,\boldsymbol{x}) := \chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\boldsymbol{x})$$

$$\lambda[W[q]](t,\boldsymbol{x}) = \begin{pmatrix} W[q](t) \\ W[q](t) \end{pmatrix}$$

$$W[q](t) = \iint_{[0,2]^2} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$



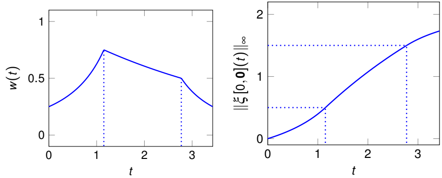
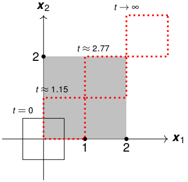


Figure 6: **Top right:** Illustration of the “moving” support of the shrinking solution q. **Bottom:** nonlocal term and the L^∞-norm of the initial characteristic.

Example 2: Moving and shrinking cuboid II/II

$$
\begin{aligned}
q_t(t,\boldsymbol{x}) + \operatorname{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) &= -\tfrac{1}{4}q(t,\boldsymbol{x})\\
q(0,\boldsymbol{x}) &:= \chi_{[-\frac{1}{2},\frac{1}{2}]^2}(\boldsymbol{x})\\
\lambda[W[q]](t,\boldsymbol{x}) &= \begin{pmatrix} W[q](t)\\ W[q](t)\end{pmatrix}\\
W[q](t) &= \iint_{[0,2]^2} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.
\end{aligned}
$$



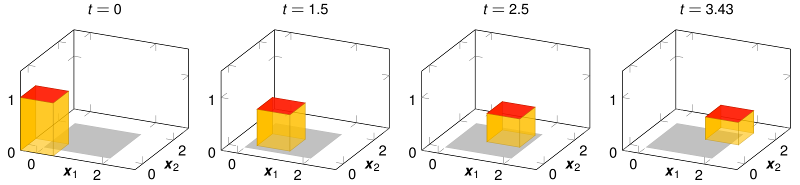


Figure 7: A time series of the solution q, gray area denotes the nonlocal area of integration, illustrated for $t \in \{0, 1.5, 2.5, 3.43\}$.

Example 3: Rotation of increasing partial annulus I/II

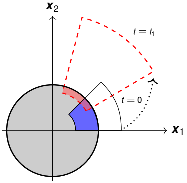
$$q_t(t,\boldsymbol{x}) + \operatorname{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) = 0$$

$$q(0,\boldsymbol{x}) := \chi_A(\boldsymbol{x})$$

$$A = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;:\; 1 \leq \|\boldsymbol{x}\|_2 \leq 3,\ 0 \leq \boldsymbol{x}_2 \leq \boldsymbol{x}_1\right\}$$

$$\lambda[W[q]](t,\boldsymbol{x}) = W[q](t)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\boldsymbol{x}$$

$$W[q](t) = \iint\limits_{B_2(\boldsymbol{0})} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$



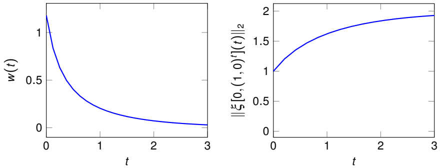


Figure 8: **Top right:** Illustration of rotating and increasing partial annulus (solution). **Bottom:** The nonlocal term $w(t)$ for $t \in [0,3]$ and the Euclidean norm of the characteristic curve starting from $(1,0)^t$ for $t \in [0,3]$.

Example 3: Rotation of increasing partial annulus II/II

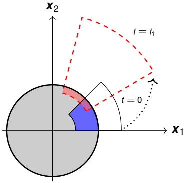
$$q_t(t,\boldsymbol{x}) + \operatorname{div}_2\left(\lambda[W[q]](t,\boldsymbol{x})q(t,\boldsymbol{x})\right) = 0$$

$$q(0,\boldsymbol{x}) := \chi_A(\boldsymbol{x})$$

$$A = \left\{\boldsymbol{x}\in\mathbb{R}^2 \;:\; 1 \le \|\boldsymbol{x}\|_2 \le 3,\ 0 \le \boldsymbol{x}_2 \le \boldsymbol{x}_1\right\}$$

$$\lambda[W[q]](t,\boldsymbol{x}) = W[q](t)\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\boldsymbol{x}$$

$$W[q](t) = \iint\limits_{B_2(\boldsymbol{0})} q(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$



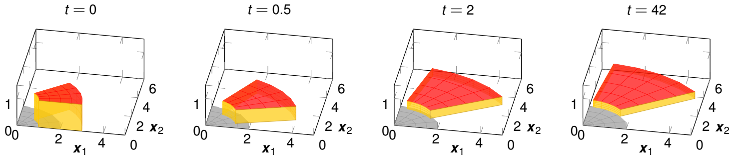


Figure 9: **Top right:** Illustration of rotating and increasing partial annulus (solution). **Bottom:** A time series of the solution q, gray area denotes the nonlocal area of integration, illustrated for $t \in \{0, 0.5, 2, 42\}$.

Conclusion

Summary

- Generalization of results in the solution theory for multi-dimensional nonlocal balance laws w.r.t. weights γ, velocity function λ and feasible integration areas $\mathscr{A}$
- Presentation of solution approach basing on solving characteristic equations and fixed-point equations in the nonlocal term
- Illustration of several analytical examples

Further research

- Integration areas $\mathscr{A}$ depending on $\boldsymbol{x} \rightarrow$ look around of individuals in a crowd
- $\mathscr{A} = \mathbb{R}^n_{>\boldsymbol{x}_{\text{min}}} \rightarrow$ relevant for ripening processes in chemical engineering
- Non-dissipative numerical scheme based on characteristics

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- Presentation of solution approach basing on solving characteristic equations and fixed-point equations in the nonlocal term
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Thanks for listening.
Any questions? Then please, feel free to ask!