Initial data identification.

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Outline of the talk

1. Entropy solutions crash course
   - Generalities
   - Blow up
   - Riemann Problem

2. Initial data identification: statement of the problem

3. Necessity

4. Sufficiency
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Conservation laws

\[ \frac{\partial u_i}{\partial t} + \sum_{j=1}^{d} \frac{\partial f_{i,j}(u_1, \ldots, u_n)}{\partial x_j} = 0, \quad 1 \leq i \leq n. \]

- Many physical systems: gas dynamics, magneto-hydrodynamic, electromagnetism, shallow water theory ...
- Scalar case: fewer physical systems (still: traffic, crowd dynamics, petroleum engineering) but first important step toward systems.
- Second order terms neglected.
Outline of the theory

- Small time solutions in $H^s$.
- Finite time blow up for most smooth initial data.
- Weak solutions in $L^\infty$ but no more uniqueness.
- Entropy solutions: taking the forgotten terms into account.
- No more reversibility in time!
- No linearization!
- No fixed point!
Bibliography for the Cauchy problem

- **Scalar**: 1-d Oleinik (59), n-d Kruzkov (70).
- **Systems 1-d**: Lax (57), Glimm (65), Bressan (92...).
- **Vanishing viscosity for systems**: Bianchini-Bressan (05).
- **Boundary conditions**: Bardos-Leroux-Nedelec (77), Otto (95)...
- **General case**: OPEN
Entropy solutions crash course
- Generalities
- Blow up
- Riemann Problem

Initial data identification: statement of the problem

Necessity

Sufficiency
Method of Characteristics: a synthetic slide

\[ \partial_t u + \partial_x (f(u)) = 0 \]

\[ \iff \partial_t u + f'(u)\partial_x u = 0 \]

\[ \iff \begin{cases} \frac{d}{dt} \psi(t, x) = f'(u(t, \psi(t, x))), & \psi(0, x) = x \\ \frac{d}{dt} u(t, \psi(t, x)) = 0 \end{cases} \]

\[ \iff \begin{cases} \frac{d}{dt} \psi(t, x) = f'(u(t, \psi(t, x))), & \psi(0, x) = x \\ u(t, \psi(t, x)) = u_0(x) \end{cases} \]

\[ \iff \begin{cases} \frac{d}{dt} \psi(t, x) = f'(u_0(x)), \\ u(t, \psi(t, x)) = u_0(x) \end{cases} \]

\[ \iff \begin{cases} \psi(t, x) = x + tf'(u_0(x)), \\ u(t, \psi(t, x)) = u_0(x) \end{cases} \]

\[ \iff u(t, x + tf'(u_0(x))) = u_0(x). \]
Multi valued solutions

- Let the plane evolve according to \((x, y) \mapsto (x + tf'(y), y)\).
- Look at the evolution of the graph of \(u\).
Alternative: gradient blow up

\[ \partial_t (u_x) + f'(u) \partial_x (u_x) = -(u_x)^2. \]
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Riemann initial data

- Simplest discontinuity and invariance by $x \mapsto x + \eta$:

$$u_0(x) := \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

- Invariance by $(t, x) \mapsto (\lambda t, \lambda x) \Rightarrow$

$$u(t, x) = v \left( \frac{x}{t} \right)$$

- Simplest case:

$$u(t, x) = \begin{cases} u_l & \text{if } x < pt \\ u_r & \text{if } x > pt \end{cases}$$

- What is $p$?
Rankine Hugoniot condition for weak solution

- Integral formulation:

\[
\frac{d}{dt} \int_{a}^{b} u(t, x)dx = f(u(a)) - f(u(b)).
\]

- Integrating on the box between the points \((0, 0)\) and \((T, pT)\) \(\Rightarrow\)

\[
pTu_l - pTu_r = Tf(u_l) - Tf(u_r),
\]

- Rankine-Hugoniot condition

\[
p = \frac{f(u_r) - f(u_l)}{u_r - u_l}
\]

- Can be localized with \(u_r\) and \(u_l\) smooth solutions and a moving discontinuity.
Characteristics: shock wave

\[ u_0(x) = \begin{cases} 
1.0 & \text{if } x < 0 \\
0.0 & \text{if } x > 0.
\end{cases} \quad \Rightarrow \quad u(t, x) = \begin{cases} 
1.0 & \text{if } x < \frac{t}{2} \\
0 & \text{if } x > \frac{t}{2}
\end{cases} \]
Characteristics: rarefaction wave

\[ u_0(x) = \begin{cases} 
0.0 & \text{if } x < 0 \\
1.0 & \text{if } x > 0.
\end{cases} \quad \Rightarrow \quad u(t, x) = \begin{cases} 
0 & \text{if } x < \frac{t}{2} \\
1.0 & \text{if } x > \frac{t}{2}
\end{cases} \]
Entropy condition

- For $\epsilon > 0$ consider

$$u_0^\epsilon(x) = \begin{cases} 
    u_l & \text{if } x < -\epsilon \\
    u_m & \text{if } -\epsilon < x < \epsilon \\
    u_r & \text{if } \epsilon < x
\end{cases}$$

- Let $\epsilon \to 0$, admissibility condition on discontinuities:

$$\forall u_m, \quad \frac{f(u_l) - f(u_m)}{u_l - u_m} \geq \frac{f(u_l) - f(u_r)}{u_l - u_r} \geq \frac{f(u_m) - f(u_r)}{u_m - u_r}$$
Characteristics: rarefaction wave the return

\[ u_0 = \begin{cases} 
0.0 & \text{if } x < 0 \\
1.0 & \text{if } x > 0. 
\end{cases} \quad \Rightarrow \quad u(t, x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{t} & \text{if } 0 < x < t \\
1.0 & \text{if } x > t 
\end{cases} \]
Simulations
Entropy solutions crash course
- Generalities
- Blow up
- Riemann Problem

Initial data identification: statement of the problem

Necessity

Sufficiency
For $T > 0$, $w \in L^\infty(\mathbb{R})$ define:

$$I_T(w) := \{ u_0 \in L^\infty(\mathbb{R}) : S_T^{CL} u_0 = w \}.$$ 

$S_t^{CL}$ : entropy solutions semigroup.

- $I_T(w) \neq \emptyset$ iff $w$?
- $u_0 \in I_T(w)$ iff $u_0$?
- Geometric description of $I_T(w)$?
Motivation

1. Sonic boom plane design (see Gosse-Zuazua 2017)
2. Traffic flow.
A simple result

To $T > 0$, $w : \mathbb{R} \mapsto \mathbb{R}$, associate

$$p : x \mapsto x - Tf'(w(x)).$$

Theorem (Colombo, P.)

Let $f$ be a $C^2$, uniformly convex flux and $T > 0$. $I_T(w)$ is non empty iff $p$ is nondecreasing.

Remark

- "There exists a representative such that..."
- Actually just Oleinik’s inequality.
- $\Rightarrow w \in BV(\mathbb{R})$. 


Geometric Properties

Theorem (Colombo, P.)

Given $T > 0$ and $f \in C^2$ uniformly convex. If $w \in SBV(\mathbb{R})$ and $I_T(w) \neq \emptyset$ then.

- $I_T(w)$ is a singleton iff $w \in C^0(\mathbb{R})$.
- Otherwise $I_T(w)$ is a convex cone with the isentropic solution for only vertex.
- $I_T(w)$ does not have finite dimensional facets.

In all cases $I_T(w)$ is always closed for the $L^1_{loc}$ topology.
Necessary and sufficient condition

Theorem (Colombo, P.)

Given $T > 0$ and $f \in C^2$ uniformly convex. Let $w$ be in $SBV_{loc}(\mathbb{R})$ such that $p$ is nondecreasing. Then $U \in \text{Lip}(\mathbb{R})$ satisfy $\partial_x U \in l_T(w)$ iff

- for all point $x$ such that $p$ is differentiable in $x$ and $p'(x) \neq 0$

$$\lim_{y \to x} \frac{U(p(y)) - U(p(x))}{p(y) - p(x)} = w(x).$$

- for all point $x$ such that $w(x^-) \neq w(x^+)$ then $\forall y \in ]p(x^-), p(x^+)[$

$$\begin{cases} 
    \frac{U(p(x^+)) - U(y)}{T} \leq f^* \left( \frac{x-y}{T} \right) - f^* \left( \frac{x-p(x^+)}{T} \right) \\
    \frac{U(p(x^-)) - U(y)}{T} \leq f^* \left( \frac{x-y}{T} \right) - f^* \left( \frac{x-p(x^-)}{T} \right)
\end{cases}$$
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Tool: generalized characteristics

- Extension of the method of characteristics to low regularity ($L^\infty$).
- Differential inclusion in the sense of Filippov (1960) (Existence but no uniqueness)
- Developed by Dafermos (1977)
- Lots of "true" characteristics $\Rightarrow$ good estimates on $u$.

**Definition**

$$t \mapsto \gamma(t) \text{ AC generalized characteristics when}$$

$$\dot{\gamma}(t) \in [f'(u(t, \gamma(t)^+)), f'(u(t, \gamma(t)^-))] \ dt \ p.p.$$
A first simple case

- Target state:
  \[ \forall x \in \mathbb{R}, \quad w(x) = \begin{cases} 
  1 & \text{if } x < 0 \\
  -1 & \text{if } x > 0 
\end{cases} \]

- First obvious remark
  \[ \forall T > 0, \quad w \in I_T(w) \Rightarrow I_T(w) \neq \emptyset. \]
Backward propagation and key zone

Figure:
Backward shock
Compressive profile
"Combo"
"Double Rarefaction"
SCAM
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Localization

Theorem (Colombo, P.)

If \( w \in \text{SBV}(\mathbb{R}) \) is such that \( p \) is nondecreasing, we define

\[
X_i := p \left( \{ x \in \mathbb{R} : p \text{ is differentiable at } x \text{ and } p'(x) \neq 0 \} \right) \\
X_{ii} := \bigcup_{x \in \mathbb{R}} p(x^-), p(x^+),
\]

we have \( \mathbb{R} \setminus (X_i \cup X_{ii}) \) is negligible.
Lax-Hopf formula

\( u \) is an entropy solution of

\[
\begin{aligned}
\partial_t u + \partial_x f(u) &= 0, \\
u(0) &= u_0.
\end{aligned}
\]

\( \Updownarrow \)

\[
\begin{aligned}
u(t, x) &= g \left( \frac{x - y(t, x)}{t} \right), \\
y(t, x) &= \arg\min \left( tf^* \left( \frac{x - y}{t} \right) + \int_0^y u_0(z) dz \right)
\end{aligned}
\]

- \( g = f'^{-1} \) and \( f^* \) Legendre transform of \( f \).
- \( f \) must be convex!
HJB Connection

- $U_0 \in \text{Lip}(\mathbb{R})$,
- $U$ viscosity solution of
  \[
  \begin{cases}
  \partial_t U + f(\partial_x U) = 0 \\
  U(0) = U_0
  \end{cases}
  \]  \hspace{2cm} \text{(HJB)}
- $u$ entropy solution of
  \[
  \begin{cases}
  \partial_t u + \partial_x f(u) = 0 \\
  u(0) = \partial_x U_0
  \end{cases}
  \]  \hspace{2cm} \text{(CL)}
- conclusion:
  \[\forall t > 0, \quad \partial_x U(t) = u(t).\]
Perspectives

1. Numerical schemes.
2. Apriori constraints for traffic flow.
3. Space dependency and source term.
5. Multi dimensional case.
THANK YOU FOR YOUR ATTENTION