Controllability of multi-d scalar conservation laws in the entropy framework and with a simple geometrical condition.

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Outline of the talk

1. The simplest toy example
2. Controllability to trajectories
The simplest toy example

Controllability to trajectories
The simplest toy example

The case of the transport equation

- **Evolution:**
  \[
  \begin{align*}
  \partial_t y + c\partial_x y &= 0, & (t, x) \in (0, T) \times (0, L) \\
  y(t, 0) &= 0, & t \in (0, T).
  \end{align*}
  \]

- **Method of characteristics ⇒**
  \[
  y(t, x) = \begin{cases} 
  y_0(x - ct) & \text{if } x > ct, \\
  0 & \text{otherwise}.
  \end{cases}
  \]

- **Conclusion:**
  \[t > \frac{L}{c} \Rightarrow y(t, \cdot) = 0.\]
A Family of Lyapunov Functionals

- Definition (for $\nu > 0$):

$$J_\nu(t) := \int_0^L y^2(t, x) e^{-\nu x} \, dx.$$ 

- Formally (at least):

$$\dot{J}_\nu(t) = \int_0^L 2y_t(t, x)y(t, x)e^{-\nu x} \, dx$$

$$= \int_0^L -2cy_x(t, x)y(t, x)e^{-\nu x} \, dx$$

$$= [-cy^2(t, x)e^{-\nu x}]_0^L - c\nu J_\nu(t)$$

$$\leq -c\nu J_\nu(t).$$

- Gronwall $\Rightarrow$

$$J_\nu(t) \leq e^{-c\nu t} J_\nu(0).$$
Return to the $L^2$ norm

- Norm equivalence
  \[ \forall t \geq 0, \quad e^{-\nu L} \|y(t,.)\|_{L^2(0,L)}^2 \leq J_\nu(t) \leq \|y(t,.)\|_{L^2(0,L)}^2. \]

- Inequality on $L^2$
  \[ \|y(t,.)\|_{L^2(0,L)}^2 \leq e^{-\nu c (t - \frac{L}{c})} \|y_0\|_{L^2(0,L)}^2, \]

- Conclusion:
  \[ t > \frac{L}{c}, \quad \nu \to +\infty \quad \Rightarrow \quad y(t,.) = 0 \]
Remarks

- Can be adapted to general "transport" type equations.
- Good for robustness estimate and perturbation:
  \[ y_t + cy_x = \epsilon g(y), \]
  \[ y_t + cy_x = \epsilon y_{xx}. \]

(Systems and source term: Gugat, P., Rosier 2018)

- In certain cases, useful for exact controllability to trajectory.
The simplest toy example

Controllability to trajectories
Short tour on controllability and entropy solutions

Why: Robustness, interesting trajectories, sampled controls, global results...


Techniques:
- No linearization possible!
- Boundary conditions are tricky!
- Lax-Hopf formula.
- Generalized characteristics.
- Wave front tracking.
- Vanishing viscosity.

Restrictions: 1d and convex flux (or genuinely nonlinear families for systems)
Kruzkov definition

Consider $\Omega \in \mathbb{R}^d$ a domain, and $f : \mathbb{R} \to \mathbb{R}^d$ a $C^1$ function. A function $u \in L^{\infty}((0, +\infty) \times \Omega)$ is an entropy solution of

$$
\begin{aligned}
\begin{cases}
\partial_t u + \text{div}(f(u)) = 0, & x \in \Omega \\
u(0, x) = u_0(x), & x \in \Omega
\end{cases}
\end{aligned}
$$

when for any $k \in \mathbb{R}$ and any positive function $\phi \in C^1_c(\mathbb{R} \times \Omega)$ we have

$$
\begin{align*}
\int_{\mathbb{R}^+ \times \Omega} |u(t, x) - k| \partial_t \phi(t, x) + \text{sign}(u(t, x) - k)\langle f(u(t, x)) - f(k)| \nabla \phi(t, x) \rangle dx dt \\
+ \int_{\Omega} |u_0(x) - k| \phi(0, x) dx \geq 0
\end{align*}
$$
A "simple" geometric condition

**Definition**

Let $I$ be a segment of $\mathbb{R}$. We say that it satisfy the replacement condition in time $T > 0$ if there exists a vector $w \in \mathbb{R}^d$ and a positive number $c$ such that

$$L := \sup_{x \in \Omega} \langle w | x \rangle - \inf_{x \in \Omega} \langle w | x \rangle < +\infty.$$ 

$$\forall u \in I, \quad \langle f'(u) | w \rangle \geq c,$$

and $T = \frac{L}{c}$. 

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Theorem (Donadello, P.)

- Let \( v \in L^\infty((0, +\infty) \times \Omega) \) be an entropy solution to
  \[
  \partial_t u + \text{div}(f(u)) = 0
  \]

  and \( u_0 \) be a function in \( L^\infty(\Omega) \).

- Suppose that both \( u_0 \) and \( v \) take value in a segment \( I \) such that \( \Omega, I \) and \( f \)
  satisfy the replacement condition in time \( T \).

- Then for any times \( T_1 \) and \( T_2 \) greater than \( T \) we have an entropy solution \( u \)
  of the previous equation satisfying

  \[
  u(0, x) = u_0(x), \quad u(T_1, x) = v(T_2, x) \quad \text{for almost all } x \in \Omega.
  \]
Idea of the proof

- Kruzkov formulation with boundary condition (Amar, Carillo, Wittbold 2006).
- Trace result of Vasseur for the boundary of the reference trajectory (Vasseur 2001).
- Doubling variable trick of Kruzkov with test function (Kruzkov 1970).
- Conclusion with Lyapunov functions.
Remarks

- No technical restriction on the flux (compared with the Cauchy-Problem).
- 1-d or n-d not different.
- Many reusable steps for the proof.
- Geometric condition clearly too restrictive, see controllability of Euler for further ideas?
THANK YOU FOR YOUR ATTENTION