First order mean-field games

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VIII PARTIAL DIFFERENTIAL EQUATIONS, OPTIMAL DESIGN AND NUMERICS

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Organizers: G. Buttazzo, O. Glass. G. Leugering, and E. Zuazua
Outline

1. Introduction to Mean Field Games

2. Mean Field Games with state constraints
   - The Lagrangian approach
   - Existence and uniqueness of relaxed equilibria
   - Regularity of relaxed solutions to constrained MFG
   - Point-wise properties of relaxed solutions

3. Concluding remarks
   - Asymptotic behaviour
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Mean field game theory is the study of strategic decision making in very large populations of small interacting agents. This class of problems was considered in the economics literature by B Jovanovic and RW Rosenthal, in the engineering literature by PE Caines and his co-workers, and independently and around the same time by mathematicians J-M Lasry and P-L Lions.

Figure: Jovanovic, Rosenthal, Caines, Lasry, and Lions
Motivations for studying MFG

Goal
To describe Nash equilibria in the collective behaviour of a large population of “small” rational agents

- large population $\rightsquigarrow$ infinite number (a continuum) of players
- rational agents $\rightsquigarrow$ each agent is controlling his/her dynamical own state

Figure: MFG impact: finance, crowd dynamics, smart grids
Lasry-Lions 1: the Hamilton-Jacobi equation

Main idea: to export the principle of statistical mechanics to interactions within rational particles by introducing a macroscopic description through a mean field model

- agents are identified with points \( x \in \bar{\Omega} \subset \mathbb{R}^n \)
- \( m(t, dx) \) is the distribution of agents at time \( t \)

Agent located in \( x \in \bar{\Omega} \) at time \( t \in [0, T] \) chooses a path \( \gamma_{t,x}(s), s \in [t, T] \), such that

\[
\begin{align*}
  u(t, x) &:= \min_{\gamma(t)=x} \left\{ \int_t^T \left[ L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] ds + G(\gamma(T), m(T)) \right\} \\
\end{align*}
\]

The value function \( u(t, x) \) satisfies the associated Hamilton-Jacobi equation

\[
\begin{cases}
  -\partial_t u + H(x, \nabla u) = F(x, m) & [0, T] \times \Omega \\
  u(T, x) = G(x, m(T))
\end{cases}
\]

where

\[
H(x, p) := \sup_{v \in \mathbb{R}^n} \left\{ -\langle p, v \rangle - L(x, v) \right\}
\]
Lasry-Lions 2: the continuity equation

The space gradient $\nabla u(t, x)$ of the solution to the Hamilton-Jacobi equation gives the optimal feedback $\gamma_{t,x}$ via the system

$$\gamma'(s) = -\partial_p H(\gamma(s), \nabla u(s, \gamma(s))) \quad (s \in [t, T])$$

for the minimization problem

$$\min_{\gamma(t) = x} \left\{ \int_t^T \left[ L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] ds + G(\gamma(T), m(T)) \right\}$$

Since $m(\cdot, dx)$ is just $m_0(dx)$ transported by such a flow, the continuity equation

$$\begin{cases}
\partial_t m - \text{div}(m \partial_p H(x, \nabla u)) = 0 & [0, T] \times \Omega \\
m(0, dx) = m_0(dx)
\end{cases}$$

must be satisfied.
Lasry-Lions 3: the MFG system

By coupling the Hamilton-Jacobi equation with the continuity equation above, one obtains the PDE system of Mean Filed Games

\[
\begin{align*}
-\partial_t u + H(x, \nabla u) - F(x, m) &= 0 \\
\partial_t m - \text{div}(m \partial_p H(x, \nabla u)) &= 0
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\]

In the last decade, system \((MFG)\) has been widely investigated for two main kinds of space domains

\[\Omega = \mathbb{T}^n, \mathbb{R}^n\]

main contributions by: Achdou, Bardi, Bensoussan, Camilli, Capuzzo Docetta, Cardaliaguet, Carmona, Delarue, Gomes, Guéant, Lachapelle, Porretta, . . .
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[0, T] \times \Omega \quad &\begin{cases}
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Mean Field Games

Solution of the MFG system

Reference: notes on Mean Field Games by P. Cardaliaguet, 2013 and 2015

- by vanishing viscosity

\[
\begin{aligned}
- \partial_t u - \epsilon \Delta u + H(x, \nabla u) &= F(x, m) \\
\partial_t m - \epsilon \Delta m - \text{div}(m \partial_p H(x, \nabla u_\mu)) &= 0
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\]

- by a fixed point argument

\[
\begin{aligned}
\mu &\rightarrow u_\mu \\
- \partial_t u + H(x, \nabla u) &= F(x, \mu) \\
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\end{aligned} \quad \quad \rightarrow \quad \quad m_\mu \\
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Very important facts:

- although nonsmooth, \( u \) is linearly semiconcave, which ensures a nice behaviour along minimizers

- if \( m_0 \) is absolutely continuous with respect to the Lebesgue measure, then \( m(t, \cdot) \) stays absolutely continuous,
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- although nonsmooth, \( u \) is linearly semiconcave, which ensures a nice behaviour along minimizers
- if \( m_0 \) is absolutely continuous with respect to the Lebesgue measure, then \( m(t, \cdot) \) stays absolutely continuous
Does all this break down under state constraints?

Our goal: To study MFG problems with state constraints: \( x \in \overline{\Omega} \)

Difficulty
Agent distribution may concentrate on small sets

Then the above methods break down
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A change of paradigm

- $\Omega \subset \mathbb{R}^n$ bounded domain with boundary of class $C^2$
- $\mathcal{P}(\overline{\Omega})$ Borel probability measures on $\overline{\Omega}$ with Katorovich-Rubinstein distance

\[
d_1(m_1, m_2) = \sup \left\{ \int_{\overline{\Omega}} f \, dm_1 - \int_{\overline{\Omega}} f \, dm_2 : |f(x) - f(y)| \leq |x - y| \right\}
\]

Recall that, given $m \in C([0, T]; \mathcal{P}(\overline{\Omega}))$, agents aim to attain

\[
\min_{\gamma(0) = x, \gamma(t) \in \overline{\Omega}} \int_0^T \left[ L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m(t)) \right] dt + G(\gamma(T), m(T)) \]

but $m$ cannot be fixed a priori as it evolves along optimal feedback.

Main idea to overcome such a difficulty:

to replace $m \in C([0, T]; \mathcal{P}(\overline{\Omega}))$

by a probability measure on the metric space $C([0, T]; \overline{\Omega})$

that is $C([0, T]; \mathcal{P}(\overline{\Omega})) \leftrightarrow \mathcal{P}(C([0, T]; \overline{\Omega}))$
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$$\mathcal{C}([0, T]; \mathcal{P}(\Omega)) \leftrightarrow \mathcal{P}(\mathcal{C}([0, T]; \Omega))$$
Lagrangian approach

References

C – Capuani (2018)
C – Capuani – Cardaliaguet (2018), C – Capuani – Cardaliaguet (2019)

Notation

- constrained arcs

\[ \Gamma = \left\{ \gamma \in AC([0, T]; \mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \forall t \in [0, T] \right\} \quad \text{with} \quad \| \cdot \|_\infty \]

\[ \Gamma[x] = \left\{ \gamma \in \Gamma : \gamma(0) = x \right\} \quad (x \in \overline{\Omega}) \]

- \( \mathcal{P}(\Gamma) \) Borel probability measures on \( \Gamma \): metric space with \( d_1 \) metric

\[ d_1(\mu_1, \mu_2) = \sup \left\{ \int_{\Gamma} f \, d\mu_1 - \int_{\Gamma} f \, d\mu_2 : |f(\gamma) - f(\xi)| \leq \|\gamma - \xi\|_\infty \right\} \]
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Relaxed MFG functional

The evaluation map $e_t : \Gamma \rightarrow \overline{\Omega}$ ($t \in [0, T]$) is defined by $e_t(\gamma) = \gamma(t)$

**Push-forward**

With any $\mu \in \mathcal{P}(\Gamma)$ and $t \in [0, T]$ one can associate the probability measure $e_t\#\mu$ on $\overline{\Omega}$ given by

$$\int_{\overline{\Omega}} f(x) \, e_t\#\mu(dx) = \int_{\Gamma} f(\gamma(t)) \, \mu(d\gamma) \quad \forall f \in C(\overline{\Omega})$$

$e_t\#\mu$ is the push-forward of $\mu$ by $e_t$

For any $\mu \in \mathcal{P}(\Gamma)$ we define

- the associated payoff functional

$$J_\mu[\gamma] = \int_0^T [L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t\#\mu)] \, dt + G(\gamma(T), e_T\#\mu) \quad \forall \gamma \in \Gamma$$

- the family of minimizing arcs for $J_\mu$ at $x \in \overline{\Omega}$

$$\Gamma^\mu[x] = \{ \gamma \in \Gamma[x] : J_\mu[\gamma] = \min_{\Gamma[x]} J_\mu \}$$
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  \[
  J_\mu[\gamma] = \int_0^T \left[ L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t\#\mu) \right] dt + G(\gamma(T), e_T\#\mu) \quad \forall \gamma \in \Gamma
  \]

- the family of **minimizing arcs** for \( J_\mu \) at \( x \in \overline{\Omega} \)

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\Gamma^\mu[x] = \{ \gamma \in \Gamma[x] : J_\mu[\gamma] = \min_{\Gamma[x]} J_\mu \}
\]
Relaxed equilibria

A Borel probability measure $\mu \in \mathcal{P}(\Gamma)$ is compatible with $m_0 \in \mathcal{P}(\overline{\Omega})$ if

$$e_0 \# \eta = m_0$$

Denote by $\mathcal{P}_{m_0}(\Gamma)$ the subspace consisting of all such measures.

**Definition**

$\mu \in \mathcal{P}_{m_0}(\Gamma)$ is called a relaxed (CMFG) equilibrium for $m_0$ if

$$\text{spt}(\mu) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^\mu [x]$$

Equivalently,

$$J_\mu [\overline{\gamma}] = \min_{\gamma \in \Gamma[\overline{\gamma}(0)]} J_\mu [\gamma] \quad \text{for } \mu \text{-a.e. } \overline{\gamma} \in \Gamma$$

where

$$J_\mu [\gamma] = \int_0^T \left[ L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t \# \mu) \right] dt + G(\gamma(T), e_T \# \mu)$$
Relaxed solutions

Let $m_0 \in \mathcal{P}(\Omega)$

Definition

$(u, m) \in C([0, T] \times \Omega) \times C([0, T]; \mathcal{P}(\Omega))$ is a relaxed solution to the CMFG problem if

$$m(t) = e^{t\#}\mu \quad \forall t \in [0, T]$$

for some relaxed equilibrium $\mu \in \mathcal{P}_{m_0}(\Gamma)$ and

$$u(t, x) = \min_{\gamma \in \Gamma, \gamma(t) = x} \left\{ \int_t^T [L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s))] \, dt + G(\gamma(T), m(T)) \right\}$$
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Existence of relaxed equilibria and solutions

**Theorem**

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed equilibrium.

**Corollary**

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed solution $(u, m)$ to the CMFG problem.
Existence of relaxed equilibria and solutions

**Theorem**

*For any* $m_0 \in \mathcal{P}(\overline{\Omega})$ *there is at least one relaxed equilibrium*

**Corollary**

*For any* $m_0 \in \mathcal{P}(\overline{\Omega})$ *there is at least one relaxed solution* $(u, m)$ *to the CMFG problem*
Proof

Kakutani’s fixed-point theorem

- $S \neq \emptyset$ compact convex subset of a locally convex Hausdorff space
- $\phi: S \Rightarrow S$ nonempty convex-valued with closed graph

$\implies \phi$ has a fixed point.

Proof of theorem: construction of a fixed point of $E: \mathcal{P}_{m_0}(\Gamma) \Rightarrow \mathcal{P}_{m_0}(\Gamma)$

$$E(\eta) = \left\{ \mu \in \mathcal{P}_{m_0}(\Gamma) \mid \text{spt}(\mu_x) \subseteq \Gamma^\eta[x] \text{ for } m_0 - \text{a.e. } x \in \overline{\Omega} \right\} \quad (\eta \in \mathcal{P}_{m_0}(\Gamma))$$

where $\{\mu_x\}_{x \in \overline{\Omega}} \subset \mathcal{P}(\Gamma)$ is the family of probability measures which disintegrates $\mu$

$$\mu = \int_{\overline{\Omega}} \mu_x \, dm_0(x) \quad \text{and} \quad \text{spt}(\mu_x) \subseteq \Gamma[x] \quad m_0 - \text{a.e. } x \in \overline{\Omega}$$

Indeed

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ relaxed equilibrium} \iff \eta \in E(\eta)$$

The existence of a fixed point of $E$ follows from Kakutani’s Theorem
Proof

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$$\mu = \int_{\Omega} \mu_x \, dm_0(x) \quad \text{and} \quad \text{spt}(\mu_x) \subseteq \Gamma[x] \quad m_0 - \text{a.e. } x \in \Omega$$

Indeed

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ relaxed equilibrium } \iff \eta \in E(\eta)$$

The existence of a fixed point of $E$ follows from Kakutani’s Theorem
Uniqueness

**Theorem**

Assume *monotonicity conditions*: for any \( m_1, m_2 \in \mathcal{P}(\Omega) \)

\[
\begin{cases}
\int_{\Omega} (G(x, m_1) - G(x, m_2)) \, d(m_1 - m_2)(x) \geq 0 \\
\int_{\Omega} (F(x, m_1) - F(x, m_2)) \, d(m_1 - m_2)(x) > 0 \quad \text{if} \ m_1 \neq m_2
\end{cases}
\]

If \((u_1, m_1)\) and \((u_2, m_2)\) are relaxed solutions to the CMFG problem, then

\[ u_1 \equiv u_2 \quad \text{and} \quad m_1 = m_2 \]

\(F\) satisfies the strict monotonicity condition if \(F: \overline{\Omega} \times \mathcal{P}(\Omega) \to \mathbb{R}\) is of the form

\[ F(x, m) = \int_{\Omega} f(y, (\phi \star m)(y)) \phi(x - y) \, dy \]

where \(\phi: \mathbb{R}^d \to \mathbb{R}\) is a smooth even kernel with compact support and

\(f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}\) is smooth and \(f(x, \cdot)\) is strictly increasing.
Uniqueness

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Assume monotonicity conditions: for any \( m_1, m_2 \in \mathcal{P}(\Omega) \)

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\int_{\Omega} (G(x, m_1) - G(x, m_2))d(m_1 - m_2)(x) \geq 0 \\
\int_{\Omega} (F(x, m_1) - F(x, m_2))d(m_1 - m_2)(x) > 0 \quad \text{if } m_1 \neq m_2
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Outline

1. Introduction to Mean Field Games

2. Mean Field Games with state constraints
   - The Lagrangian approach
   - Existence and uniqueness of relaxed equilibria
   - Regularity of relaxed solutions to constrained MFG
   - Point-wise properties of relaxed solutions

3. Concluding remarks
   - Asymptotic behaviour
More notation and assumptions

Recall $\Omega \subset \mathbb{R}^n$ is bounded with $\partial \Omega \in C^2$. Consequently

- **distance** $d_\Omega(x) = \min_{y \in \overline{\Omega}} |x - y|$ of class $C^2(\Omega_\delta^+)$ for some $\delta > 0$ with $\Omega_\delta^+ = \{ x \in \mathbb{R}^n \setminus \Omega : d_\Omega(x) < \delta \}$

- **oriented boundary distance** $b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^n \setminus \Omega}(x)$ of class $C^2(\Omega_\delta)$ on $\Omega_\delta = \{ x \in \mathbb{R}^n : |b_\Omega(x)| < \delta \}$
References

- Dubovitskii – Milyutin (1964)
- Malanowski (1978)
- Hager (1979)
- Vinter (2000)
- Frankowska (2006, 2009)
- Bettiol – Khalil – Vinter (2016)
Necessary conditions for smooth state constraints

**Theorem**

*Given* $x \in \overline{\Omega}$ *let* $\gamma^*$ *minimize over* $\Gamma[x]$ *the functional*

$$
\gamma \mapsto \int_0^T \left[ L(\gamma(s), \dot{\gamma}(s)) + f(s, \gamma(s)) \right] dt + g(\gamma(T))
$$

*where* $g \in C^1(\overline{\Omega})$ *and* $f : [0, T] \times \overline{\Omega} \to \mathbb{R}$ *satisfies* $|f_t| + |\nabla f| \leq C$

*Then there exist*

- $p^* : [0, T] \to \mathbb{R}^n$ *Lipschitz*
- $\nu \in \mathbb{R}$ *and* $\Lambda \in C_b([0, T] \times \Omega_\delta \times \mathbb{R}^n)$ *independent of* $\gamma^*, p^*$

*such that* $(\mathbb{I}_\partial \Omega = \text{characteristic function of} \, \partial \Omega)$

$$
\begin{align*}
\dot{\gamma}^* &= -\partial_p H(\gamma^*, p^*) \\
\dot{p}^* &= \nabla H(\gamma^*, p^*) - \nabla f(t, \gamma^*) - \Lambda(t, \gamma^*, p^*) \mathbb{I}_{\partial \Omega}(\gamma^*) \nabla b_\Omega(\gamma^*) \quad \forall t \in [0, T] \\
p^*(T) &= \nabla g(\gamma^*(T)) + \nu \mathbb{I}_{\partial \Omega}(\gamma^*(T)) \nabla b_\Omega(\gamma^*(T))
\end{align*}
$$

*Consequently,* $\gamma^* \in C^1_{\text{Lip}}([0, T]; \mathbb{R}^n)$ *and* $\|\dot{\gamma}^*\|_{\text{Lip}} \leq C(\Omega, H, f, g)$
Necessary conditions for smooth state constraints

**Theorem**

Given \( x \in \overline{\Omega} \) let \( \gamma^* \) minimize over \( \Gamma[x] \) the functional

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\gamma \mapsto \int_0^T \left[ L(\gamma(s), \dot{\gamma}(s)) + f(s, \gamma(s)) \right] dt + g(\gamma(T))
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where \( g \in C^1(\overline{\Omega}) \) and \( f : [0, T] \times \overline{\Omega} \to \mathbb{R} \) satisfies \( |f_t| + |\nabla f| \leq C \)

Then there exist

- \( p^* : [0, T] \to \mathbb{R}^n \) Lipschitz
- \( \nu \in \mathbb{R} \) and \( \Lambda \in C_b([0, T] \times \Omega_\delta \times \mathbb{R}^n) \) (independent of \( \gamma^*, p^* \))

such that \( (\mathbb{I}_{\partial \Omega} = \text{characteristic function of } \partial \Omega) \)

\[
\begin{aligned}
\dot{\gamma}^* &= -\partial_p H(\gamma^*, p^*) \\
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Consequently, \( \gamma^* \in C^1_{\text{Lip}}([0, T]; \mathbb{R}^n) \) and \( \|\dot{\gamma}^*\|_{\text{Lip}} \leq C(\Omega, H, f, g) \)
Necessary conditions for smooth state constraints

**Theorem**

Given $x \in \overline{\Omega}$ let $\gamma^*$ minimize over $\Gamma[x]$ the functional

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where $g \in C^1(\overline{\Omega})$ and $f : [0, T] \times \overline{\Omega} \to \mathbb{R}$ satisfies $|f_t| + |\nabla f| \leq C$

Then there exist

- $p^* : [0, T] \to \mathbb{R}^n$ Lipschitz
- $\nu \in \mathbb{R}$ and $\Lambda \in C^b_b([0, T] \times \Omega_\delta \times \mathbb{R}^n)$ (independent of $\gamma^*, p^*$)

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\dot{p}^* = \nabla H(\gamma^*, p^*) - \nabla f(t, \gamma^*) - \Lambda(t, \gamma^*, p^*) \mathbb{1}_{\partial \Omega}(\gamma^*) \nabla b_\Omega(\gamma^*) \quad \forall t \in [0, T] \\
p^*(T) = \nabla g(\gamma^*(T)) + \nu \mathbb{1}_{\partial \Omega}(\gamma^*(T)) \nabla b_\Omega(\gamma^*(T))
\end{cases}
$$

Consequently, $\gamma^* \in C^1_{Lip}([0, T]; \mathbb{R}^n)$ and $\|\dot{\gamma}^*\|_{Lip} \leq C(\Omega, H, f, g)$
Existence of Lipschitz solutions

Theorem

Let \( m_0 \in \mathcal{P}(\Omega) \) and suppose

\[
|F(x_1, m_1) - F(x_2, m_2)| + |G(x_1, m_1) - G(x_2, m_2)| \leq C \left( |x_1 - x_2| + d_1(m_1, m_2) \right)
\]

Then there exists at least one relaxed solution of CMFG problem \((u, m)\) such that

\[
 u \in Lip([0, T] \times \Omega) \quad \text{and} \quad m \in Lip([0, T]; \mathcal{P}(\Omega))
\]

Such a solution will be called a \textit{Lipschitz relaxed solution} of the CMFG problem.

The proof applies necessary conditions to construct a relaxed CMFG equilibrium

\[
\eta \in \mathcal{P}_{m_0}(\Gamma) \quad \text{such that} \quad m(t) := e_t \# \eta \quad \text{belongs to} \quad Lip([0, T]; \mathcal{P}(\Omega))
\]

and uses the Lipschitz continuity of \( m \) to deduce that \( u \in Lip([0, T] \times \Omega) \)
Existence of Lipschitz solutions

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Then there exists at least one relaxed solution of CMFG problem \((u, m)\) such that

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u \in \text{Lip}([0, T] \times \Omega) \quad \text{and} \quad m \in \text{Lip}([0, T]; \mathcal{P}(\Omega))\]

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and uses the Lipschitz continuity of \( m \) to deduce that \( u \in \text{Lip}([0, T] \times \Omega) \)
A quick look at semiconcave functions

\( \Omega \subseteq \mathbb{R}^n \) open
\( v : \Omega \to \mathbb{R} \) semiconcave with modulus \( \omega : [0, \infty[ \to [0, \infty[ \) if

\[
\lambda v(x) + (1 - \lambda) v(y) - v(\lambda x + (1 - \lambda)y) \leq \lambda(1 - \lambda)|x - y|\omega(|x - y|)
\]

for all \( x, y \) such that \([x, y] \subset \Omega\) and \( \lambda \in [0, 1] \)

Special cases:

- \( \omega(s) \equiv 0 \quad \rightarrow \quad \text{concave} \)

- \( \omega(s) = Cs \ (C > 0) \quad \rightarrow \quad \text{linearly semiconcave} \)
  
  In this case

\[
x \mapsto v(x) - \frac{C}{2} |x|^2
\]

(is concave on all convex subsets of \( \Omega \))

- \( \omega(s) = Cs^\alpha \ (C > 0, 0 < \alpha < 1) \quad \rightarrow \quad \text{fractionally semiconcave} \)
  
  In this case, (⋆) is no longer valid
Some references on semiconcave functions

- **control theory and sensitivity analysis**
  - Fleming – McEneaney 2000
  - Rifford 2000, 2002

- **nonsmooth and variational analysis**
  - Rockafellar 1982
  - Colombo – Marigonda 2006, Colombo – Nguyen 2010

- **differential geometry**
  - Perelman 1995, Petrunin 2007

- **monographs**
  - C – Sinestrari (Birkhäuser 2004)
  - Villani (Springer 2009)
Semiconcavity & nonsmooth analysis

For any semiconcave $v : \Omega \to \mathbb{R}$

- the superdifferential at $x \in \Omega$ coincides with Clarke’s gradient
  
  $$D^+ v(x) = \text{co } D^* v(x) = \partial v(x)$$

  where $D^* v(x) = \{ \lim_{i \to \infty} Dv(x_i) \mid x_i \to x \}$ reachable gradients

- $D^+ v(x) = \{ p \} \iff v$ differentiable
Semiconcavity of relaxed Lipschitz solution

**Theorem**

Any Lipschitz relaxed solution \((u, m)\) of CMFG problem is *locally semiconcave* on \([0, T[\times\bar{\Omega}\) with a *fractional modulus*:

\[
\forall \rho \in ]0, T[\text{ there exists } C_\rho \geq 0 \text{ such that }
\]

\[
u(t + \tau, x + h) + u(t - \tau, x - h) - 2u(t, x) \leq C_\rho (|\tau| + |h|)^{3/2}
\]

for all \(t, t \pm \tau \in [0, T - \rho]\) and \(x, x \pm h \in \bar{\Omega}\)

Several proofs of the above result can be given.
An interesting method of proof uses *sensitivity relations* that we discuss next.
Adjoint state inclusion / sensitivity relations

Given

- a Lipschitz relaxed solution $\mathbf{(u, m)}$ of the CMFG problem
- $(t, x) \in [0, T] \times \bar{\Omega}$ and a solution $\gamma^\ast \in \Gamma$ to

$$\min_{\gamma \in \Gamma, \gamma(t) = x} \left\{ \int_t^T \left[ L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] dt + G(\gamma(T), m(T)) \right\}$$

- the adjoint state $p^\ast : [t, T] \to \mathbb{R}^n$ associated with $\gamma^\ast$

we have that

$$\left( H(\gamma^\ast(s), p^\ast(s)) - F(\gamma^\ast(s), m(s)), p^\ast(s) \right) \in D^+ u(s, \gamma^\ast(s)) \quad \forall s \in [t, T]$$

and $\forall \rho \in ]0, T[$ there exists $C_\rho \geq 0$ such that $\forall t, t + \tau \in [0, T - \rho]$ and all $x + h \in \bar{\Omega}$

$$u(t + \tau, x + h) - u(t, x) - \tau (H(x, p^\ast(t)) - F(x, m(t))) - \langle p^\ast(t), h \rangle \leq C_\rho (|\tau| + |h|)^{3/2}$$
Adjoint state inclusion / sensitivity relations

Given

- a Lipschitz relaxed solution \((u, m)\) of the CMFG problem
- \((t, x) \in [0, T] \times \Omega\) and a solution \(\gamma^* \in \Gamma\) to

\[
\min_{\gamma \in \Gamma, \gamma(t) = x} \left\{ \int_t^T \left[ L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] dt + G(\gamma(T), m(T)) \right\}
\]

- the adjoint state \(p^* : [t, T] \rightarrow \mathbb{R}^n\) associated with \(\gamma^*\)

we have that

\[
\left( H(\gamma^*(s), p^*(s)) - F(\gamma^*(s), m(s)), p^*(s) \right) \in D^+ u(s, \gamma^*(s)) \quad \forall s \in [t, T[\]

and \(\forall \rho \in ]0, T[\) there exists \(C_\rho \geq 0\) such that \(\forall t, t + \tau \in [0, T - \rho]\) and all \(x + h \in \overline{\Omega}\)

\[
u(t + \tau, x + h) - u(t, x) - \tau (H(x, p^*(t)) - F(x, m(t))) - \langle p^*(t), h \rangle \leq C_\rho (|\tau| + |h|)^{3/2}\]

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Proof of sensitivity relation for $\tau = 0$

We want to show that $\forall t \in [0, T - \rho]$ and all $x + h \in \Omega$

$$u(t, x + h) - u(t, x) - \langle p(t), h \rangle \leq C\rho |h|^{3/2}$$

Let $0 < \sigma \leq \rho$ to be fixed later and define for all $s \in [t, T]$

$$\gamma_h(s) = \gamma^*(s) + \left(1 + \frac{t - s}{\sigma}\right)_+ h$$

$$\tilde{\gamma}_h(s) = \gamma_h(s) - d_{\partial\Omega}(\gamma_h(s)) Dd_{\partial\Omega}(\gamma_h(s))$$
Proof of sensitivity relation (continued)

By dynamic programming

\[ u(t, x + h) - u(t, x) - \langle p(t), h \rangle \leq \int_t^{t+\sigma} \left[ L(\hat{\gamma}_h, \dot{\gamma}_h) - L(\gamma^*, \dot{\gamma}^*) \right] ds \]

\[ + \int_t^{t+\sigma} \left[ F(\hat{\gamma}_h, m) - F(\gamma^*, m) \right] ds - \langle p(t), h \rangle \]

We want to express \( \langle p(t), h \rangle \) so we expand

\[ -\langle p(t), h \rangle = -\langle p(t + \sigma), \hat{\gamma}_h(t + \sigma) - \gamma^*(t + \sigma) \rangle + \int_t^{t+\sigma} \frac{d}{ds} \langle p, \hat{\gamma}_h - \gamma^* \rangle ds \]

\[ = \int_t^{t+\sigma} \langle \dot{p}, \hat{\gamma}_h - \gamma^* \rangle ds + \int_t^{t+\sigma} \langle p, \dot{\hat{\gamma}}_h - \dot{\gamma}^* \rangle ds \]

By appealing to PMP to represent \( \langle \dot{p}, \hat{\gamma}_h - \gamma^* \rangle \) and \( \langle p, \dot{\hat{\gamma}}_h - \dot{\gamma}^* \rangle \) we obtain

\[ u(t, x + h) - u(t, x) - \langle p(t), h \rangle \leq \ldots \]

\[ \leq C \int_t^{t+\sigma} |\hat{\gamma}_h - \gamma^*|^2 ds + C \int_t^{t+\sigma} |\dot{\hat{\gamma}}_h - \dot{\gamma}^*|^2 ds + C \int_t^{t+\sigma} |\hat{\gamma}_h - \gamma^*| ds \]
Proof of sensitivity relation (completed)

Recalling

\[
\begin{align*}
\gamma_h(s) &= \gamma^*(s) + \left(1 + \frac{t-s}{\sigma}\right) h \\
\hat{\gamma}_h(s) &= \gamma_h(s) - d_\Omega(\gamma_h(s)) \text{D}d_\partial\Omega(\gamma_h(s))
\end{align*}
\]

we have that

\[|\hat{\gamma}_h(s) - \gamma^*(s)| \leq 2|h| \quad \forall s \in [t, t+\sigma]\]

Using the regularity of the distance functions one can also prove (technical)

\[
\int_t^{t+\sigma} |\dot{\gamma}_h(s) - \dot{\gamma}^*(s)|^2 \, ds \leq C \frac{|h|^2}{\sigma} + C|h|\sigma
\]

Therefore

\[
u(t, x + h) - u(t, x) - \langle p(t), h \rangle \leq C|h| \left( \frac{|h|}{\sigma} + \sigma \right) \leq 2C|h|^{3/2}
\]

by taking \(\sigma = |h|^{1/2}\)
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3. Concluding remarks
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Point-wise solutions of the HJ equation

Given a Lipschitz relaxed solution \((u, m)\) to CMFG problem, we have that

(I) \(u\) is a constrained viscosity solution of
\[
\begin{cases}
  -\partial_t u + H(x, \nabla u) = F(x, m) & \text{in } ]0, T[ \times \Omega \\
  u(T, x) = G(x, m(T)) & \forall x \in \Omega 
\end{cases}
\]

Moreover, defining
\[
Q_m = \left\{ (t, x) \in ]0, T[ \times \Omega : x \in \text{spt}(m(t)) \right\}
\]
\[
\partial Q_m = \left\{ (t, x) \in ]0, T[ \times \partial \Omega : x \in \text{spt}(m(t)) \right\}
\]
the following holds true

(II) \(u\) is differentiable on \(Q_m\) and \(-\partial_t u + H(x, \nabla u) = F(x, m)\) on \(Q_m\)

(III) \(u\) has

time derivative, one-sided normal derivative, and tangential gradient on \(\partial Q_m\)

(IV) the tangential gradient \(\nabla^\tau u\) satisfies
\[
-\partial_t u + H^\tau(x, \nabla^\tau_x u) = F(x, m) \text{ on } \partial Q_m
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where \(H^\tau(x, p) = \sup \left\{ -\langle p, v \rangle - L(x, v) \mid \langle v, \nu(x) \rangle = 0 \right\}\)
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\[-\partial_t u + H^\tau(x, \nabla^\tau_x u) = F(x, m) \quad \text{on } \partial Q_m\]

where \(H^\tau(x, p) = \sup \left\{ -\langle p, v \rangle - L(x, v) : \langle v, \nu(x) \rangle = 0 \right\}\)
Point-wise solutions of the HJ equation

Given a Lipschitz relaxed solution \((u, m)\) to CMFG problem, we have that

(I) \(u\) is a constrained viscosity solution of

\[
\begin{aligned}
-\partial_t u + H(x, \nabla u) &= F(x, m) \quad \text{in } ]0, T[ \times \overline{\Omega} \\
u(T, x) &= G(x, m(T)) \quad \forall x \in \overline{\Omega}
\end{aligned}
\]

Moreover, defining

\[
Q_m = \left\{ (t, x) \in ]0, T[ \times \Omega : x \in spt(m(t)) \right\}
\]
\[
\partial Q_m = \left\{ (t, x) \in ]0, T[ \times \partial \Omega : x \in spt(m(t)) \right\}
\]

the following holds true

(II) \(u\) is differentiable on \(Q_m\) and \(-\partial_t u + H(x, \nabla u) = F(x, m)\) on \(Q_m\)

(III) \(u\) has
time derivative, one-sided normal derivative, and tangential gradient on \(\partial Q_m\)

(IV) the tangential gradient \(\nabla^\tau u\) satisfies

\[
-\partial_t u + H^\tau(x, \nabla_x^\tau u) = F(x, m) \quad \text{on } \partial Q_m
\]

where \(H^\tau(x, p) = \sup \left\{ -\langle p, v \rangle - L(x, v) \mid \langle v, \nu(x) \rangle = 0 \right\}\)
Given a Lipschitz relaxed solution \((u, m)\) to CMFG problem, we have that

(I) there exists a bounded continuous vector field \(V : ]0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^n\) such that \(m\) satisfies the continuity equation

\[
\partial_t m + \text{div}(mV) = 0 \quad \text{in } ]0, T[ \times \overline{\Omega}
\]

in the sense of distributions: \(\forall \phi \in C^1_c(]0, T[ \times \overline{\Omega})\)

\[
\int^T_0 \int_{\overline{\Omega}} (\phi_t + \langle V, \nabla \phi \rangle) \, dm(t, dx) \, dt = 0
\]

(II) \(V\) is given by the optimal feedback on \(Q_m\), that is,

\[
V(t, x) = \begin{cases}
-\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\
-\partial_p H(x, \nabla^\top_x u(t, x) + \partial^+_{\nu_i} u(t, x) \nu_i(x)) & \forall (t, x) \in \partial Q_m
\end{cases}
\]
Analysis of the continuity equation

Given a Lipschitz relaxed solution \((u, m)\) to CMFG problem, we have that

(I) there exists a bounded continuous vector field \(V : ]0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^n\) such that \(m\) satisfies the continuity equation

\[
\partial_t m + \text{div}(mV) = 0 \quad \text{in } ]0, T[ \times \Omega
\]

in the sense of distributions: \(\forall \phi \in C^1_c(]0, T[ \times \overline{\Omega})\)

\[
\int_0^T \int_{\Omega} \left( \phi_t + \langle V, \nabla \phi \rangle \right) dm(t, dx) dt = 0
\]

(II) \(V\) is given by the optimal feedback on \(Q_m\), that is,

\[
V(t, x) = \begin{cases} 
-\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\
-\partial_p H(x, \nabla_x u(t, x) + \partial_{\nu_i}^+ u(t, x) \nu_i(x)) & \forall (t, x) \in \partial Q_m 
\end{cases}
\]
Analysis of the continuity equation

Given a Lipschitz relaxed solution \((u, m)\) to CMFG problem, we have that

(I) there exists a \textbf{bounded continuous} vector field \(V : ]0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^n\) such that \(m\) satisfies the continuity equation

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in the sense of distributions: \(\forall \phi \in C^1_c(]0, T[ \times \overline{\Omega})\)

\[
\int_0^T \int_{\overline{\Omega}} (\phi_t + \langle V, \nabla \phi \rangle) \, dm(t, dx) \, dt = 0
\]

(II) \(V\) is given by the \textbf{optimal feedback} on \(Q_m\), that is,

\[
V(t, x) = \begin{cases} 
-\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\
-\partial_p H(x, \nabla^\tau_x u(t, x) + \partial^+_{\nu_i} u(t, x) \nu_i(x)) & \forall (t, x) \in \partial Q_m
\end{cases}
\]
Proof

Consider the continuous map \( V_m : Q_m \cup \partial Q_m \to \mathbb{R}^n \)

\[
V_m(t, x) = \begin{cases} 
-\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\
-\partial_p H(x, \nabla^\tau u(t, x) + \partial_{\nu_i} u(t, x)\nu_i(x)) & \forall (t, x) \in \partial Q_m 
\end{cases}
\]

and extend it to a continuous vector field \( V : ]0, T[ \times \overline{\Omega} \to \mathbb{R}^n \) by Tietze theorem

Let \( \eta \) be a constrained equilibrium associated with \((u, m)\): then

\[
(t, \gamma(t)) \in Q_m \cup \partial Q_m \quad \text{and} \quad \dot{\gamma}(t) = V(t, \gamma(t)) \quad \forall t \in ]0, T[ 
\]

for \( \eta \)-a.e. \( \gamma \in \Gamma \)

So, \( \forall \phi \in C^1_\text{c}(]0, T[ \times \overline{\Omega}) \) we use the change of variables \( m(t) = e^{t \# \eta} \) to compute

\[
\frac{d}{dt} \int_{\Omega} \phi(t, x)m(t, dx) = \frac{d}{dt} \int_{\Gamma} \phi(t, \gamma(t)) \eta(d\gamma)
\]

\[
= \int_{\Gamma} (\partial_t \phi(t, \gamma(t)) + \langle D\phi(t, \gamma(t)), \dot{\gamma}(t) \rangle) \eta(d\gamma) = V(t, \gamma(t))
\]

\[
= \int_{\overline{\Omega}} (\partial_t \phi(t, x) + \langle D\phi(t, x), V(t, x) \rangle) m(t, dx)
\]
Proof

Consider the continuous map $V_m : Q_m \cup \partial Q_m \rightarrow \mathbb{R}^n$

$$V_m(t, x) = \begin{cases} 
-\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\
-\partial_p H(x, \nabla_x u(t, x) + \partial_{\nu_i} u(t, x)\nu_i(x)) & \forall (t, x) \in \partial Q_m 
\end{cases}$$

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Let $\eta$ be a constrained equilibrium associated with $(u, m)$: then

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for $\eta$-a.e. $\gamma \in \Gamma$

So, $\forall \phi \in C_c^1(]0, T[ \times \Omega)$ we use the change of variables $m(t) = e_t\#\eta$ to compute

$$\frac{d}{dt} \int_{\Omega} \phi(t, x)m(t, dx) = \frac{d}{dt} \int_{\Gamma} \phi(t, \gamma(t))\eta(d\gamma)$$

$$= \int_{\Gamma} (\partial_t \phi(t, \gamma(t)) + \langle D\phi(t, \gamma(t)), \dot{\gamma}(t) \rangle)\eta(d\gamma)$$

$$= \int_{\Omega} (\partial_t \phi(t, x) + \langle D\phi(t, x), V(t, x) \rangle)m(t, dx)$$
Proof

- Consider the continuous map $V_m : Q_m \cup \partial Q_m \to \mathbb{R}^n$

$$V_m(t, x) = \begin{cases} -\partial_p H(x, \nabla u(t, x)) & \forall (t, x) \in Q_m \\ -\partial_p H(x, \nabla_x^\tau u(t, x) + \partial_{\nu_i}^+ u(t, x)\nu_i(x)) & \forall (t, x) \in \partial Q_m \end{cases}$$

and extend it to a continuous vector field $V : [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ by Tietze theorem.

- Let $\eta$ be a constrained equilibrium associated with $(u, m)$: then

$$(t, \gamma(t)) \in Q_m \cup \partial Q_m \quad \text{and} \quad \dot{\gamma}(t) = V(t, \gamma(t)) \quad \forall t \in ]0, T[$$

for $\eta$-a.e. $\gamma \in \Gamma$.

- So, $\forall \phi \in C^1_c([0, T] \times \overline{\Omega})$ we use the change of variables $m(t) = e^{t\#\eta}$ to compute

$$\frac{d}{dt} \int_{\Omega} \phi(t, x)m(t, dx) = \frac{d}{dt} \int_{\Gamma} \phi(t, \gamma(t)))\eta(d\gamma)$$

$$= \int_{\Gamma} (\partial_t \phi(t, \gamma(t)) + \langle D\phi(t, \gamma(t)), \dot{\gamma}(t) \rangle)\eta(d\gamma)$$

$$= \int_{\Omega} (\partial_t \phi(t, x) + \langle D\phi(t, x), V(t, x) \rangle)m(t, dx)$$
We have shown how to recover a fairly complete theory for the
- existence and uniqueness
- regularity
- pointwise behaviour
of solutions to constrained MFG systems

This opens the way to the study of at least two main problems
- Since constrained equilibria may develop singular parts (Dirac masses) induced by the presence of state constraints, are such singularities stable or do they disappear if constraints become inactive?
- How to describe the behaviour of the solution \((u^T, m^T)\) of the constrained Mean Field Games system

\[
\begin{aligned}
-\partial_t u^T(t, x) + \nabla_x u^T(t, x) &= F(x, m^T(t)), \quad \text{in } ]0, T[ \times \bar{\Omega} \\
\partial_t m^T(t) - \text{div} \left( m^T(t) D_p H(x, \nabla_x u^T(t, x)) \right) &= 0, \quad \text{in } ]0, T[ \times \bar{\Omega} \\
u^T(T, x) &= u^f(x), \quad m^T(0) = m_0, \quad \text{in } \bar{\Omega}.
\end{aligned}
\]

as \(T \to +\infty\)?
Conclusions

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\[
\begin{aligned}
-\partial_t u^T(t, x) + H(x, \nabla_x u^T(t, x)) &= F(x, m^T(t)), \quad \text{in } ]0, T[ \times \Omega \\
\partial_t m^T(t) - \text{div} \left( m^T(t) D_p H(x, \nabla_x u^T(t, x)) \right) &= 0, \quad \text{in } ]0, T[ \times \Omega \\
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\begin{aligned}
- \partial_t u^T(t, x) + H(x, \nabla_x u^T(t, x)) &= F(x, m^T(t)), \quad \text{in } [0, T] \times \Omega \\
\partial_t m^T(t) - \text{div} (m^T(t) D_p H(x, \nabla_x u^T(t, x))) &= 0, \quad \text{in } [0, T] \times \Omega \\
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\partial_t m^T(t) - \text{div} \left( m^T(t) D_p H(x, \nabla_x u^T(t, x)) \right) = 0, & \text{in } ]0, T[ \times \Omega \\
u^T(T, x) = u^f(x), \quad m^T(0) = m_0, & \text{in } \Omega.
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\]

as \( T \to +\infty \)?
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\partial_t m^T(t) - \text{div} \left( m^T(t) D_p H(x, \nabla_x u^T(t, x)) \right) = 0, & \text{in } ]0, T[ \times \bar{\Omega} \quad (CMFG) \\
u^T(T, x) = u^f(x), \quad m^T(0) = m_0, & \text{in } \bar{\Omega}.
\end{cases}
\]

as \( T \rightarrow +\infty \)?
Outline

1. Introduction to Mean Field Games

2. Mean Field Games with state constraints
   - The Lagrangian approach
   - Existence and uniqueness of relaxed equilibria
   - Regularity of relaxed solutions to constrained MFG
   - Point-wise properties of relaxed solutions

3. Concluding remarks
   - Asymptotic behaviour
Asymptotic behaviour: the unconstrained case

References

(i) P. Cardaliaguet (2013) on $\mathbb{T}^n$

(ii) joint work with W. Cheng, C. Mendico, and K. Wang (2019) in $\mathbb{R}^n$
under the following assumptions

(F1) There is a constant $C > 0$ such that for every $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2) \geq C \int_{\mathbb{R}^n} (F(x, m_1) - F(x, m_2))^2 dx$$

(F2) There exist a compact set $K_0 \subset \mathbb{R}^n$ and a constant $\delta_0 > 0$ such that

$$\min_{x \in K_0} \left\{ L(x, 0) + F(x, m) \right\} \leq \inf_{x \in \mathbb{R}^n \setminus K_0} \left\{ L(x, 0) + F(x, m) \right\} - \delta_0, \quad \forall m \in \mathcal{P}_1(\mathbb{R}^n)$$
Asymptotic behaviour: the unconstrained case

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(i) P. Cardaliaguet (2013) on $\mathbb{T}^n$

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under the following assumptions

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The limit behaviour of solutions is captured by the Ergodic Mean Field Games (EMFG) system

\[
\begin{aligned}
H(x, Du(x)) &= c_H(m) + F(x, m) \quad \text{in} \quad \mathbb{R}^n \\
\text{div}(m \nabla_p H(x, Du(x))) &= 0 \quad \text{in} \quad \mathbb{R}^n \\
\int_{\mathbb{R}^n} m(dx) &= 1
\end{aligned}
\]

where Mañé’s critical value \( c_H(m) \) is defined by

\[
c_H(m) := \inf \{ c \in \mathbb{R} : \exists u \in C(\mathbb{R}^n) \text{ viscosity solution of } H(x, Du) = c + F(x, m) \}
\]

see A. Fathi, ”Weak KAM Theorem in Lagrangian dynamics”
The limit behaviour of solutions is captured by

**Ergodic Mean Field Games (EMFG) system**

\[
\begin{align*}
H(x, Du(x)) &= c_H(m) + F(x, m) \quad \text{in} \quad \mathbb{R}^n \\
\text{div} \left( m \nabla_p H(x, Du(x)) \right) &= 0 \quad \text{in} \quad \mathbb{R}^n \\
\int_{\mathbb{R}^n} m(dx) &= 1
\end{align*}
\]

where Mañé’s critical value \( c_H(m) \) is defined by

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\]

see A. Fathi, ”Weak KAM Theorem in Lagrangian dynamics”
Solution of \((EMFG)\)

\[
\begin{cases}
H(x, D\bar{u}(x)) = c_H(\bar{m}) + F(x, \bar{m}), & \text{in } \mathbb{R}^n \\
\text{div}(\bar{m} \nabla \rho H(x, D\bar{u}(x))) = 0, & \text{in } \mathbb{R}^n \\
\int_{\mathbb{R}^n} \bar{m}(dx) = 1.
\end{cases}
\]

Theorem (existence of solutions – uniqueness of critical values)

\[(i)\] There exists at least one solution \((\bar{u}, \bar{m}, c_H(\bar{m}))\) of system \(EMFG\)

\[(ii)\] Let \((\bar{u}_1, \bar{m}_1, c_H(\bar{m}_1)), (\bar{u}_2, \bar{m}_2, c_H(\bar{m}_2))\) solve \((EMFG)\). Then,

\[c_H(\bar{m}_1) = c_H(\bar{m}_2) \text{ and } F(x, \bar{m}_1) = F(x, \bar{m}_2), \forall x \in \mathbb{R}^n\]
Conclusions

Asymptotic behaviour

Convergence of MFG solution

Theorem

Let \((\bar{u}, \bar{m}, c_H(\bar{m}))\) be any solution of

\[
\begin{aligned}
H(x, Du(x)) &= c_H(\bar{m}) + F(x, \bar{m}), \quad \text{in} \quad \mathbb{R}^n \\
\text{div}\left(\bar{m} \nabla_p H(x, Du(x))\right) &= 0, \quad \text{in} \quad \mathbb{R}^n \\
\int_{\mathbb{R}^n} \bar{m}(dx) &= 1.
\end{aligned}
\]  

\((EMFG)\)

Then, for any sufficiently large \(R > 0\) there exists a constant \(C(R) > 0\) such that for every \(T \geq 1\) the solution \((u^T, m^T)\) of the MFG system satisfies

\[
\sup_{t \in [0, T]} \frac{\|u^T(t, \cdot) - c_H(\bar{m})(t - T)\|_{\infty, B_R}}{T} \leq \frac{C(R)}{T^{1/(n+2)}},
\]  

\((2)\)

\[
\frac{1}{T} \int_0^T \|F(\cdot, m^T(s)) - F(\cdot, \bar{m})\|_{\infty, B_R} ds \leq \frac{C(R)}{T^{1/(n+2)}}.
\]  

\((3)\)
Thank you for your attention!

Figure: Rational agents at work, Benasque 2019