First order mean-field games

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VIII PARTIAL DIFFERENTIAL EQUATIONS, OPTIMAL DESIGN AND NUMERICS

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Outline

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A quote from Wikipedia

Mean field game theory is the study of strategic decision making in very large populations of small interacting agents. This class of problems was considered in the economics literature by B Jovanovic and RW Rosenthal, in the engineering literature by PE Caines and his co-workers, and independently and around the same time by mathematicians J-M Lasry and P-L Lions

Figure: Jovanovic, Rosenthal, Caines, Lasry, and Lions

Motivations for studying MFG

Goal

To describe Nash equilibria in the collective behaviour of a large population of “small” rational agents

- large population $\rightsquigarrow$ infinite number (a continuum) of players
- rational agents $\rightsquigarrow$ each agent is controlling his/her dynamical own state

Figure: MFG impact: finance, crowd dynamics, smart grids

Lasry-Lions 1: the Hamilton-Jacobi equation

Main idea: to export the principle of statistical mechanics to interactions within rational particles by introducing a macroscopic description through a mean field model

- agents are identified with points $x \in \overline{\Omega} \subset \mathbb{R}^n$
- $m(t, dx)$ is the distribution of agents at time t

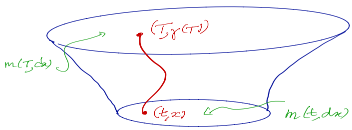
Agent located in $x \in \overline{\Omega}$ at time $t \in [0, T]$ chooses a path $\gamma_{t,x}(s)$, $s \in [t, T]$, such that

$$u(t,x) := \min_{\gamma(t)=x} \left\{ \int_t^T \left[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s)) \right] ds + G(\gamma(T), m(T)) \right\}$$

The value function $u(t, x)$ satisfies the associated Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u + H(x, \nabla u) = F(x, m) & [0, T] \times \Omega \\ u(T, x) = G(x, m(T)) \end{cases}$$

where
$H(x,p) := \sup_{v \in \mathbb{R}^n} \{ -\langle p, v\rangle - L(x, v) \}$



Lasry-Lions 2: the continuity equation

The space gradient $\nabla u(t,x)$ of the solution to the Hamilton-Jacobi equation gives the optimal feedback $\gamma_{t,x}$ via the system

$$\gamma'(s) = -\partial_p H(\gamma(s), \nabla u(s,\gamma(s))) \quad (s \in [t,T])$$

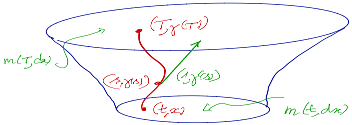
for the minimization problem

$$\min_{\gamma(t)=x} \left\{ \int_t^T \big[L(\gamma(s),\dot\gamma(s)) + F(\gamma(s),m(s))\big]\, ds + G(\gamma(T),m(T)) \right\}$$

Since $m(\cdot,dx)$ is just $m_0(dx)$ transported by such a flow, the continuity equation

$$\begin{cases} \partial_t m - \operatorname{div}(m\,\partial_p H(x,\nabla u)) = 0 & [0,T]\times\Omega \\ m(0,dx) = m_0(dx) \end{cases}$$

must be satisfied



Lasry-Lions 3: the MFG system

By coupling the Hamilton-Jacobi equation with the continuity equation above, one obtains the PDE system of Mean Filed Games

$$\begin{cases} -\partial_t u + H(x, \nabla u) - F(x, m) = 0 \\ \partial_t m - \operatorname{div}(m\, \partial_p H(x, \nabla u)) = 0 \end{cases} \quad]0, T[\times\Omega \quad \begin{cases} u(T, x) = G(x, m(T)) \\ m(0, dx) = m_0(dx) \end{cases} \qquad (MFG)$$

In the last decade, system (MFG) has been widely investigated for two main kinds od space domains

$$\Omega = \mathbb{T}^n,\ \mathbb{R}^n$$

main contributions by: Achdou, Bardi, Bensoussan, Camilli, Capuzzo Docetta, Cardaliaguet, Carmona, Delarue, Gomes, Guéant, Lachapelle, Porretta, . . .

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Solution of the MFG system

Reference: notes on Mean Field Games
by P. Cardaliaguet, 2013 and 2015

- by vanishing viscosity

$$\begin{cases} -\partial_t u - \epsilon\Delta u + H(x,\nabla u) = F(x,m) \\ \partial_t m - \epsilon\Delta m - \operatorname{div}(m\,\partial_p H(x,\nabla u_\mu)) = 0 \end{cases}$$

- by a fixed point argument

$$\mu \longrightarrow u_\mu \begin{cases} -\partial_t u + H(x,\nabla u) = F(x,\mu) \\ u(T,x) = G(x,\mu(T)) \end{cases} \longrightarrow m_\mu \begin{cases} \partial_t m - \operatorname{div}(m\,\partial_p H(x,\nabla u_\mu)) = 0 \\ m(0,dx) = m_0(x)dx \end{cases}$$

Very important facts:

- although nonsmooth, u is linearly semiconcave, which ensures a nice behaviour along minimizers
- if m_0 is absolutely continuous with respect to the Lebesgue measure, then $m(t,\cdot)$ stays absolutely continuous

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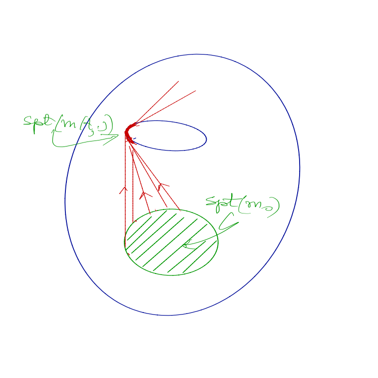
Does all this break down under state constraints?

Our goal To study MFG problems with state constraints: $x \in \overline{\Omega}$

Difficulty

Agent distribution may concentrate on small sets

Then the above methods break down



Outline

A change of paradigm

- $\Omega \subset \mathbb{R}^n$ bounded domain with boundary of class $\mathcal{C}^2$
- $\mathcal{P}(\overline{\Omega})$ Borel probability measures on $\overline{\Omega}$ with Katorovich-Rubinstein distance

$$d_1(m_1, m_2) = \sup\left\{ \int_{\overline{\Omega}} f\, dm_1 - \int_{\overline{\Omega}} f\, dm_2 \ : \ \big|f(x) - f(y)\big| \leqslant |x - y| \right\}$$

Recall that, given $m \in \mathcal{C}([0,T];\mathcal{P}(\overline{\Omega}))$, agents aim to attain

$$\min_{\gamma(0)=x,\gamma(t)\in\overline{\Omega}} \left\{ \int_0^T \big[L(\gamma(t),\dot\gamma(t)) + F(\gamma(t),m(t))\big]\,dt + G(\gamma(T),m(T)) \right\}$$

but m cannot be fixed a priori as it evolves along optimal feedback
Main idea to overcome such a difficulty:

to replace $\quad m \in \mathcal{C}([0,T];\mathcal{P}(\overline{\Omega}))$

by a probability measure on the metric space $\mathcal{C}([0,T];\overline{\Omega})$

that is $\quad \mathcal{C}([0,T];\mathcal{P}(\overline{\Omega})) \quad \longleftrightarrow \quad \mathcal{P}(\mathcal{C}([0,T];\overline{\Omega}))$

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Lagrangian approach

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Notation

- constrained arcs

$$\begin{aligned} \Gamma &= \left\{ \gamma \in AC([0,T];\mathbb{R}^n) \;:\; \gamma(t) \in \overline{\Omega}\,,\ \forall t \in [0,T] \right\} \quad \text{with} \quad \|\cdot\|_\infty \\ \Gamma[x] &= \left\{ \gamma \in \Gamma \;:\; \gamma(0) = x \right\} \quad (x \in \overline{\Omega}) \end{aligned}$$

- $\mathcal{P}(\Gamma)$ Borel probability measures on Γ: metric space with d_1 metric

$$d_1(\mu_1,\mu_2) = \sup\left\{ \int_\Gamma f\,d\mu_1 - \int_\Gamma f\,d\mu_2 \;:\; |f(\gamma) - f(\xi)| \leqslant \|\gamma - \xi\|_\infty \right\}$$

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Relaxed MFG functional

The evaluation map $e_t : \Gamma \to \overline{\Omega}$ $(t \in [0, T])$ is defined by $e_t(\gamma) = \gamma(t)$

Push-forward

With any $\mu \in \mathcal{P}(\Gamma)$ and $t \in [0, T]$ one can associate the probability measure $e_t \sharp \mu$ on $\overline{\Omega}$ given by

$$\int_{\overline{\Omega}} f(x)\, e_t \sharp \mu(dx) = \int_{\Gamma} f(\gamma(t))\, \mu(d\gamma) \qquad \forall f \in \mathcal{C}(\overline{\Omega})$$

$e_t \sharp \mu$ is the push-forward of μ by e_t

For any $\mu \in \mathcal{P}(\Gamma)$ we define

- the associated payoff functional

$$J_\mu[\gamma] = \int_0^T \Big[L(\gamma(t), \dot\gamma(t)) + F(\gamma(t), e_t \sharp \mu)\Big]\, dt + G(\gamma(T), e_T \sharp \mu) \qquad \forall \gamma \in \Gamma$$

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Relaxed equilibria

A Borel probability measure $\mu \in \mathcal{P}(\Gamma)$ is compatible with $m_0 \in \mathcal{P}(\overline{\Omega})$ if

$$e_0 \sharp \eta = m_0$$

Denote by $\mathcal{P}_{m_0}(\Gamma)$ the subspace consisting of all such measures

Definition

$\mu \in \mathcal{P}_{m_0}(\Gamma)$ is called a relaxed (CMFG) equilibrium for m_0 if

$$\mathrm{spt}(\mu) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^{\mu}[x]$$

Equivalently,

$$J_\mu[\overline{\gamma}] = \min_{\gamma \in \Gamma[\overline{\gamma}(0)]} J_\mu[\gamma] \quad \text{for } \mu\text{–a.e. } \overline{\gamma} \in \Gamma$$

where

$$J_\mu[\gamma] = \int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), e_t \sharp \mu) \right] dt + G(\gamma(T), e_T \sharp \mu)$$

Relaxed solutions

Let $m_0 \in \mathcal{P}(\overline{\Omega})$

Definition

$(u, m) \in \mathcal{C}([0, T] \times \overline{\Omega}) \times \mathcal{C}\big([0, T]; \mathcal{P}(\overline{\Omega})\big)$ *is a relaxed solution to the CMFG problem if*

$$m(t) = e_t \sharp \mu \qquad \forall t \in [0, T]$$

for some relaxed equilibrium $\mu \in \mathcal{P}_{m_0}(\Gamma)$ *and*

$$u(t, x) = \min_{\gamma \in \Gamma, \gamma(t) = x} \left\{ \int_t^T \big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s))\big] dt + G(\gamma(T), m(T)) \right\}$$

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Existence of relaxed equilibria and solutions

Theorem

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed equilibrium

Corollary

For any $m_0 \in \mathcal{P}(\overline{\Omega})$ there is at least one relaxed solution (u, m) to the CMFG problem

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Proof

Kakutani's fixed-point theorem

- $S \neq \varnothing$ compact convex subset of a locally convex Hausdorff space
- $\phi : S \rightrightarrows S$ nonempty convex-valued with closed graph

$\Longrightarrow \quad \phi$ has a fixed point.

Proof of theorem: construction of a fixed point of $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$

$$E(\eta) = \left\{ \mu \in \mathcal{P}_{m_0}(\Gamma) \mid \mathrm{spt}(\mu_x) \subseteq \Gamma^\eta[x] \text{ for } m_0\text{−a.e. } x \in \overline{\Omega} \right\} \quad (\eta \in \mathcal{P}_{m_0}(\Gamma))$$

where $\{\mu_x\}_{x \in \overline{\Omega}} \subset \mathcal{P}(\Gamma)$ is the family of probability measures which disintegrates μ

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Indeed

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ relaxed equilibrium} \quad \Longleftrightarrow \quad \eta \in E(\eta)$$

The existence of a fixed point of E follows from Kakutani's Theorem

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$$\mu = \int_{\overline{\Omega}} \mu_x dm_0(x) \quad \text{and} \quad \operatorname{spt}(\mu_x) \subseteq \Gamma[x] \ \ m_0 - \text{a.e. } x \in \overline{\Omega}$$

Indeed

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \ \text{ relaxed equilibrium} \quad \Longleftrightarrow \quad \eta \in E(\eta)$$

The existence of a fixed point of E follows from Kakutani's Theorem

Uniqueness

Theorem

Assume monotonicity conditions: for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$

$$\begin{cases} \displaystyle\int_{\overline{\Omega}} (G(x,m_1) - G(x,m_2))\, d(m_1 - m_2)(x) \geqslant 0 \\ \displaystyle\int_{\overline{\Omega}} (F(x,m_1) - F(x,m_2))\, d(m_1 - m_2)(x) > 0 \quad \text{if } m_1 \neq m_2 \end{cases}$$

If (u_1, m_1) *and* (u_2, m_2) *are relaxed solutions to the CMFG problem, then*

$$u_1 \equiv u_2 \quad \text{and} \quad m_1 = m_2$$

F satisfies the strict monotonicity condition if $F : \overline{\Omega} \times P(\overline{\Omega}) \to \mathbb{R}$ is of the form

$$F(x,m) = \int_{\Omega} f(y, (\phi \star m)(y))\, \phi(x - y)\, dy$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a smooth even kernel with compact support and

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Outline

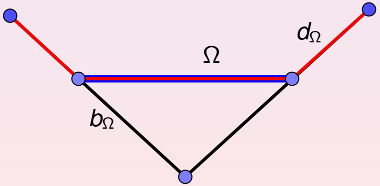
More notation and assumptions

Recall $\Omega \subset \mathbb{R}^n$ is bounded with $\partial\Omega \in C^2$. Consequently

- distance $\boxed{d_\Omega(x) = \min_{y\in\overline{\Omega}} |x-y|}$

  of class $\mathcal{C}^2(\Omega_\delta^+)$ for some $\delta > 0$ with $\Omega_\delta^+ = \{x \in \mathbb{R}^n \setminus \Omega \ : \ d_\Omega(x) < \delta\}$
- oriented boundary distance $\boxed{b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^n\setminus\Omega}(x)}$

  of class $\mathcal{C}^2(\Omega_\delta)$ on $\Omega_\delta = \{x \in \mathbb{R}^n \ : \ |b_\Omega(x)| < \delta\}$



References

- Dubovitskii – Milyutin (1964)
- Malanowski (1978)
- Hager (1979)
- Vinter (2000)
- Galbraith – Vinter (2003)
- Frankowska (2006, 2009)
- Bettiol – Frankowska (2007, 2008)
- Bettiol – Khalil – Vinter (2016)

Necessary conditions for smooth state constraints

Theorem

Given $x \in \overline{\Omega}$ *let* γ^* *minimize over* $\Gamma[x]$ *the functional*

$$\gamma \mapsto \int_0^T \big[L(\gamma(s),\dot\gamma(s)) + f(s,\gamma(s))\big]\,dt + g(\gamma(T))$$

where $g \in \mathcal{C}^1(\overline{\Omega})$ *and* $f : [0,T]\times\overline{\Omega} \to \mathbb{R}$ *satisfies* $|f_t| + |\nabla f| \le C$

Then there exist

- $p^* : [0,T] \to \mathbb{R}^n$ *Lipschitz*
- $\nu \in \mathbb{R}$ *and* $\Lambda \in \mathcal{C}_b([0,T]\times\Omega_\delta\times\mathbb{R}^n)$ *(independent of* γ^*, p^**)*

such that ($\mathbb{I}_{\partial\Omega}$ = *characteristic function of* $\partial\Omega$)

$$\begin{cases} \dot\gamma^* = -\partial_p H(\gamma^*, p^*) \\ \dot p^* = \nabla H(\gamma^*, p^*) - \nabla f(t,\gamma^*) - \Lambda(t,\gamma^*,p^*)\,\mathbb{I}_{\partial\Omega}(\gamma^*)\,\nabla b_\Omega(\gamma^*) \qquad \forall t \in [0,T] \\ p^*(T) = \nabla g(\gamma^*(T)) + \nu\,\mathbb{I}_{\partial\Omega}(\gamma^*(T))\,\nabla b_\Omega(\gamma^*(T)) \end{cases}$$

Consequently, $\gamma^* \in C^1_{Lip}([0,T];\mathbb{R}^n)$ *and* $\|\dot\gamma^*\|_{Lip} \leqslant C(\Omega, H, f, g)$

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Existence of Lipschitz solutions

Theorem

Let $m_0 \in \mathcal{P}(\overline{\Omega})$ and suppose

$$|F(x_1, m_1) - F(x_2, m_2)| + |G(x_1, m_1) - G(x_2, m_2)| \leqslant C\Big(|x_1 - x_2| + d_1(m_1, m_2)\Big)$$

Then there exists at least one relaxed solution of CMFG problem (u, m) such that

$$u \in Lip\big([0, T] \times \overline{\Omega}\big) \quad \text{and} \quad m \in Lip\big([0, T]; \mathcal{P}(\overline{\Omega})\big)$$

Such a solution will be called a Lipschitz relaxed solution of the CMFG problem

The proof applies necessary conditions to construct a relaxed CMFG equilibrium

$$\eta \in \mathcal{P}_{m_0}(\Gamma) \text{ such that } m(t) := e_t \sharp \eta \text{ belongs to } Lip\big([0, T]; \mathcal{P}(\overline{\Omega})\big)$$

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A quick look at semiconcave functions

$\Omega \subseteq \mathbb{R}^n$ open
$v : \Omega \to \mathbb{R}$ semiconcave with modulus $\omega : [0, \infty[\to [0, \infty[$ if

$$\lambda v(x) + (1 - \lambda)v(y) - v\big(\lambda x + (1 - \lambda)y\big) \leqslant \lambda(1 - \lambda)|x - y|\omega\big(|x - y|\big)$$

for all x, y such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$

Special cases:

- $\omega(s) \equiv 0 \quad \longrightarrow \quad$ concave
- $\omega(s) = Cs \ (C > 0) \quad \longrightarrow \quad$ linearly semiconcave
  In this case
  $$x \mapsto v(x) - \frac{C}{2}|x|^2 \qquad (\star)$$
  is concave on all convex subsets of Ω
- $\omega(s) = Cs^\alpha \ (C > 0, 0 < \alpha < 1) \ \longrightarrow$ fractionally semiconcave
  In this case, $(\star)$ is no longer valid

Some references on semiconcave functions

- control theory and sensitivity analysis
  Hrustalev 1978, C – Frankowska 1991
  Fleming – McEneaney 2000
  Rifford 2000, 2002
- nonsmooth and variational analysis
  Rockafellar 1982
  Colombo – Marigonda 2006, Colombo – Nguyen 2010
- differential geometry
  Perelman 1995, Petrunin 2007
- monographs
  C – Sinestrari (Birkhäuser 2004)
  Villani (Springer 2009)

Semiconcavity & nonsmooth analysis

For any semiconcave $v : \Omega \to \mathbb{R}$

- the superdifferential at $x \in \Omega$ coincides with Clarke's gradient

$$D^+ v(x) = \text{ co } D^* v(x) = \partial v(x)$$

  where $D^* v(x) = \left\{ \lim_{i\to\infty} Dv(x_i) \;\middle|\; x_i \to x \right\}$ reachable gradients

- $D^+ v(x) = \{p\} \quad \Longleftrightarrow \quad v \quad$ differentiable

Semiconcavity of relaxed Lipschitz solution

Theorem

Any Lipschitz relaxed solution (u, m) of CMFG problem is locally semiconcave on $[0, T[\times\overline{\Omega}$ with a fractional modulus:

$\forall \rho \in]0, T[$ there exists $C_\rho \geq 0$ such that

$$u(t+\tau, x+h) + u(t-\tau, x-h) - 2u(t, x) \leqslant C_\rho(|\tau| + |h|)^{3/2}$$

for all $t, t \pm \tau \in [0, T-\rho]$ and $x, x \pm h \in \overline{\Omega}$

Several proofs of the above result can be given
An interesting method of proof uses sensitivity relations that we discuss next

Adjoint state inclusion / sensitivity relations

Given

- *a Lipschitz relaxed solution* (u, m) *of the CMFG problem*
- $(t, x) \in [0, T[\times\overline{\Omega}$ *and a solution* $\gamma^* \in \Gamma$ *to*

$$\min_{\gamma\in\Gamma,\gamma(t)=x} \left\{ \int_t^T \big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m(s))\big]\, dt + G(\gamma(T), m(T)) \right\}$$

- *the adjoint state* $p^* : [t, T] \to \mathbb{R}^n$ *associated with* γ^*

we have that

$$\Big(H(\gamma^*(s), p^*(s)) - F(\gamma^*(s), m(s)) \,,\, p^*(s) \Big) \in D^+ u(s, \gamma^*(s)) \quad \forall s \in [t, T[$$

and $\forall \rho \in]0, T[$ *there exists* $C_\rho \geqslant 0$ *such that* $\forall\, t, t+\tau \in [0, T-\rho]$ *and all* $x + h \in \overline{\Omega}$

$$u(t+\tau, x+h) - u(t, x) - \tau\big(H(x, p^*(t)) - F(x, m(t))\big) - \langle p^*(t), h\rangle \leqslant C_\rho(|\tau| + |h|)^{3/2}$$

Adjoint state inclusion / sensitivity relations

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and $\forall \rho \in]0, T[$ *there exists* $C_\rho \geqslant 0$ *such that* $\forall\, t, t + \tau \in [0, T - \rho]$ *and all* $x + h \in \overline{\Omega}$

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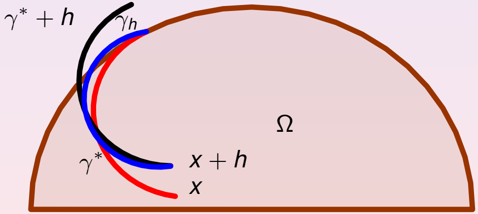
Proof of sensitivity relation for $\tau = 0$

We want to show that $\forall\, t \in [0, T-\rho]$ and all $x + h \in \overline{\Omega}$

$$u(t, x+h) - u(t,x) - \langle p(t), h \rangle \leqslant C_\rho |h|^{3/2}$$

Let $0 < \sigma \leqslant \rho$ to be fixed later and define for all $s \in [t, T]$

$$\gamma_h(s) = \gamma^*(s) + \left(1 + \frac{t-s}{\sigma}\right)_+ h$$



$$\widehat{\gamma}_h(s) = \gamma_h(s) - d_{\overline{\Omega}}\big(\gamma_h(s)\big)\, Dd_{\partial\Omega}\big(\gamma_h(s)\big)$$

Proof of sensitivity relation (continued)

By dynamic programming

$$u(t,x+h)-u(t,x)-\langle p(t),h\rangle \leqslant \int_t^{t+\sigma}\left[L(\widehat{\gamma}_h,\dot{\widehat{\gamma}}_h)-L(\gamma^*,\dot{\gamma}^*)\right]ds + \int_t^{t+\sigma}\left[F(\widehat{\gamma}_h,m)-F(\gamma^*,m)\right]ds-\langle p(t),h\rangle \qquad (1)$$

We want to express $\langle p(t),h\rangle$ so we expand

$$-\langle p(t),h\rangle = -\langle p(t+\sigma),\underbrace{\widehat{\gamma}_h(t+\sigma)-\gamma^*(t+\sigma)}_{=0}\rangle + \int_t^{t+\sigma}\frac{d}{ds}\langle p,\widehat{\gamma}_h-\gamma^*\rangle\,ds$$

$$= \int_t^{t+\sigma}\langle \dot{p},\widehat{\gamma}_h-\gamma^*\rangle\,ds + \int_t^{t+\sigma}\langle p,\dot{\widehat{\gamma}}_h-\dot{\gamma}^*\rangle\,ds$$

By appealing to PMP to represent $\langle \dot{p},\widehat{\gamma}_h-\gamma^*\rangle$ and $\langle p,\dot{\widehat{\gamma}}_h-\dot{\gamma}^*\rangle$ we obtain

$$u(t,x+h)-u(t,x)-\langle p(t),h\rangle \leqslant \dots$$

$$\leqslant C\int_t^{t+\sigma}|\widehat{\gamma}_h-\gamma^*|^2\,ds + C\int_t^{t+\sigma}|\dot{\widehat{\gamma}}_h-\dot{\gamma}^*|^2\,ds + C\int_t^{t+\sigma}|\widehat{\gamma}_h-\gamma^*|\,ds$$

Proof of sensitivity relation (completed)

Recalling

$$\begin{cases} \gamma_h(s) = \gamma^*(s) + \left(1 + \frac{t-s}{\sigma}\right)_+ h \\ \widehat{\gamma}_h(s) = \gamma_h(s) - d_{\overline{\Omega}}(\gamma_h(s))\, Dd_{\partial\Omega}(\gamma_h(s)) \end{cases}$$

we have that

$$|\widehat{\gamma}_h(s) - \gamma^*(s)| \leq 2|h| \quad \forall s \in [t, t+\sigma]$$

Using the regularity of the distance functions one can also prove (technical)

$$\int_t^{t+\sigma} |\dot{\widehat{\gamma}}_h(s) - \dot{\gamma}^*(s)|^2\, ds \leqslant C\,\frac{|h|^2}{\sigma} + C|h|\sigma$$

Therefore

$$u(t, x+h) - u(t,x) - \langle p(t), h\rangle \leqslant C|h|\left(\frac{|h|}{\sigma} + \sigma\right) \leqslant 2C|h|^{3/2}$$

by taking $\sigma = |h|^{1/2}$

Outline

Point-wise solutions of the HJ equation

Given a Lipschitz relaxed solution (u, m) to CMFG problem, we have that

(I) u is a constrained viscosity solution of

$$\begin{cases} -\partial_t u + H(x, \nabla u) = F(x, m) & \text{in }]0, T[\times\overline{\Omega} \\ u(T, x) = G(x, m(T)) & \forall x \in \overline{\Omega} \end{cases}$$

Moreover, defining

$$Q_m = \Big\{(t, x) \in]0, T[\times\Omega \ : \ x \in spt\big(m(t)\big)\Big\}$$
$$\partial Q_m = \Big\{(t, x) \in]0, T[\times\partial\Omega \ : \ x \in spt\big(m(t)\big)\Big\}$$

the following holds true

(II) u is differentiable on Q_m and $-\partial_t u + H(x, \nabla u) = F(x, m)$ on Q_m

(III) u has time derivative, one-sided normal derivative, and tangential gradient on ∂Q_m

(IV) the tangential gradient $\nabla^\tau u$ satisfies

$$-\partial_t u + H^\tau(x, \nabla_x^\tau u) = F(x, m) \ \text{ on } \ \partial Q_m$$

where $H^\tau(x, p) = \sup\Big\{ -\langle p, v\rangle - L(x, v) \mid \langle v, \nu(x)\rangle = 0\Big\}$

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$$-\partial_t u + H^\tau(x, \nabla_x^\tau u) = F(x, m) \text{ on } \partial Q_m$$

where $H^\tau(x, p) = \sup\big\{ - \langle p, v\rangle - L(x, v) \mid \langle v, \nu(x)\rangle = 0\big\}$

Analysis of the continuity equation

Given a Lipschitz relaxed solution (u, m) to CMFG problem, we have that

(I) there exists a bounded continuous vector field $V :]0, T] \times \overline{\Omega} \to \mathbb{R}^n$ such that m satisfies the continuity equation

$$\partial_t m + div(mV) = 0 \qquad \text{in }]0, T[\times\overline{\Omega}$$

in the sense of distributions: $\forall\, \phi \in \mathcal{C}^1_c(]0, T[\times\overline{\Omega})$

$$\int_0^T \int_{\overline{\Omega}} \left(\phi_t + \langle V, \nabla\phi\rangle\right) dm(t, dx)dt = 0$$

(II) V is given by the optimal feedback on Q_m, that is,

$$V(t,x) = \begin{cases} -\partial_p H(x, \nabla u(t,x)) & \forall (t,x) \in Q_m \\ -\partial_p H(x, \nabla^\tau_x u(t,x) + \partial^+_{\nu_i} u(t,x)\nu_i(x)) & \forall (t,x) \in \partial Q_m \end{cases}$$

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Proof

- Consider the continuous map $V_m : Q_m \cup \partial Q_m \to \mathbb{R}^n$

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  and extend it to a continuous vector field $V :]0,T[\times\overline{\Omega} \to \mathbb{R}^n$ by Tietze theorem
- Let η be a constrained equilibrium associated with (u,m): then

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  for η-a.e. $\gamma \in \Gamma$
- So, $\forall \phi \in \mathcal{C}_c^1(]0,T[\times\overline{\Omega})$ we use the change of variables $m(t) = e_t \sharp \eta$ to compute

$$\begin{aligned} \frac{d}{dt}\int_\Omega \phi(t,x)m(t,dx) &= \frac{d}{dt}\int_\Gamma \phi(t,\gamma(t))\eta(d\gamma) \\ &= \int_\Gamma \big(\partial_t\phi(t,\gamma(t)) + \langle D\phi(t,\gamma(t)), \underbrace{\dot\gamma(t)}_{=V(t,\gamma(t))} \rangle\big)\eta(d\gamma) \\ &= \int_\Omega \big(\partial_t\phi(t,x) + \langle D\phi(t,x), V(t,x)\rangle\big)m(t,dx) \end{aligned}$$

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Conclusions

We have shown how to recover a fairly complete theory for the

- existence and uniqueness
- regularity
- pointwise behaviour

of solutions to constrained MFG systems

This opens the way to the study of at least two main problems

- Since constrained equilibria may develop singular parts (Dirac masses) induced by the presence of state constraints, are such singularities stable or do they disappear if constraints become inactive?
- How to describe the behaviour of the solution (u^T, m^T) of the constrained Mean Field Games system

$$\begin{cases} -\partial_t u^T(t,x) + H(x, \nabla_x u^T(t,x)) = F(x, m^T(t)), & \text{in} \quad]0,T[\times\overline{\Omega} \\ \partial_t m^T(t) - \operatorname{div}\Big(m^T(t) D_p H(x, \nabla_x u^T(t,x))\Big) = 0, & \text{in} \quad]0,T[\times\overline{\Omega} \\ u^T(T,x) = u^f(x), \ m^T(0) = m_0, & \text{in} \quad \overline{\Omega}. \end{cases} \qquad (CMFG)$$

as $T \to +\infty$?

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Outline

Asymptotic behaviour: the unconstrained case

References

(*i*) P. Cardaliaguet (2013) on $\mathbb{T}^n$

(*ii*) joint work with W. Cheng, C. Mendico, and K. Wang (2019) in $\mathbb{R}^n$ under the following assumptions

(F1) There is a constant $C > 0$ such that for every $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (F(x,m_1) - F(x,m_2))d(m_1 - m_2) \geqslant C \int_{\mathbb{R}^n} (F(x,m_1) - F(x,m_2))^2 \, dx$$

(F2) There exist a compact set $K_0 \subset \mathbb{R}^n$ and a constant $\delta_0 > 0$ such that

$$\min_{x \in K_0} \Big\{ L(x,0) + F(x,m) \Big\} \leqslant \inf_{x \in \mathbb{R}^n \setminus K_0} \Big\{ L(x,0) + F(x,m) \Big\} - \delta_0, \quad \forall m \in \mathcal{P}_1(\mathbb{R}^n)$$

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Ergodic Mean Field Games system

The limit behaviour of solutions is captured by

Ergodic Mean Field Games (EMFG) system

$$\begin{cases} H(x, D\overline{u}(x)) = c_H(\overline{m}) + F(x, \overline{m}) & \text{in} \quad \mathbb{R}^n \\ \operatorname{div}\Big(\overline{m}\,\nabla_p H(x, D\overline{u}(x))\Big) = 0 & \text{in} \quad \mathbb{R}^n \\ \int_{\mathbb{R}^n} \overline{m}(dx) = 1 \end{cases}$$

where Mañé's critical value $c_H(\overline{m})$ is defined by

$$c_H(\overline{m}) := \inf\left\{c \in \mathbb{R} : \ \exists u \in C(\mathbb{R}^n) \text{ viscosity solution of } H(x, Du) = c + F(x, \overline{m})\right\}$$

see A. Fathi, "Weak KAM Theorem in Lagrangian dynamics"

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Solution of (*EMFG*)

$$\begin{cases} H(x, D\overline{u}(x)) = c_H(\overline{m}) + F(x, \overline{m}), & \text{in} \quad \mathbb{R}^n \\ \operatorname{div}\Big(\overline{m}\,\nabla_p H(x, D\overline{u}(x))\Big) = 0, & \text{in} \quad \mathbb{R}^n \\ \int_{\mathbb{R}^n} \overline{m}(dx) = 1. \end{cases} \tag{EMFG}$$

Theorem (existence of solutions – uniqueness of critical values)

(*i*) *There exists at least one solution* $\big(\overline{u}, \overline{m}, c_H(\overline{m})\big)$ *of system EMFG*

(*ii*) *Let* $\big(\overline{u}_1, \overline{m}_1, c_H(\overline{m}_1)\big)$, $\big(\overline{u}_2, \overline{m}_2, c_H(\overline{m}_2)\big)$ *solve* (*EMFG*)*. Then,*

$$c_H(\overline{m}_1) = c_H(\overline{m}_2) \quad \text{and} \quad F(x, \overline{m}_1) = F(x, \overline{m}_2), \quad \forall x \in \mathbb{R}^n$$

Convergence of MFG solution

Theorem

Let $(\overline{u}, \overline{m}, c_H(\overline{m}))$ *be any solution of*

$$\begin{cases} H(x, D\overline{u}(x)) = c_H(\overline{m}) + F(x, \overline{m}), & in \quad \mathbb{R}^n \\ div\Big(\overline{m}\,\nabla_p H(x, D\overline{u}(x))\Big) = 0, & in \quad \mathbb{R}^n \\ \int_{\mathbb{R}^n} \overline{m}(dx) = 1. \end{cases} \qquad (EMFG)$$

Then, for any sufficiently large $R > 0$ *there exists a constant* $C(R) > 0$ *such that for every* $T \geq 1$ *the solution* (u^T, m^T) *of the MFG system satisfies*

$$\sup_{t \in [0,T]} \frac{\left\| u^T(t, \cdot) - c_H(\overline{m})(t - T) \right\|_{\infty, \overline{B}_R}}{T} \leq \frac{C(R)}{T^{\frac{1}{n+2}}}, \qquad (2)$$

$$\frac{1}{T} \int_0^T \left\| F(\cdot, m^T(s)) - F(\cdot, \overline{m}) \right\|_{\infty, \overline{B}_R} ds \leq \frac{C(R)}{T^{\frac{1}{n+2}}}. \qquad (3)$$

Thank you for your attention!

Figure: Rational agents at work, Benasque 2019