One-side boundary controllability of the $p$-system

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Partial Differential Equations, Optimal Design and Numerics VIII
Benasque 2019
I. Introduction

One-dimensional isentropic Euler equations:

- The $p$-system (compressible Euler equation in Lagrangian coordinates):

\[
\begin{align*}
\frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(\kappa \tau - \gamma) &= 0.
\end{align*}
\]

- In original Eulerian coordinates:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} &= 0, \\
\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m^2}{\rho} + \kappa \rho^\gamma\right) &= 0.
\end{align*}
\]

where

- $\rho = \rho(t, x) \geq 0$ is the density of the fluid,
- $m(t, x)$ is the momentum ($v(t, x) = \frac{m(t, x)}{\rho(t, x)}$ is the velocity of the fluid),
- $\tau := 1/\rho$ is the specific volume,
- the pressure law is $p(\rho) = \kappa \rho^\gamma$, $\gamma \in (1, 3]$. 
Controllability problem

- **Domain**: $(t, x) \in [0, T] \times [0, L]$.

- **State of the system**: $u = (\tau, \nu)$.

- **Control**: the “boundary data”: here, on one side, say $x = 0$, while there is a fixed boundary law at $x = L$.

- **Controllability problem**: given $u_0$ and $u_1$, can we find boundary data $x = 0$ driving the state from $u_0$ to $u_1$?

- **Equivalently**: given $u_0$ and $u_1$, can we find a solution of the system satisfying the boundary condition and driving $u_0$ to $u_1$?
Both systems enter the class of **hyperbolic systems of conservation laws**:

\[ U_t + f(U)_x = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \]

satisfying the (strict) hyperbolicity condition that at each point \( df \) has \( n \) distinct real eigenvalues \( \lambda_1 < \cdots < \lambda_n \).

Hyperbolic systems of conservation laws develop **singularities in finite time**.

This easy to see for instance for the Burgers equation:

\[ u_t + (u^2)_x = 0. \]
Class of solutions

- One can either work with regular solutions \((C^1)\) with small \(C^1\)-norm (for small time), or with discontinuous (weak) solutions.

- For the latter case, is natural for the sake of uniqueness to consider weak solutions which satisfy entropy conditions (entropy solutions).

- More precisely, the solutions will be of bounded variation, with small total variation in \(x\) ("à la Glimm").

- Note that there exist weaker solutions (DiPerna, Lions-Perthame-Souganidis-Tadmor, etc.)
Entropy conditions

Definition
An entropy/entropy flux couple for a hyperbolic system of conservation laws (SCL) is defined as a couple of regular functions \((\eta, q) : \Omega \to \mathbb{R}\) satisfying:

\[
\forall U \in \Omega, \quad D\eta(U) \cdot Df(U) = Dq(U).
\]

Definition
A function \(U \in L^\infty(0, T; BV(0, L)) \cap Lip(0, T; L^1(0, L))\) is called an entropy solution of (SCL) when, for any entropy/entropy flux couple \((\eta, q)\), with \(\eta\) convex, one has in the sense of measures

\[
\eta(U)_t + q(U)_x \leq 0,
\]

that is, for all \(\varphi \in \mathcal{D}((0, T) \times (0, L))\) with \(\varphi \geq 0\),

\[
\int_{(0, T) \times (0, L)} \left( \eta(U(t, x))\varphi_t(t, x) + q(U(t, x))\varphi_x(t, x) \right) dx \, dt \geq 0.
\]
Boundary condition

- Our boundary condition will take the following form at $x = L$:

$$b(u(t, L)) = 0 \text{ for a.e. } t,$$

where $b = b(\rho, v) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a function satisfying some non-degeneracy conditions (to be specified later).

- Examples:
  - $v = 0$: zero-speed on the right boundary,
  - $\rho = \bar{\rho}$: constant density (or constant pressure) at $x = L$. 
Main result

Theorem
Let $b$ satisfy the non-degeneracy condition.

Let $\bar{u}_0 := (\bar{\tau}_0, \bar{\nu}_0) \in \mathbb{R}^2$ with $\bar{\tau}_0 > 0$ and $b(\bar{u}_0) = 0$ and let $\bar{u}_1 = (\bar{\tau}_1, \bar{\nu}_1)$ with $\bar{\tau}_1 > 0$ and $b(\bar{u}_1) = 0$.

There exist $\varepsilon > 0$ and $T > 0$ such that for any $u_0 = (\tau_0, \nu_0)$ in $BV(0, L; \mathbb{R}^2)$ such that

$$\|u_0 - \bar{u}_0\|_{L^\infty(0, L)} + TV(u_0) \leq \varepsilon,$$

and $b(u_0(L^-)) = 0$, there is

$$u \in L^\infty(0, T; BV(0, L)) \cap Lip([0, T]; L^1(0, L)),$$

a weak entropy solution of the $p$-system such that

$$u|_{t=0} = u_0 \quad \text{and} \quad u|_{t=T} = \bar{u}_1.$$
Theorem
Let $b$ satisfy the non-degeneracy condition.

Let $\overline{u}_0 := (\overline{\tau}_0, \overline{v}_0) \in \mathbb{R}^2$ with $\overline{\tau}_0 > 0$ and $b(\overline{u}_0) = 0$ and let $\overline{u}_1 = (\overline{\tau}_1, \overline{v}_1)$ with $\overline{\tau}_1 > 0$ and $b(\overline{u}_1) = 0$.

Let $\eta > 0$. There exist $\varepsilon > 0$ and $T > 0$ such that for any $u_0 = (\tau_0, v_0)$ in $BV(0, L; \mathbb{R}^2)$ such that

$$\|u_0 - \overline{u}_0\|_{L^\infty(0, L)} + TV(u_0) \leq \varepsilon,$$

and $b(u_0(L^-)) = 0$, there is

$$u \in L^\infty(0, T; BV(0, L)) \cap Lip([0, T]; L^1(0, L)),$$

a weak entropy solution of the $p$-system such that

$$u_{|t=0} = u_0 \text{ and } u_{|t=T} = \overline{u}_1,$$

and

$$TV(u(t, \cdot)) \leq \eta, \quad \forall t \in (0, T).$$
II. Two connected results

▶ Bressan and Coclite (2002) : for a class of systems containing Di Perna’s system :

\[
\begin{align*}
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) &= 0, \\
\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + \frac{K^2}{\gamma-1} \rho^{\gamma-1}\right) &= 0,
\end{align*}
\]

there are initial conditions \( \varphi \in BV([0, 1]) \) of arbitrary small total variation such that any entropy solution \( u \) remaining of small total variation satisfies : for any \( t \), \( u(t, \cdot) \) is not constant. \( \neq C^1 \) case!

▶ G. (2007) : A sufficient condition concerning the isentropic Euler equation

\[
\begin{align*}
(E) : \left\{ \begin{array}{l}
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0, \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + \kappa \rho^{\gamma}) = 0,
\end{array} \right. \\
(P) : \left\{ \begin{array}{l}
\frac{\partial}{\partial t}\tau - \frac{\partial}{\partial x}\nu = 0, \\
\frac{\partial}{\partial t}\nu + \frac{\partial}{\partial x}(\kappa \tau^{\gamma-1}) = 0,
\end{array} \right.
\]

for final states to be reachable. For instance, all constant states are reachable.
III. Basic facts on systems of conservation laws

- **Systems of conservations laws**: 
  \[ u_t + f(u)_x = 0, \quad f : \mathbb{R}^n \to \mathbb{R}^n, \]
  
  \[ A(u) := df(u) \] has \( n \) real distinct eigenvalues \( \lambda_1 < \cdots < \lambda_n \), which are characteristic speeds of the system with corresponding eigenvectors \( r_i(u) \).

- **Genuinely non-linear fields in the sense of Lax**: 
  \[ \nabla \lambda_i \cdot r_i \neq 0 \quad \text{for all} \quad u. \]
  
  ⇒ we normalize \( \nabla \lambda_i \cdot r_i = 1 \).

- In the case of (P) we have  
  \[ \lambda_1 = -\sqrt{\kappa \gamma \tau - \gamma - 1} \quad \text{and} \quad \lambda_2 = \sqrt{\kappa \gamma \tau - \gamma - 1}. \]
Boundary conditions

- We can now express our non-degeneracy condition on the boundary law $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$.

We ask that $b$ satisfies the two following conditions:

- **Standard** condition for the Cauchy problem:

  $$ r_1 \cdot \nabla b \neq 0 \text{ on } \Omega, $$

- **Condition** for the backward in time Cauchy problem:

  $$ r_2 \cdot \nabla b \neq 0 \text{ on } \Omega, $$
The Riemann problem

- Find autosimilar solutions $u = \bar{u}(x/t)$ to

$$\begin{cases} u_t + (f(u))_x = 0 \\ u|_{\mathbb{R}^-} = u_l \text{ and } u|_{\mathbb{R}^+} = u_r. \end{cases}$$

- Solved by introducing Lax’s curves which consist of points that can be joined starting from $u_l$ either by a shock or a rarefaction wave.
Shocks and rarefaction waves

Discontinuities satisfying:

- Rankine-Hugoniot (jump) relations
\[
[f(u)] = s \begin{bmatrix} u \end{bmatrix},
\]

- Lax’s inequalities:
\[
\lambda_i(u_r) < s < \lambda_i(u_l)
\]

Propagates at speed \( s \sim \int_{u_l}^{u_r} \lambda_i \)

Regular solutions, obtained with integral curves of \( r_i \):

\[
\begin{cases}
\frac{d}{d\sigma} R_i(\sigma) = r_i(R_i(\sigma)), \\
R_i(0) = u_l,
\end{cases}
\]

with \( \sigma \geq 0 \).

Propagates at speed \( \lambda_i(R_i(\sigma)) \)
Lax’s Theorem proves that one can solve (at least locally) the Riemann problem by first following the 1-curve (gathering states connected to \( u_l \) by a 1-rarefaction/1-shock), then the 2-curve.
Boundary Riemann problem

The same principle applies on the boundary (both forward and backward in time)
Front-tracking algorithm (Dafermos, Di Perna, Bressan, Risebro, ...)

- Approximate initial condition by piecewise constant functions
- Solve the Riemann problems and replace rarefaction waves by rarefaction fans

- One obtains a piecewise constant function, with straight discontinuities (**fronts**)
- Iterate the process at each **interaction point** (points where fronts meet)
Estimates, convergence, etc.

- One shows that this defines a piecewise constant function, with a finite number of fronts and discrete interaction points.

- A central argument is due to Glimm: analyzing interactions of fronts $\alpha + \beta \rightarrow \alpha' + \beta' + \gamma'$ and the evolution of the strength of waves across an interaction, one proves that:

  \[
  \text{if } TV(u_0) \text{ is small enough, then } TV(u(t)) \leq C \ TV(u_0) \text{ for some } C > 0.
  \]

- One deduces bounds in $L^\infty_t BV_x$, then in $\text{Lip}_t L^1_x$, so we have compactness...
IV. A light idea of the construction when the control acts on both sides

- **Bressan & Coclite’s counterexample.** DiPerna’s system is a $2 \times 2$ hyperbolic system with GNL fields, and which satisfies

  *the interaction of two shocks of the same family generates a shock in this family (normal) and a shock in the other family.*

Hence starting from an initial date with a dense set of shocks, this propagates over time, even with control on both sides.

- A basic idea (even to control on both sides) is to use the fact that for the $p$-system :

  *the interaction of two shocks of the same family generates a shock in this family (normal) and a rarefaction in the other family.*
Some ideas, control from both boundaries, 1

- To begin with, sends a strong (large) shock of the second family from the boundary.
Some ideas, control from both boundaries, 2

- Then one sends additional 2-shocks from the boundary and one relies on cancellations to prevent 1-shocks to cross.

To make the construction, use $L - x$ as time variable.

- Since only 1-rarefactions cross and since they do not interact, the system reaches a constant state after a finite time.
V. A light idea of the construction, one-side controls

- When one controls only from one side (say, from the left), there are two differences:
  - One has to take into account the reflections at $x = L$ below the strong shock. Not an issue.
  - One has to take into account the reflections at $x = L$ of the strong shock. There are two situations, one of which changes everything.

- **Situation 1.** The strong 2-shock is reflected as a 1-rarefaction when
  
  $$(r_1 \cdot \nabla b)(r_2 \cdot \nabla b) < 0.$$ 

  In this case, since this adds a rarefaction to the picture, the above construction still works.

- **Situation 2.** The strong 2-shock is reflected as a 1-shock when
  
  $$(r_1 \cdot \nabla b)(r_2 \cdot \nabla b) > 0.$$ 

  In this case, one needs an additional construction.

**Example :** $v = 0$ at $x = L$. 
When the strong 2-shock is reflected as a 1-shock, it can then interact with 1-rarefactions, and one does not reach a constant state.
When there is a reflected strong shock, then the idea is to send again more small 2-shocks from the boundary (or here more precisely compression fronts) and rely on the reflection at $x = L$ to cancel the rarefactions fronts that interact with the reflected strong 1-shock.
Main difficulty

▶ There is no “good” direction of time to make the construction. Whether you use $t$, $T - t$, $x$, $L - x$ and so on as a time direction, the construction depends on the future.

▶ Hence to reach the previous picture of the construction, we rely on a fixed-point scheme.

▶ Problem: the front-tracking approach makes the scheme discontinuous...
Thank you for your attention!