Null controllability of a penalized Stokes problem in dimension two with one scalar control.

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Contextualization

Theorem (Coron,Guerrero;2009)

Let $\Omega \subset \mathbb{R}^2$ be a regular domain, $\omega \subset \Omega$ a subdomain, $T > 0$ and $e \in \mathbb{R}^2$. Then, there is $C > 0$ such that for any $y^0 \in \mathcal{H}(\Omega)$ there is a scalar function $f \in L^2((0,T) \times \omega)$ such that the regular solution of:

$$\begin{cases} y_t - \Delta y + \nabla p = f 1_\omega e & \text{in } (0,T) \times \Omega, \\ \nabla \cdot y = 0 & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y^0 & \text{on } \Omega, \end{cases}$$

satisfies $y(T,0) = 0$ and such that:

$$\|f\|_{L^2((0,T)\times\omega)} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)}.$$

Our objective

Throughout this talk we study if we have null controllability uniformly with respect to ε for the following penalized Stokes system:

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_\omega e & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(0,\cdot) = y^0 & \text{in } \Omega. \end{cases}$$

We take $\Omega \subset \mathbb{R}^2$, $Q := (0,T) \times \Omega$, $\Sigma := (0,T) \times \partial\Omega$ and $y^0 \in \mathbf{L}^2(\Omega)$. We expect $f^\varepsilon \in L^2((0,T) \times \omega)$ such that there is $\varepsilon_0 > 0$ and $C > 0$ such that $\|f^\varepsilon\|_{L^2((0,T)\times\omega)} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)}$ for $\varepsilon \in (0, \varepsilon_0]$. The most interesting cases are $y^0 \in \mathcal{H}(\Omega)$, even if we study the more general case $y^0 \in \mathbf{L}^2(\Omega)$.

The observability inequality when $e = (1,0)$

Proving the null-controllability is equivalent to proving

$$\int_{\Omega} |\varphi^{\varepsilon}(0,\cdot)|^2 \leq C \iint_{(0,T)\times\omega} |\varphi_1^{\varepsilon}|^2,$$

for φ^{ε} any solution of the adjoint system:

$$\begin{cases} -\varphi_t^{\varepsilon} - \Delta\varphi^{\varepsilon} + \nabla\pi^{\varepsilon} = 0 & \text{in } Q, \\ \varepsilon\pi^{\varepsilon} + \nabla\cdot\varphi^{\varepsilon} = 0 & \text{in } Q, \\ \varphi^{\varepsilon} = 0 & \text{on } \Sigma, \\ \varphi^{\varepsilon}(T,\cdot) = \varphi^T & \text{in } \Omega, \end{cases}$$

for $\varphi^T \in \mathbf{L}^2(\Omega)$. The equivalence is a consequence of the Lax-Milgram theorem.

The key point: the coupling

Let us take a closer look to the equations of φ^ε:

$$\begin{cases} -\partial_t\varphi_1^\varepsilon - \partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\partial_t\varphi_2^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon, \\ \varphi_\Sigma^\varepsilon = 0. \end{cases}$$

The main difficulty is to make sure that $\partial_{xy}\varphi_2^\varepsilon$ and φ_1^ε small implies φ_2^ε small (and to quantify it). If we had $\Delta\varphi_2^\varepsilon$ instead of $\partial_{xy}\varphi_2^\varepsilon$ it would be a well-known result and we would not need any information at all from φ_1^ε.

An important difficulty: a negative case in a domain that is just Lipschitz

Let $\varepsilon > 0$. We have that the function

$$\varphi^\varepsilon(x,y) := \left(0, e^{\lambda t}\left[\sin\left(\sqrt{\lambda}x\right) - \sin\left(\sqrt{\frac{\varepsilon\lambda}{1+\varepsilon}}y\right)\right]\right)$$

is a solution of the adjoint system for Ω_ε limited by the lines:

$$\begin{cases} x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y, \\ x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{2\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y - \frac{\pi}{\sqrt{\lambda}}. \end{cases}$$

In particular, for those rhombus there is $\varepsilon > 0$ such that the penalized Stokes system is not controllable, even when $\omega = \Omega$.

The assumption

We suppose that $\Omega \subset \mathbb{R}^2$ is a regular domain which satisfies the following:

Hypothesis (1)

Let Ω be a C^2 domain, of boundary $\partial\Omega$ parametrized by functions σ^i, for $i = 1, \ldots, k$. For any $i \in \{1, \ldots, k\}$ and for any θ such that $(\sigma_1^i)'(\theta) = 0$ or $(\sigma_2^i)'(\theta) = 0$, we have $\kappa^i(\theta) \neq 0$.

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Lemma

Let Ω be a C^2 domain. Then, there is an orthogonal $\mathbb{R}^2$-endomorphism U such that the domain $\widetilde{\Omega} := U(\Omega)$ satisfies Hypothesis 1. In fact, if we denote U_ψ the endomorphism characterized by $e_1 := (1,0) \mapsto (\cos(\psi), \sin(\psi))$ and $e_2 := (0,1) \mapsto (-\sin(\psi), \cos(\psi))$, then, for almost every ψ in $[-\pi, \pi]$, $U_\psi(\Omega)$ satisfies Hypothesis 1.

Since our system is invariant with respect to rotations, the previous lemma will imply that for a given domain Ω the penalized Stokes system is null-controllable for almost every direction e.

An elliptic estimate

Let us consider the operator:

$$L_a u = -a\partial_{xx} u - \partial_{yy} u.$$

Theorem

Let Ω be a C^4 domain that satisfies Hypothesis 1. Then, for $a_0 > 0$ small enough, there is $C > 0$ such that for any function $u \in H^4(\Omega) \cap H_0^1(\Omega)$ and for any $a \in (0, a_0]$ we have that:

$$\|\partial_x u\|_{C^0(\overline{\Omega})} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right).$$

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We first prove it for Ω strictly convex, and then we explain how to generalize the proof to any domain that satisfies Hypothesis 1.

The main result

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a regular domain satisfyig Hypothesis 1, $\omega \subset \Omega$ a subdomain and $T > 0$. Then, there is $C > 0$ and $\varepsilon_0 > 0$ such that for any $y^0 \in L^2(\Omega)$ and any $\varepsilon \in (0, \varepsilon_0)$ there is a scalar function $f^\varepsilon \in L^2((0,T)\times\omega)$ such that the regular solution of:

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = (f^\varepsilon 1_\omega, 0) & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(0,\cdot) = y^0 & \text{in } \Omega. \end{cases}$$

satisfies $y^\varepsilon(T,0) = 0$ and such that:

$$\|f^\varepsilon\|_{L^2((0,T)\times\omega)} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)}.$$

Getting an equation on the boundary.

First of all, we consider that, using the definition of L_a and Dirichlet boundary conditions:

$$-\partial_x u + A\partial_{xx} u = -\frac{2\sigma_1'(\sigma_2')^2}{\kappa}\partial_{xy} u + \frac{(\sigma_2')^3}{\kappa} L_a u \qquad \forall \theta \in [0, |\partial\Omega|].$$

for

$$A(\theta) := \frac{\sigma_2'(\theta)}{\kappa(\theta)}\left((\sigma_1'(\theta))^2 - a(\sigma_2'(\theta))^2\right) = \frac{\sigma_2'(\theta)}{\kappa(\theta)}\left(1 - (a+1)(\sigma_2'(\theta))^2\right).$$

We remark that $A = 0$ if $\sigma_2' = 0$ or if $\sigma_2' = (a+1)^{-1/2}$.

Defining an auxiliary function: the source of an ODE

We define,

$$g(x,y) := -\partial_x u(x,y) + A(\Theta_h(x))\partial_{xx} u(x,y),$$

for $\Theta_h(x)$ the value such that $\sigma_1(\Theta_h(x)) = x$ and such that $\sigma_2'(\Theta_h(x)) \geq 0$.

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for $\Theta_h(x)$ the value such that $\sigma_1(\Theta_h(x)) = x$ and such that $\sigma_2'(\Theta_h(x)) \geq 0$. Using the equation on the boundary, we get for any horizontal segment $I \subset \overline{\Omega}$:

$$\|g\|_{C^0(I)} + \|\partial_x g\|_{L^1(I,dx)} \leq C\left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}\right).$$

Estimation of $\partial_x u$ in the segments of Ω

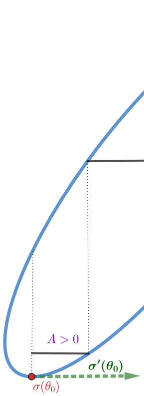

Figure: Convex case: estimation in the right of $\sigma(\theta_0)$

*Immediate consequences of Hypothesis 1 (1)

Let Ω be a domain that satisfies Hypothesis 1. We have:

- If $\sigma_1^i(\theta) = 0$ or if $\sigma_2^i(\theta) = 0$, then, for some $\delta(\theta) > 0$, κ^i does not change of sign in $(\theta - \delta(\theta), \theta + \delta(\theta))$.

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- Given any $c \in \mathbb{R}$, the number of points in $\partial\Omega \cap \{x = c\}$ or in $\partial\Omega \cap \{y = c\}$ is finite.

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- Given any $c \in \mathbb{R}$, the number of points in $\partial\Omega \cap \{x = c\}$ or in $\partial\Omega \cap \{y = c\}$ is finite.
- Given any $c \in \mathbb{R}$, there is $\delta(c) > 0$ such that:
  - We have
  $$([c - \delta(c), c + \delta(c)] \times \mathbb{R}) \cap \partial\Omega = \bigcup_{p = \sigma^{i_p}(\theta_p) \in \partial\Omega \cap \{x = c\}} \sigma^{i_p}(I_p),$$
  for $I_p = (\theta_p^1, \theta_p^2)$, for some $\theta_p^1 < \theta_p < \theta_p^2$.
  - In the set
  $$\left(\left([c - \delta(c), c + \delta(c)] \setminus \{c\}\right) \times \mathbb{R}\right) \cap \partial\Omega,$$
  we do not have $p = \sigma^i(\theta)$ with $(\sigma^i)'(\theta) = \pm e_2$.

*Immediate consequences of Hypothesis 1 (2)

- There is some $\eta > 0$ such that for all points $p = \sigma^i(\theta_p) \in \partial\Omega$ with $(\sigma^i)'(\theta_p) = \pm e_1$, there exists a neighbourhood $V_p = \sigma^i(I_p) \subset \partial\Omega$ ($I_p = (\theta_p^1, \theta_p^2)$, for some $\theta_p^1 < \theta_p < \theta_p^2$) such that $\sigma_2^i(\theta_p^1) = \sigma_2^i(\theta_p^2)$ and such that $|\kappa^i| > \eta$.
- There exists $a_0 > 0$ small enough such that, for all $a \in (0, a_0)$, for each point $p = \sigma^i(\theta) \in \partial\Omega$ with $(\sigma^i(\theta))' = \pm e_2$ there is a neighbourhood $U_p \subset \partial\Omega$ which has exactly a point of tangent vector $\pm\left(\sqrt{\frac{a}{1+a}}, \sqrt{\frac{1}{1+a}}\right)$ and exactly another one of tangent vector $\pm\left(\sqrt{\frac{a}{1+a}}, -\sqrt{\frac{1}{1+a}}\right)$.
  Reciprocally, if $p_a = \sigma^i(\theta^a) \in \partial\Omega$ satisfies $(\sigma^i)'(\theta^a) = \left(\pm\sqrt{\frac{a}{1+a}}, \pm\sqrt{\frac{1}{1+a}}\right)$, then $p_a \in U_p$, for U_p one of the above defined neighbourhoods. Finally, we can suppose that for some $\eta > 0$, $|\kappa^i| > \eta$ on those neighbourhoods.

*Decomposing in segments

We define Γ as the subset of $\partial\Omega$ such that $p = \sigma^i(\theta) \in \Gamma$ if and only if at least one of the following properties is satisfied:

- $\exists \delta_0(p) > 0 : \forall \delta \in (0, \delta_0(p)),\ p + \delta e_2 \in \overline{\Omega}$,
- $(\sigma^i)'(\theta) = \pm e_2$.

Moreover, let $(x, y) \in \overline{\Omega}$. We define:

$$\mathbb{P}_h(x, y) := (x, y) - \lambda e_2 \text{ such that } \lambda := \min\{\lambda \in \mathbb{R}^+ : (x, y) - \lambda e_2 \in \Gamma\}.$$

Lemma

Let Ω be a domain that satisfies Hypothesis 1. Then, there is a subset $S \subset \overline{\Omega}$ such that:

- *S is a finite union of horizontal segments $l_i := [x_l^i, x_r^i] \times \{y^i\}$.*
- *$\mathbb{P}_h(S) = \Gamma$.*
- *$\mathbb{P}_h$ is continuous in the relative interior of each segment l_i.*

*An example

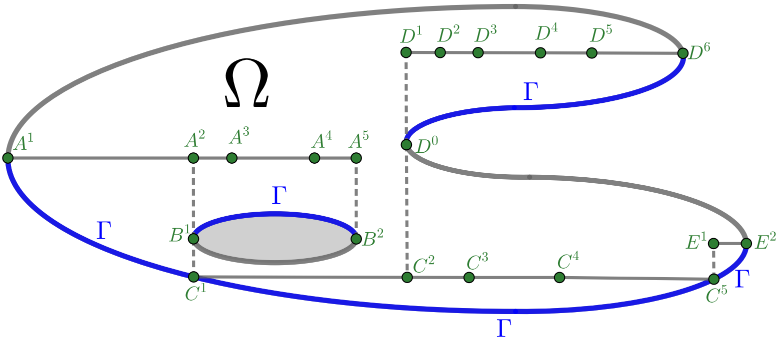

Figure: A typical example on how to construct S.

*Estimate at the left endpoint of each segment

It is just to consider that the left endpoint of each segment I_i is either a point $p = \sigma^i(\theta) \in \Gamma$ with $(\sigma^i)'(\theta) = \pm e_2$ (the case of A^1 in the previous figure) or it can be joined by a vertical segment (including degenerated segments) inside Ω with some other segment I_j such that $x_l^j < x_l^i \leq x_r^j$.

*Four different type of segments depending on $\mathbb{P}_h(I)$

1. $P_h(I_i)$ is the intersection of Γ with one of the neighbourhoods $\overline{U_p}$.
2. $P_h(I_i)$ has null intersection with all the neighbourhoods U_p and V_p.
3. $P_h(I_i)$ is one of the neighbourhoods $\overline{V_p}$ which has a positive curvature.
4. $P_h(I_i)$ is one of the neighbourhoods $\overline{V_p}$ which has a negative curvature.

*Situation 1

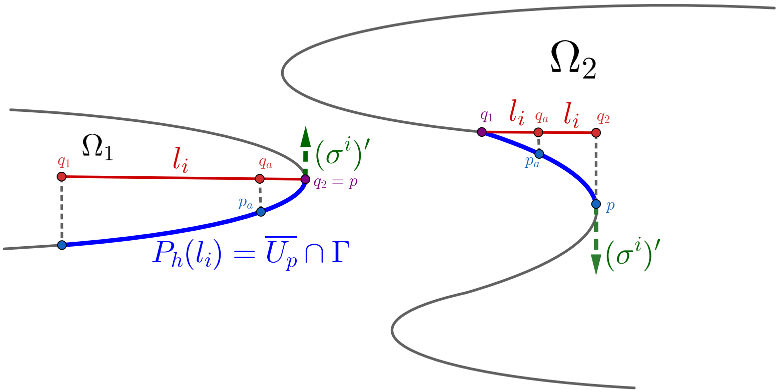

Figure: Situation 1

*Situation 2

We define in the segment a function g^l as before. Of course, we have a function A^l as before. Due to our hypothesis in $\mathbb{P}_h(l)$, there is $\delta > 0$ such that $|A^l(l)| > \delta$. Consequently, we just get the estimate by calculating explicitly the solution as a linear ODE and then applying usual estimates.

*Situation 3

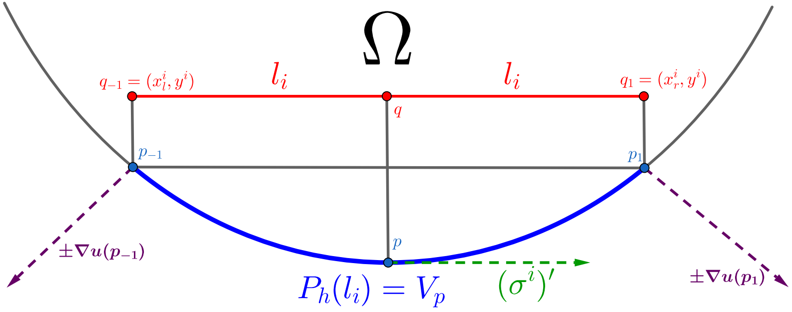

Figure: Situation 3

*Situation 4

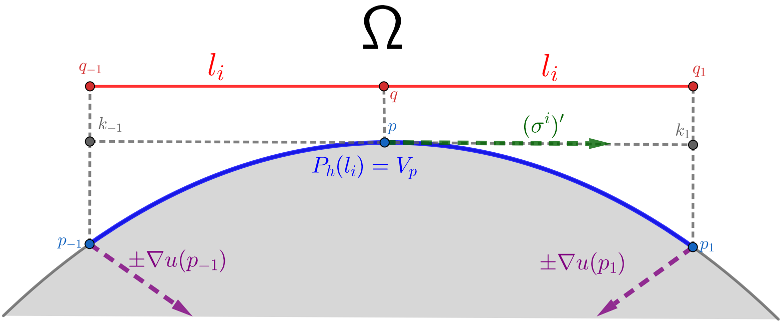

Figure: Situation 4

The coupling estimate

We remark that, on $\partial\Omega$, for all $t \in [0, T)$:

$$\begin{cases} -\partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon. \end{cases}$$

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Thus, there is $C > 0$ and $\varepsilon_0 > 0$ such that for all $t \in [0, T)$ and $\varepsilon \in (0, \varepsilon_0]$:

$$\|\varphi^\varepsilon(t,\cdot)\|_{\mathbf{L}^2(\Omega)} \leq C\|\partial_{xy}\varphi^\varepsilon(t,\cdot)\|_{\mathbf{H}^2(\Omega)}.$$

Weights and a Carleman estimate

We consider the following weights:

$$\alpha(t,x) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0}}{(t(T-t))^m}, \quad \xi(t,x) = \frac{e^{\lambda\eta^0}}{(t(T-t))^m},$$
$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(t,x), \qquad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(t,x),$$

for η^0 an Imanuvilov's function.

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for η^0 an Imanuvilov's function. We have that:

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 \leq Cs^{15}\lambda^{16} \sum_{i=0}^{2} \iint_Q e^{-2s\alpha}\xi^{15}|D^i\partial_{xy}\varphi^\varepsilon|^2.$$

*Estimating with higher derivatives (1)

Lemma (Coron,Guerrero,2009)

Let $\Omega \in C^4$ *and* $r \in \mathbb{R}$. *Then, there is* $C > 0$ *and* $\lambda_0 \geq 1$ *such that if* $T > 0$, $\lambda \geq \lambda_0$, $s \geq CT^{2m}$ *and* $u \in L^2(0, T; H^1(\Omega))$, *we have that:*

$$s^{2+r}\lambda^{3+r}\iint_Q e^{-2s\alpha}\xi^{2+r}|u|^2 \leq C\left(s^r\lambda^{1+r}\iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2 + s^{2+r}\lambda^{3+r}\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2+r}|u|^2\right).$$

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$$s^{2+r}\lambda^{3+r}\iint_Q e^{-2s\alpha}\xi^{2+r}|u|^2 \leq C\left(s^r\lambda^{1+r}\iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2 + s^{2+r}\lambda^{3+r}\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2+r}|u|^2\right).$$

Applying this lemma seven times and using known bounds of the weights we get that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 + \sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$

$$\leq C\left(s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2 + \sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2\right).$$

*Estimating with higher derivatives (2)

We use a technical result proven in the annex of the paper:

Proposition

Let Ω *be a* C^4 *domain, let* $\tilde{\omega}$ *be an open subset* Ω *such that* $\overline{\omega_0} \subset \widetilde{\omega}$ *and let* $m \geq 8$*. Then, there is* $\varepsilon_0 > 0$*,* $C > 0$ *and* $\lambda_0 \geq 1$ *such that if* $T > 0$*,* $\varepsilon \in (0, \varepsilon_0)$*,* $\varphi^T \in \mathbf{L}^2(\Omega)$*,* $h \in \mathbf{H}^{2,5/2}(\Sigma)$*,* $\lambda \geq \lambda_0$ *and* $s \geq e^{C\lambda}(T^m + T^{2m})$*, we have:*

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2$$

$$\leq C\left(s^4\lambda^5 \iint_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|\varphi^\varepsilon|^2 + (1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)} + \|\tilde{\eta} h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\right),$$

for $\eta(t) := (s\xi^*(t))^{1/4+1/m}e^{-s\alpha^*(t)}$, $\tilde{\eta}(t) := (s\xi^*(t))^{-3/4}e^{-s\alpha^*(t)}$ *and* φ^ε *the solution of:*

$$\begin{cases} -\varphi^\varepsilon_t - \Delta\varphi^\varepsilon + \nabla\pi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon\pi^\varepsilon + \nabla\cdot\varphi^\varepsilon = 0 & \text{in } Q, \\ \partial_n\varphi^\varepsilon - \pi^\varepsilon n = h & \text{on } \Sigma, \\ \varphi^\varepsilon(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases}$$

*Estimating with higher derivatives (3)

Thus, applying the previous proposition with each term of $D^8\partial_{xy}\varphi^\varepsilon$:

$$s\lambda^2 \iint_Q e^{-2s\alpha}\xi|D^9\partial_{xy}\varphi^\varepsilon|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2$$

$$\leq C\left(s^4\lambda^5 \iint\limits_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|D^8\partial_{xy}\varphi^\varepsilon|^2 + (1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)} + \|\tilde{\eta} h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\right),$$

for $h := \partial_n D^8\varphi^\varepsilon + \varepsilon^{-1}\nabla\cdot D^8\varphi^\varepsilon$.

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$$s\lambda^2\iint_Q e^{-2s\alpha}\xi|D^9\partial_{xy}\varphi^\varepsilon|^2+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2$$

$$\leq C\left(s^4\lambda^5\iint_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|D^8\partial_{xy}\varphi^\varepsilon|^2+(1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)}+\|\tilde{\eta}h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\right),$$

for $h:=\partial_n D^8\varphi^\varepsilon+\varepsilon^{-1}\nabla\cdot D^8\varphi^\varepsilon$. Using interpolation estimates we recall that:

$$\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}\leq C\left(\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{6,12}(Q)}+\varepsilon^{-1}\|\nabla\cdot(\eta\varphi^\varepsilon)\|_{\mathbf{H}^{5,11}(Q)}\right),$$

and

$$\|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)}\leq C\left(\|\tilde{\eta}\varphi^\varepsilon\|_{\mathbf{H}^{7,14}(Q)}+\varepsilon^{-1}\|\nabla\cdot(\tilde{\eta}\varphi^\varepsilon)\|_{\mathbf{H}^{6,13}(Q)}\right).$$

*Using estimates on the Cauchy problem

Lemma

Let $i \in \mathbb{N}$, $\Omega \in C^{2i}$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$ and $f \in \mathbf{H}^{i-1,2i-2}(Q)$ satisfying $\partial_{t^m} f(t,\cdot) = 0$ for all $m \in \mathbb{N} \cap [0, i-2]$, we have that the solution v^ε of the Penalized Stokes problem with Dirichlet boundary conditions satisfies $v^\varepsilon \in \mathbf{H}^{i,2i}(Q)$ with the estimate:

$$\|v^\varepsilon\|_{\mathbf{H}^{i,2i}(Q)} + \varepsilon^{-1}\|\nabla \cdot v^\varepsilon\|_{H^{i-1,2i-1}(Q)} \leq C\|f\|_{\mathbf{H}^{i-1,2i-2}(Q)}.$$

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Lemma

Let $i \in \mathbb{N}$, $\Omega \in C^{2i}$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$ and $f \in \mathbf{H}^{i-1,2i-2}(Q)$ satisfying $\partial_{t^m} f(t, \cdot) = 0$ for all $m \in \mathbb{N} \cap [0, i-2]$, we have that the solution v^ε of the Penalized Stokes problem with Dirichlet boundary conditions satisfies $v^\varepsilon \in \mathbf{H}^{i,2i}(Q)$ with the estimate:

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In particular, for any real-valued function $g(t)$ that decays exponentially in T, $g\varphi^\varepsilon$ is the solution of the backwards penalized Stokes system of force $g'(t)\varphi^\varepsilon$. Consequently, by induction, we have that:

$$\|g\varphi^\varepsilon\|_{\mathbf{H}^{i,2i}(Q)} + \varepsilon^{-1} \|\nabla \cdot (g\varphi^\varepsilon)\|_{H^{i-1,2i-1}(Q)} \leq C \|g^{i)} \varphi^\varepsilon\|_{\mathbf{L}^2(Q)}.$$

*Summing up

So, after absorbing the trace term, we have that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2+\sum_{i=0}^{9}s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$

$$\leq C\left(\sum_{i=0}^{7}s^{19-2i}\lambda^{20-2i}\iint\limits_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2+s^4\lambda^5\iint\limits_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|D^8\partial_{xy}\varphi^\varepsilon|^2\right)$$

Leaving just $\partial_{xy}\varphi^\varepsilon$ as a local term

We consider a cut-off function $\chi \geq 0$ satisfying $\operatorname{supp}(\chi) \subset \omega$ and $\chi = 1$ in $\widetilde{\omega}$. We have that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 + \sum_{i=0}^{9} s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$+\sum_{i=1}^{8} s^{28-3i}\lambda^{29-3i}\iint_{(0,T)\times\omega}\chi^{4+2i}e^{-2s\alpha}\xi^{28-3i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$\leq Cs^{28}\lambda^{29}\iint_{(0,T)\times\omega}\chi^4 e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi^\varepsilon|^2.$$

Indeed, it is just integrations by parts and usual Cauchy-Schwarz.

Dealing with the local norm of $\partial_{xy}\varphi^\varepsilon$

We can deal with the local norm of $\partial_{xy}\varphi_1^\varepsilon$ as before.

Dealing with the local norm of $\partial_{xy}\varphi^\varepsilon$

We can deal with the local norm of $\partial_{xy}\varphi_1^\varepsilon$ as before. As for the term $\partial_{xy}\varphi_2^\varepsilon$, we have to consider that:

$$s^{28}\lambda^{29}\iint_{(0,T)\times\omega}\chi^4 e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi_2^\varepsilon|^2$$
$$= s^{28}\lambda^{29}\iint_{(0,T)\times\omega}\chi^4 e^{-2s\alpha}\xi^{28}\partial_{xy}\varphi_2^\varepsilon(-\varepsilon\partial_t\varphi_1^\varepsilon-(1+\varepsilon)\partial_{xx}\varphi_1^\varepsilon-\varepsilon\partial_{yy}\varphi_1^\varepsilon).$$

In order to deal with the term of $\varepsilon\partial_{txy}\varphi_2^\varepsilon$ that appears after the integration by parts, we have to consider that:

$$\varepsilon\partial_{txy}\varphi_2^\varepsilon=-\left(\varepsilon\partial_{xxxy}\varphi_2^\varepsilon+(1+\varepsilon)\partial_{xyyy}\varphi_2^\varepsilon+\partial_{xxyy}\varphi_1^\varepsilon\right).$$

For the other terms we deal as before.

Summing up

Let Ω be a regular domain that satisfies our Hypothesis, let $\omega \subset \Omega$ be an open set, and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 \leq Cs^{34}\lambda^{35} \iint\limits_{(0,T)\times\omega} e^{-2s\alpha}\xi^{34}|\varphi_1^\varepsilon|^2,$$

for φ^ε the solution of the adjoint penalized Stokes problem presented before.

Summing up

Let Ω be a regular domain that satisfies our Hypothesis, let $\omega \subset \Omega$ be an open set, and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 \leq Cs^{34}\lambda^{35} \iint_{(0,T)\times\omega} e^{-2s\alpha}\xi^{34}|\varphi_1^\varepsilon|^2,$$

for φ^ε the solution of the adjoint penalized Stokes problem presented before. From here we can get the observability inequality through parabolic estimates in the Cauchy problem.

Open problems

- The analogue problem for $\Omega \subset \mathbb{R}^3$.
- To remove the Hypothesis, at least for ε small enough.
- To study if the control obtained by the Riesz representation theorem for the penalized Stokes system converges to the control obtained by the Riesz representation theorem for the Stokes system.
- The study of the local null controllability of the penalized Navier-Stokes system.

Thank you for your attention!

Is there any question?