Null-controllability of parabolic-hyperbolic systems

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VIII Partial differential equations, optimal design and numerics
Parabolic-transport systems
Parabolic-transport systems

Equation we are interested in:

\[ \partial_t y(t, x) + A \partial_x y(t, x) - B \partial_{xx} y(t, x) = f(t, x)1_\omega, \quad (t, x) \in [0, +\infty) \times \mathbb{T} \]

\[ B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad D + D^* \text{ definite-positive}; \quad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A' = A'^*. \]
Parabolic-transport systems

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Question
Are these systems null-controllable (equivalently: observable) in \( \omega \subset \mathbb{T} \)?

if \( u = 0 \), \( |y(T, \cdot)|_{L^2(\mathbb{T})} \leq C|y|_{L^2([0,T] \times \omega)} \) ?
Theorem (Beauchard-K-Le Balc’h 2019)

$\omega$ an open interval of $\mathbb{T}$.

$$T^* = \frac{2\pi - \text{length}(\omega)}{\min_{\mu \in \text{Sp}(A')} |\mu|}$$

Then

1. the system is not null-controllable on $\omega$ in time $T < T^*$,
2. the system is null-controllable on $\omega$ in any time $T > T^*$. 
Parabolic frequencies, Hyperbolic frequencies
Fourier components

If \( y(t, x) = \sum y_n(t)e^{inx} \)

\[
\partial_t y_n(t) + n^2 \left( B + \frac{i}{n}A \right) y_n(t) = 0
\]
Fourier components, well-posedness

Fourier components
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\[ \partial_t y_n(t) + n^2 \left( B + \frac{i}{n} A \right) y_n(t) = 0 \]

Perturbation theory
\( \lambda_{nk} \) eigenvalues of \( B + \frac{i}{n} A \). Perturbation of \( B \): \( \lambda_{nk} \rightarrow \lambda_k \in \text{Sp}(B) \)

- If \( \lambda_k \neq 0 \), \( \lambda_{nk} \sim \lambda_k \) as \( n \rightarrow +\infty \): parabolic frequencies
- If \( \lambda_k = 0 \), \( \lambda_{nk} \sim i\mu_k/n \): hyperbolic frequencies
Fourier components, well-posedness

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Perturbation theory
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- If \( \lambda_k \neq 0 \), \( \lambda_{nk} \xrightarrow{n \to +\infty} \lambda_k \): parabolic frequencies
- If \( \lambda_k = 0 \), \( \lambda_{nk} \xrightarrow{n \to +\infty} i\mu_k/n \): hyperbolic frequencies
- Well posed if all \( \Re(\lambda_k) > 0 \) and \( \mu_k \in \mathbb{R} \)

\[
y(t, x) = \sum_{n, k} a_{nk}e^{inx-n^2\lambda_{nk}t}y_{nk}
\]

\[
\approx \sum_{n, k} a_{nk}e^{inx-n^2\lambda_k t}y_k + \sum a_{nk}e^{i(x-\mu_k t)}y_k
\]

parabolic frequencies  hyperbolic frequencies
Lack of null-controllability in small time
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Transport-like solutions
If \( \lambda_{nk} \sim i\mu_k/n \), and \( y_{nk} \) is an associated eigenvector

\[
y(t, x) = \sum_n a_n e^{in(x-n^2\lambda_{nk}t)} y_{nk} \approx \sum_n a_n e^{in(x-\mu_k t)} y_k
\]

Not observable in time \( T < \frac{2\pi - \text{length}(\omega)}{|\mu_k|} \).

Minimal time = minimal time for transport equation
In the case

\[
\partial_t y_h + A' \partial_x y_h = f_h 1_\omega
\]

Solutions = sum of solutions moving at speed \( \mu_k \in \text{Sp}(A') \).
Null-controllability in large time
The control strategy

Decoupling the system and controlling

Given $f_h$, find $f_p$ that steers parabolic frequencies to $0$ in time $T$

Given $f_p$, find $f_h$ that steers hyperbolic frequencies to $0$ in time $T$

If both steps agree, OK (except a finite number of frequencies)

Make steps agree by choosing $f_p$ smooth and using Fredholm’s alternative

First step: parabolic null-controllability problem in time $T - T' > 0$

Second step: hyperbolic exact controllability problem in time $T'$. Ok if $T' > T^*$
The control strategy

Decoupling the system and controlling

• Given $f_h$, find $f_p$ that steers parabolic frequencies to 0 in time $T$

0  $T' < T$
The control strategy

Decoupling the system and controlling

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What is new in our work

Dealing with a system of arbitrary size

- Previous strategy: Lebeau-Zuazua (1998) for linear thermolasticity systems (coupled wave-heat system)
- What we did: generalize to systems of arbitrary size

\[ \frac{1}{2} i \pi \oint_{\Gamma} (M - z)^{-1} dz = \text{Projection on eigenspaces associated with eigenvalues of } M \text{ inside } \Gamma \]
What is new in our work

Dealing with a system of arbitrary size

- Previous strategy: Lebeau-Zuazua (1998) for linear thermolasticity systems (coupled wave-heat system)
- What we did: generalize to systems of arbitrary size
- Issue: eigenvalues and eigenvectors not nice as $n \to +\infty$
- Solution: we don’t need either of these
- We only need total eigenprojections: sums of eigenprojections on eigenvalues close to each other (Kato’s perturbation theory...)

\[-\frac{1}{2i\pi} \oint_{\Gamma} (M - z)^{-1} \, dz = \text{Projection on eigenspaces associated with eigenvalues of } M \text{ inside } \Gamma\]
What we (don’t) know
Open problems

We don’t know

• Unique continuation in small time?
• Less controls than equations? (partial results)
• Higher dimensions?
• Non-constant coefficients?
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- Unique continuation in small time?
- Less controls than equations? (partial results)
- Higher dimensions?
- Non-constant coefficients?

That’s all folks!