Carleman estimate and null controllability for a fourth order parabolic equation

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Overview

Assumptions

In this presentation, we consider $\Omega \subset \mathbb{R}^N$ with $(N \geq 2)$ a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$.

Section 1: Introduction

Control problem

Let us introduce the following control system :

$$\begin{cases} \partial_t y + \Delta^2 y = \chi_\omega v & \text{in } Q \ , \\ y = \Delta y = 0 & \text{on } \Sigma \ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{1}$$

where $y_0 \in L^2(\Omega)$ is the initial condition and $v \in L^2((0,T) \times \omega)$ is the control function. Let us notice that y may represent a scaled film height and the term $\Delta^2\, y$ represents the capillarity-driven surface diffusion. Let us start by giving the definition of null controllability for (1) :

Definition 1.1

It is said that (1) is null controllable at time $T > 0$ if for each $y_0 \in L^2(\Omega)$, there exists $v \in L^2((0,T) \times \omega)$ such that the corresponding initial problem (1) admits a solution $y \in C^0([0,T]; L^2(\Omega))$ satisfying

$$y(T,\cdot) = 0 \ \text{ in } \Omega.$$

Control problem

Let us introduce the non-homogeneous adjoint system associated to (1) :

$$\begin{cases} -\partial_t \varphi + \Delta^2 \varphi = f & \text{in } Q \ , \\ \varphi = \Delta \varphi = 0 & \text{on } \Sigma \ , \\ \varphi(T, \cdot) = \varphi_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{2}$$

where $\varphi_0 \in L^2(\Omega)$ and $f \in L^2(Q)$. It is very well-known by now that the null controllability (and continuous dependence of $\|v\|_{L^2((0,T)\times\omega)}$ with respect to $\|y_0\|_{L^2(\Omega)}$) is equivalent to the observability inequality :

$$\exists C > 0 : \int_\Omega |\varphi(0,x)|^2 dx \leq C \iint_{(0,T)\times\omega} |\varphi|^2 dx\, dt, \ \forall \varphi_0 \in L^2(\Omega),$$

where φ is the solution of (2) with $f \equiv 0$.

State of art

In higher dimensions, there has been limited publications on the controllability of fourth order parabolic equations. By using the idea introduced by Lebeau, G. and Robbiano, a null controllability result was proved for the following system :

$$\begin{cases} \partial_t y + \Delta^2 y = \chi_\omega v & \text{in } Q \ , \\ y = \Delta y = 0 & \text{on } \Sigma \ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{3}$$

where $y_0 \in L^2(\Omega)$.

State of art

There is an extended literature in dimension 1. A null controllability result was proved for the following semi-linear system

$$\begin{cases} \partial_t u + \partial_{xxxx} u + f(u) = \chi_\omega h & \text{in } (0,T) \times (0,1) \ , \\ u(\cdot,0) = u(\cdot,1) = 0 & \text{in } (0,T) \ , \\ \partial_x u(\cdot,0) = \partial_x u(\cdot,1) = 0 & \text{in } (0,T) \ , \\ u(0,\cdot) = u_0(\cdot) & \text{in } (0,1) \ , \end{cases}$$

where $u_0 \in L^2(0,1)$ and f is a globally Lipschitz continuous function. Many works were done related to the previous system by using a Carleman inequality that will be extended in the next section. As we said, in higher dimensions, there has been limited publications on the controllability of fourth order parabolic equations. As far as we know, a Carleman inequality for a fourth order parabolic equation was an open problem whenever $N \geq 2$.

Section 2: Carleman inequality for a fourth order parabolic equation

In order to state our Carleman inequality, we will need some weight functions :

$$\alpha(x,t) = \frac{e^{4\lambda||\eta||_\infty} - e^{\lambda(2||\eta||_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}}, \; \xi(x,t) = \frac{e^{\lambda(2||\eta||_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}},$$

where $\eta \in C^4(\bar{\Omega})$ satisfies:

$$\eta > 0 \text{ in } \Omega, \quad \eta_{|\partial\Omega} = 0, \quad |\nabla\eta| \geq C_0 > 0 \text{ in } \Omega \setminus \overline{\omega'},$$

with $\omega' \subset \omega$ an open set.

Carleman inequality

The main objective is to prove the following theorem :

Theorem 2.1

There exists a positive constant $\tilde{C}_0 = \tilde{C}_0(\Omega, \omega)$ *such that*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\Big(s^6\lambda^8\xi^6|\varphi|^2 + s^4\lambda^6\xi^4|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\Delta\varphi|^2 \\ &+ s^2\lambda^4\xi^2|\nabla^2\varphi|^2 + s\lambda^2\xi|\nabla\Delta\varphi|^2 + s^{-1}\xi^{-1}(|\partial_t\varphi|^2 + |\Delta^2\varphi|^2)\Big)dxdt \\ &\leq \tilde{C}_0\Big(s^7\lambda^8 \iint_{(0,T)\times\omega} e^{-2s\alpha}\xi^7|\varphi|^2 dxdt + \iint_Q e^{-2s\alpha}|f|^2 dxdt \Big) \end{aligned} \tag{4}$$

for any $\lambda \geq \tilde{C}_0$, *any* $s \geq \tilde{C}_0(T^{1/2} + T)$ *and where* φ *fulfills (2).*

Proof

We present the 5 steps of the proof :

1. Change of variable and first computations.
2. Computation of the product $(P_1\psi, P_2\psi)_{L^2(Q)}$.
3. Simplifications in the computation of $(P_1\psi, P_2\psi)_{L^2(Q)}$ and first main estimate.
4. Estimate of the boundary terms.
5. Last arrangements and conclusion.

Proof (Change of variable and first computations)

In order to prove this, let us set

$$\psi(t,x) = e^{-s\alpha(t,x)}\varphi(t,x), \ \forall (t,x) \in Q. \tag{5}$$

By replacing φ in the equation $-\partial_t \varphi + \Delta^2 \varphi = f$ by $e^{s\alpha}\psi$, we have

$$P(\psi) \triangleq -s\alpha_t \psi - \partial_t \psi + e^{-s\alpha}(\Delta^2 e^{s\alpha}\psi + 4\nabla\Delta e^{s\alpha}\cdot\nabla\psi + 2\Delta e^{s\alpha}\Delta\psi$$
$$+4\nabla^2 e^{s\alpha} : \nabla^2\psi + 4\nabla e^{s\alpha}\cdot\nabla\Delta\psi + e^{s\alpha}\Delta^2\psi) = e^{-s\alpha} f \ \text{ in } Q,$$

Proof (Change of variable and first computations)

Let us consider the following functionals :

$$P_1(\psi) \triangleq -4s^3\lambda^3\xi^3|\nabla\eta|^2\nabla\eta\cdot\nabla\psi - \partial_t\psi - 4s\lambda\xi(\nabla\eta\cdot\nabla\Delta\psi)$$
$$-2s\lambda^2\xi|\nabla\eta|^2\Delta\psi - 12s^3\lambda^3\xi^3(\nabla^2\eta\nabla\eta\nabla\eta)\psi - 2s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta\psi$$
$$-2s\lambda\xi\Delta\eta\Delta\psi + 4s\lambda\xi\nabla^2\psi : \nabla^2\eta - 4s\lambda^2\xi(\nabla^2\psi\nabla\eta\nabla\eta)$$
$$-6s^3\lambda^4\xi^3|\nabla\eta|^4\psi \text{ in } Q,$$

$$P_2(\psi) \triangleq s^4\lambda^4\xi^4|\nabla\eta|^4\psi + \Delta^2\psi + 4s^2\lambda^2\xi^2(\nabla^2\psi\nabla\eta\nabla\eta))$$
$$+8s^2\lambda^2\xi^2(\nabla^2\eta\nabla\eta\nabla\psi) + 4s^2\lambda^2\xi^2\Delta\eta(\nabla\psi\cdot\nabla\eta)$$
$$+2s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi + 12s^2\lambda^3\xi^2|\nabla\eta|^2(\nabla\eta\cdot\nabla\psi \text{ in } Q$$

Proof (Change of variable and first computations)

and

$$R(\psi) \triangleq (P - P_1 - P_2)(\psi) = e^{-s\alpha} f - (P_1 + P_2)(\psi) \ \text{ in } Q. \tag{6}$$

Then, we deduce

$$\|P_1\psi\|^2_{L^2(Q)} + \|P_2\psi\|^2_{L^2(Q)} + 2(P_1\psi, P_2\psi)_{L^2(Q)} = \|R\psi - e^{-s\alpha} f\|^2_{L^2(Q)}.$$

Proof (Computation of the product $(P_1\psi, P_2\psi)_{L^2(Q)}$)

The idea is to prove that

$$(P_1\psi, P_2\psi)_{L^2(Q)} \geq X(\psi) \backsimeq \iint_Q ...|D^\tau\psi|^2 dxdt > 0,$$

where $\tau \in \{0,1,2,3,4\}$. Let us start by the computations. For example, we consider

$$\begin{aligned} ((P_1\psi)_1, (P_2\psi)_1)_{L^2(Q)} &= -4s^7\lambda^7 \iint_Q \xi^7 |\nabla\eta|^6 \nabla\eta \cdot \nabla\psi\psi dxdt \\ &= 14s^7\lambda^8 \iint_Q |\nabla\eta|^8 \xi^7 |\psi|^2 dxdt \\ &\quad +2s^7\lambda^7 \iint_Q \xi^7 |\nabla\eta|^6 \Delta\eta |\psi|^2 dxdt \\ &\quad +12s^7\lambda^7 \iint_Q \xi^7 |\nabla\eta|^4 (\nabla^2\eta\nabla\eta\nabla\eta) |\psi|^2 dxdt. \end{aligned}$$

Proof (First main estimate)

At the end, we deduce

$$\begin{aligned}((P_1\psi)_2,(P_2\psi)_1)_{L^2(Q)} &= -s^4\lambda^4\iint_Q \xi^4|\nabla\eta|^4\psi\partial_t\psi\,dxdt\\ &= -\frac{1}{2}s^4\lambda^4\iint_Q \xi^4|\nabla\eta|^4\partial_t|\psi|^2\,dxdt\\ &= \frac{1}{2}s^4\lambda^4\iint_Q \partial_t(\xi^4)|\nabla\eta|^4|\psi|^2\,dxdt.\end{aligned}$$

By using the fact that $|\partial_t\,\xi| \leq \dfrac{T}{2}\xi^3$ and $|\nabla\,\eta|^4 \leq C_0$, we deduce

$$\left|((P_1\psi)_2,(P_2\psi)_1)_{L^2(Q)}\right| \leq 4\,TC_0\,s^4\lambda^4\iint_Q \xi^6|\psi|^2\,dxdt.$$

We can deduce that for $s \geq C_0\,T^{1/2}$ this term can be easily absorbed by

$$s^6\lambda^8\iint_Q \xi^6|\psi|^2\,dxdt.$$

Proof (First main estimate)

$$
\begin{aligned}
&(P_1(\psi), P_2(\psi))_{L^2(Q)} \geq 8s^7\lambda^8 \iint_Q \xi^6 |\nabla\eta|^8 |\psi|^2 dxdt \\
&-16s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^6 |\nabla\psi|^2 dxdt + 8s^3\lambda^4 \iint_Q \xi^3 |\nabla\eta|^4 |\Delta\psi|^2 dxdt \\
&+22s^5\lambda^6 \iint_Q \xi^5 |\nabla\eta|^4 |\nabla\eta\cdot\nabla\psi|^2 dxdt + 32s^3\lambda^4 \iint_Q \xi^3 |\nabla^2\psi\nabla\eta\nabla\eta|^2 dxdt \\
&+3s\lambda^2 \iint_Q \xi |\nabla\Delta\psi\cdot\nabla\eta|^2 dxdt - C_0 s^5\lambda^5 \iint_{\omega'\times(0,T)} \xi^5 \left|\nabla\psi\right|^2 dxdt,
\end{aligned}
\tag{7}
$$

for $s \geq C_0(T^{1/2} + T)$ and $\lambda \geq C_0$.

Proof (Last arrangement)

Notice that,

$$
\begin{aligned}
&8s^7\lambda^8 \iint_Q \xi^6|\nabla\eta|^8|\psi|^2 dxdt - 16s^5\lambda^6 \iint_Q \xi^5|\nabla\eta|^6|\nabla\psi|^2 dxdt \\
&+8s^3\lambda^4 \iint_Q \xi^3|\nabla\eta|^4|\Delta\psi|^2 dxdt \\
&\geq 8s^{-1} \iint_Q \xi^{-1}\left(s^4\lambda^4\xi^4|\nabla\eta|^4\psi + s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi\right)^2 \\
&-C_0 s^5\lambda^8 \iint_Q \xi^5|\psi|^2 dxdt,
\end{aligned}
\tag{8}
$$

for $\lambda \geq C_0$.

Proof (Last arrangement)

On the other hand, we have

$$\varphi = e^{s\alpha}\psi \Rightarrow \Delta\varphi = e^{s\alpha}\Big(\Delta\psi + s^2\lambda^2\xi^2|\nabla\eta|^2\psi - 2s\lambda\xi\nabla\eta\cdot\nabla\psi$$
$$-s\lambda^2\xi|\nabla\eta|^2\psi - s\lambda\xi\Delta\eta\psi\Big).$$

Proof (Last arrangement)

we deduce that

$$
\begin{aligned}
&s^3\lambda^4 \iint_Q \xi^3 e^{-2s\alpha}|\nabla\eta|^4|\Delta\varphi|^2 dxdt = s^{-1}\iint_Q \xi^{-1}\left(e^{-s\alpha}s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\varphi\right)^2 d \\
&= s^{-1}\iint_Q \xi^{-1}\Big(s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi + s^4\lambda^4\xi^4|\nabla\eta|^4\psi \\
&-2s^3\lambda^3\xi^3|\nabla\eta|^2\nabla\eta\cdot\nabla\psi - s^3\lambda^4\xi^3|\nabla\eta|^4\psi - s^3\lambda^3\xi^3|\nabla\eta|^2\Delta\eta\psi\Big)^2 dxdt \\
&\leq 8s^{-1}\iint_Q \xi^{-1}\left(s^2\lambda^2\xi^2|\nabla\eta|^2\Delta\psi + s^4\lambda^4\xi^4|\nabla\eta|^4\psi\right)^2 dxdt \\
&+8s^5\lambda^6\iint_Q \xi^5|\nabla\eta|^4|\nabla\eta\cdot\nabla\psi|^2 dxdt + C_0 s^5\lambda^8\iint_Q \xi^5|\psi|^2 dxdt.
\end{aligned}
$$

Proof (Last arrangement)

At the end, we obtain

$$(P_1(\psi), P_2(\psi))_{L^2(Q)} \geq s^3\lambda^4 \iint_Q \xi^3 e^{-2s\alpha}|\nabla\eta|^4|\Delta\varphi|^2 dxdt$$

$$+14s^5\lambda^6 \iint_Q \xi^5|\nabla\eta|^4|\nabla\eta\cdot\nabla\psi|^2 dxdt$$

$$+3s\lambda^2 \iint_Q \xi|\nabla\eta\cdot\nabla\Delta\psi|^2 dxdt - s^5\lambda^5 \iint_\Sigma \xi^5 \left(\frac{\partial\eta}{\partial\vec{n}}\right)^5 \left|\frac{\partial\psi}{\partial\vec{n}}\right|^2 d\sigma dt - A,$$

for $\lambda \geq C_0$ and $s \geq C_0(T + T^{1/2})$.

Remarks

At the end, we can deduce the following remarks :

Remark 2.2

System (1) is also exact controllable, i.e, for any $\bar{y}_0 \in L^2(\Omega)$ *and any* $\bar{y} \in C^0([0,T]; L^2(\Omega))$ *solution of*

$$\begin{cases} \partial_t \bar{y} + \Delta^2 \bar{y} = 0 & \text{in } Q \ , \\ \bar{y} = \Delta \bar{y} = 0 & \text{on } \Sigma \ , \\ \bar{y}(0,\cdot) = \bar{y}_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{9}$$

there exists $v \in L^2(Q)$ *such that*

$$y(T,\cdot) = \bar{y}(T,\cdot) \ \text{ in } \Omega.$$

Remarks

Remark 2.3

The following system :

$$\begin{cases} \partial_t y + \Delta^2 y = 0 & \text{in } Q \ , \\ y = \chi_{\gamma_1} v_1 & \text{on } \Sigma \ , \\ \Delta y = \chi_{\gamma_2} v_2 & \text{on } \Sigma \ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{10}$$

is boundary null controllable if $\gamma_1 \cap \gamma_2 \neq \emptyset$.

Remarks

Remark 2.4

The following system

$$\begin{cases} \partial_t y + \Delta^2 y + a_0 y + \nabla \cdot (B_0\, y) + \sum_{ij=1}^{N} \partial_{ij}(D_{ij}\, y) + \Delta(a_1\, y) = \chi_\omega v & \text{in } Q\ , \\ y = \Delta y = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases}$$

is null and exact controllable, where $a_0,\, a_1 \in L^\infty(Q;\mathbb{R})$, $B_0 \in L^\infty(Q;\mathbb{R}^N)$, $D \in L^\infty(Q;\mathbb{R}^{N^2})$ *and* $y_0 \in L^2(\Omega)$.

Remarks

Remark 2.5

The Carleman inequality (4) also holds when φ satisfies the boundary conditions

$$\varphi = \frac{\partial \varphi}{\partial \vec{n}} = 0 \ \text{ on } \Sigma.$$

Section 3: Null and exact controllability of semi-linear fourth order parabolic equations

Control problem

We consider parabolic systems of the form

$$\begin{cases} \partial_t y + \Delta^2 y + f(y, \nabla y, \nabla^2 y) = \chi_\omega v & \text{in } Q \ , \\ y = \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma \ , \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{11}$$

where

$$f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to \mathbb{R}.$$

In addition we will suppose that f is a locally Lipschitz-continuous function and

$$f(0_{\mathbb{R}}, 0_{\mathbb{R}^N}, 0_{\mathbb{R}^{N^2}}) = 0. \tag{12}$$

Control problem

Observe that, under the hypothesis above, we can write

$$f(s,p,q) = g(s,p,q)s + G(s,p,q)\cdot p + E(s,p,q) : q \ ,$$

for all $(s,p,q) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}$ and where $g \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})$, $G \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^N$ and $E \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^{N^2}$. We will assume also the following conditions on g, G and E :

$$\lim_{|s|,|p|,|q|\to\infty} \frac{|g(s,p,q)|}{\log^3(1+|s|+|p|+|q|)} = 0,$$
$$\lim_{|s|,|p|,|q|\to\infty} \frac{|G(s,p,q)|}{\log^2(1+|s|+|p|+|q|)} = 0,$$
$$\lim_{|s|,|p|,|q|\to\infty} \frac{|E(s,p,q)|}{\log(1+|s|+|p|+|q|)} = 0.$$

Control problem

Let us give the definition of null controllability for (11).

Definition 3.1

It is said that (11) is null controllable at time $T > 0$ if for each $y_0 \in W^{2,\infty}(\Omega) \cap H_0^2(\Omega)$, there exists $v \in L^\infty((0,T) \times \omega)$ such that the corresponding initial problem (11) admits a solution $y \in C^0([0,T]; L^2(\Omega))$ satisfying

$$y(T,\cdot) = 0 \ \text{ in } \Omega.$$

Using a new Carleman inequality and Kakutani's fixed point theorem, we deduce the following theorem :

Theorem 3.2

Assume that f verifies the conditions below. Then (11) is null controllable at any time $T > 0$.

Section 4: Insensitizing controls for a fourth order parabolic equation in dimension $N \geq 2$

Control problem

We consider the following system of the form

$$\begin{cases} \partial_t w + \Delta^2 w + f(w, \nabla w, \nabla^2 w) = \chi_\omega v & \text{in } Q \ , \\ w = \dfrac{\partial w}{\partial \vec{n}} = 0 & \text{on } \Sigma \ , \\ w(0, \cdot) = y_0(\cdot) + \tau \tilde{y}_0 & \text{in } \Omega \ , \end{cases} \tag{13}$$

where $w_0 \in L^2(\Omega)$ is the initial condition, $\tau \in \mathbb{R}$ unknown and small enough, f is a C^1 globally Lipschitz-continuous function defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^2 = N^2$, $\tilde{y}_0 \in L^2(\Omega)$ is unknown and $||\tilde{y}_0||_{L^2(\Omega)} = 1$ and $v \in L^2(Q)$ is the control function.

Control problem

Our objective is to establish the existence of insensitizing controls for this equation. Let us introduce the following functional

$$\phi(w) = \frac{1}{2}\iint_{O\times(0,T)} |w|^2 dxdt, \tag{14}$$

where $O \cap \omega \neq \emptyset$.

Definition 4.1

We say that the control v insensitizes ϕ if

$$\left| \frac{\partial\phi\left(w(x,t,h,\tau)\right)}{\partial\tau}\Big|_{\tau=0} \right| = 0. \tag{15}$$

Control problem

Let us notice that the existence of a control v such that (15) holds is equivalent to the null controllability of a coupled system. This result is given in the following lemma.

Lemma 4.2

The existence of control v such that (15) holds true if the following system :

$$\begin{cases} \partial_t y + \Delta^2 y + f(y, \nabla y, \nabla^2 y) = \chi_\omega v & in\ Q\ , \\ -\partial_t g + \Delta^2 g + \partial_s f(y, \nabla y, \nabla^2 y) g - \nabla \cdot (\partial_p f(y, \nabla y, \nabla^2 y) g) & in\ Q\ , \\ \qquad + \displaystyle\sum_{i,j=1}^{N} \partial_{ij}(\partial_q f(y, \nabla y, \nabla^2 y) g) = \chi_O y & \\ y = \dfrac{\partial y}{\partial \vec{n}} = g = \dfrac{\partial g}{\partial \vec{n}} = 0 & on\ \Sigma\ , \\ w(0, \cdot) = y_0(\cdot) & in\ \Omega\ , \\ g(T, \cdot) = 0 & in\ \Omega\ , \end{cases}$$

verifies

$$g(0, \cdot) = 0 \quad in\ \Omega\ .$$

Using Carleman inequality and Schauder's fixed point theorem we deduce the following theorem :

Theorem 4.3

Assume that $y_0 \equiv 0$. Then, there exist insensitizing controls in $L^2(Q)$ for system (13).

Control problem

Another complicated problem that i am studying now is the proof of the existence of insensitizing controls in $L^2(Q)$ for system (13) with the following functional :

$$\phi(w) = \frac{1}{2} \iint_{O\times(0,T)} |\Delta\, w|^2 dxdt, \tag{16}$$

where $O \cap \omega \neq \emptyset$. Let us notice that this problem, can be easily solved if we find a new Carleman estimate with a local term depending only on $\Delta\phi$.