QCD, Jets and Monte Carlo techniques

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Lecture 1 - Basics of QCD
Lecture 2 - Higher orders and Monte Carlos
Lecture 3 - Jets
Strong interactions are complicated

1. High-$Q^2$ Scattering
2. Parton Shower
3. Hadronization
4. Underlying Event
“We are driven to the conclusion that the Hamiltonian method for strong interactions is dead and must be buried, although of course with deserved honor”

Lev Landau

“The correct theory [of strong interactions] will not be found in the next hundred years”

Freeman Dyson

We have come a long way towards disproving these predictions
A hadronic process

PDFs

Final state

Hard process

Initial state

hadronisation

τ⁺, τ⁻, π, K, p, etc.

hard proc.

proton

underlying event

proton

u

u

τ⁺ → H → τ⁻, u

u

Matteo Cacciari - LPTHE

2018 Taller de Altas Energías - Benasque
Books and “classics”...

- **R.D. Field**, *Applications of perturbative QCD*, Addison Wesley (1989)
  - Great for specific examples of detailed calculations
  - Phenomenology-oriented
  - A QFT book, but applications tilted towards QCD
- **Dokshitzer, Khoze, Muller, Troyan**, *Basics of perturbative QCD*, http://www.lpthe.jussieu.fr/~yuri
  - For the brave ones
  - One of the most recent QCD books
- **S. Catani**, *Introduction to QCD*, CERN Summer School Lectures 1999
Bibliography

...and recent lectures, slides and...videos

- Gavin Salam,
  - [http://gsalam.web.cern.ch/gsalam/teaching/PhD-courses.html](http://gsalam.web.cern.ch/gsalam/teaching/PhD-courses.html)

- Peter Skands

- Fabio Maltoni
  - “QCD and collider physics”, GGI lectures,
    - [https://www.youtube.com/playlist?list=PL1CFLtxeIrQqvt-e8C5pwBKG4PljSyouP](https://www.youtube.com/playlist?list=PL1CFLtxeIrQqvt-e8C5pwBKG4PljSyouP)
Outline of ‘Basics of QCD’

• strong interactions
• QCD lagrangian, colour, ghosts
• running coupling
• radiation
• calculations of observables
  • theoretical uncertainties estimates
  • power corrections
  • infrared divergencies and IRC safety
• factorisation
QED has a wonderfully simple lagrangian, determined by local gauge invariance

$$\mathcal{L} = \bar{\psi}(i\not\!D - m)\psi - e\bar{\psi}\not\!A\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

In the same spirit, we build **QCD**: a non abelian local gauge theory, based on $SU(3)_{\text{colour}}$, with 3 quarks (for each flavour) in the fundamental representation of the group and 8 gluons in the adjoint

\[ F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \]
I. Colour

quark-gluon interaction

colour matrix (generator of $SU(3)_{\text{colour}}$)

Index of the *adjoint* representation

Indices of the *fundamental* representation

$-ig \left( t^A \right)_{cb} \left( \gamma^\alpha \right)_{ji}$
A fundamental colour relation

\[ ij \, ik = \frac{1}{N} \, jk + 2 \, l \]

\[ \delta_{ij} \delta_{lk} = \frac{1}{N} \, \delta_{ik} \delta_{lj} + 2 \, t^A_{ik} \, t^A_{lj} \]
Take \( i=j \) in

\[
\delta_{ij} \delta_{lk} = \frac{1}{N} \delta_{ik} \delta_{lj} + 2 t^{A}_{ik} t^{A}_{lj}
\]

\[\downarrow\]

\[
N \delta_{lk} = \frac{1}{N} \delta_{lk} + 2 t^{A}_{ik} t^{A}_{li}
\]

\[\downarrow\]

\[
(t^{A} t^{A})_{lk} = \frac{1}{2} \left(N - \frac{1}{N}\right) \delta_{lk} = \frac{N^2 - 1}{2N} \delta_{lk} \equiv C_F \delta_{lk}
\]

This defines \( C_F \).

It is the Casimir of the fundamental representation of SU(N).

What is it, physically?
Gluon emission from a quark

\[ \text{Prob} \sim \left| \sum_{jA} t^A_{ij} t^A_{ji} \right|^2 \sim \sum_{jA} t^A_{ij} t^A_{ji} = \sum_A (t^A t^A)_{ii} = C_F \delta_{ii} \]

\( C_F = (N^2 - 1)/(2N) \) is therefore the ‘colour charge’ of a quark, i.e. its probability of emitting a gluon (except for the strong coupling, of course).
Analogously, one can show that

\[ \text{Prob} \sim \sum_{BC} \left| A \right|_{A}^{C} \right|_{B}^{2} \sim C_A \delta_{AA} \]

\( C_A = N \) is the ‘colour charge’ of a gluon, i.e. its probability of emitting a gluon (except for the strong coupling, of course).

It is also the Casimir of the adjoint representation.
2. Gauge bosons self couplings

In QCD the gluons interact among themselves:

\[ \mathcal{L}_{YM} = -\frac{1}{4} \sum_a F_{\mu\nu}^a F^{a\mu\nu} \]

\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \]

New Feynman diagrams, in addition to the ‘standard’ QED-like ones

Direct consequence of non-abelianity of theory
3. Need for ghosts

Cancel unphysical degrees of freedom that would otherwise propagate in covariant gauges.

Table 1: Feynman rules for QCD in a covariant gauge.
Ghosts: an example

\[ gg \rightarrow qq \]

In QED (i.e. replacing gluons with photons) we’d only have the second and third diagram, and we would sum over the photon polarisations using

\[ \sum_{\text{pol}} \epsilon^\mu_i \epsilon^{*\nu}_i = -g_{\mu\nu} \]

In QCD this would give the wrong result

We must use instead

\[ \sum_{\text{phys pol}} \epsilon^\mu_i \epsilon^{*\nu}_i = -g_{\mu\nu} + \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k \cdot \bar{k}} \]

\( \bar{k} \) is a light-like vector, we can use \((k_0, 0, 0, -k_0)\)
An **alternative** approach is to include the ghosts in the calculation.

Now we can safely use

\[ \sum_{\text{pol}} \epsilon_i^\mu \epsilon_i^{*\nu} = -g_{\mu\nu} \]
Macroscopic differences

1. Confinement (probably -- no proof in QCD)
   We never observe the fundamental degrees of freedom (quarks and gluons). They are always confined into hadrons.

2. Asymptotic Freedom
   The running coupling of the theory, $\alpha_s$, decreases at large energies
Start with $\gamma^* \rightarrow q\bar{q}$:

$$\mathcal{M}_{q\bar{q}} = -\bar{u}(p_1)ie_q\gamma_\mu v(p_2)$$

Emit a gluon:

$$\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1)ig_s\gamma^\mu \frac{i}{p_1 + k}ie_q\gamma_\mu v(p_2)$$

$$-\bar{u}(p_1)ie_q\gamma_\mu \frac{i}{p_2 + k}ig_s\gamma^\mu v(p_2)$$

In the **soft** limit, $k \ll p_{1,2}$

$$\mathcal{M}_{q\bar{q}g} \approx \bar{u}(p_1)ie_q\gamma_\mu t^A v(p_2) g_s \left( \frac{p_1 \cdot \epsilon}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon}{p_2 \cdot k} \right)$$
Squared amplitude, including phase space

\[ d\Phi_{q\bar{q}g} |M_{q\bar{q}g}^2| \sim \left( d\Phi_{q\bar{q}} |M_{q\bar{q}}^2| \right) \frac{d^3 \vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)} \]

**Factorisation:** Born $\times$ radiation

Changing variables (use energy of gluon $E$ and emission angle $\theta$) we get for the radiation part:

\[ dS = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin \theta} \frac{d\phi}{2\pi} \]
QCD emission probability

\[ \frac{dP_{k \rightarrow ij}}{dE_i d\theta_{ij}} \sim \frac{\alpha_s}{\min(E_i, E_j) \theta_{ij}} \]

**Singular** in the soft \((E_{i,j} \rightarrow 0)\) and in the collinear \((\theta_{ij} \rightarrow 0)\) limits. **Divergent** upon integration.

The divergences can be cured by the addition of virtual corrections and/or if the definition of an observable is appropriate.
Using the variables $E=(1-z)p$ and $k_t = E\theta$ we can rewrite

$$dS = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin \theta} \frac{d\phi}{2\pi} \rightarrow \frac{\alpha_s C_F}{\pi} \frac{1}{1 - z} \frac{dk_t^2}{k_t^2} \frac{d\phi}{2\pi}$$

‘almost’ the Altarelli-Parisi splitting function $P_{qq}$
If the quark is massive the collinear singularity is screened.

\[
\frac{\alpha_s C_F}{\pi} \frac{1}{1-z} d\phi \frac{d^2k^2_t}{2\pi} \rightarrow \frac{\alpha_s C_F}{\pi} \frac{1}{1-z} d\phi \frac{d^2k^2_t}{k_t^2} + (1-z)^2 m^2 \frac{d^2k^2_t}{2\pi} + \cdots
\]
The universal soft and collinear spectrum is not the only relevant characteristic of radiation. **Angular ordering** is another.

Angular ordering means \( \theta < \theta_{ee} \)

Soft radiation emitted by a dipole is restricted to cones smaller than the angle of the dipole.
Angular ordering is a manifestation of **coherence**, a phenomenon typical of gauge theories.

Coherence leads to the **Chudakov effect**, suppression of soft bremsstrahlung from an $e^+e^-$ pair.

“Quasi-classical” explanation: a soft photon cannot resolve a small-sized pair, and only sees its total electric charge (i.e. zero)

The phenomenon of coherence is preserved also in QCD. Soft guon radiation off a coloured pair can be described as being emitted coherently by the colour charge of the parent of the pair.
Easiest higher order calculation in QCD. Calculate $e^+e^- \rightarrow qq\bar{q}+X$ in pQCD

- **Born**
- **Virtual**
- **Real**
Regularize with dimensional regularization, expand in powers of $\varepsilon$

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_s}{\pi} H(\varepsilon) \left[ \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\varepsilon) \right]$$

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{ 1 + \frac{2\alpha_s}{3\pi} H(\varepsilon) \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + \mathcal{O}(\varepsilon) \right] \right\}$$

$$R = 3 \sum_q Q_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

Real and virtual, separately divergent, ‘conspire’ to make total cross section finite
In higher orders $\alpha_s$ must be renormalised and acquires a scale dependence.

$$K_{QCD} = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n \geq 2} C_n \left( \frac{s}{\mu^2} \right) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n$$

$C_n$ known up to $C_3$

Cross section prediction varies with renormalisation scale choice. Which value do we pick for $\mu$?

$\mu$ cannot be uniquely fixed. It can however be exploited to estimate the theoretical uncertainty of the calculation.
Theoretical uncertainties

We wrote before: \( \frac{d}{d \ln \mu^2} \ln \sigma^{\text{phys}} = 0 \)

i.e. independence of cross sections on artificial scales

Would only hold for all-orders calculations.

In real life: residual dependence at one order higher than the calculation

\[ \frac{d}{d \log \mu} \sum_{n=1}^{N} c_n(\mu) \alpha_S^n(\mu) \sim \mathcal{O}(\alpha_S^n(\mu)^{N+1}(\mu)) \]

Vary scales (around a physical one) to \textbf{ESTIMATE} the uncalculated higher order
Non-perturbative contributions

We have calculated $\sum_q \sigma(e^+ e^- \rightarrow q\bar{q})$ in **perturbative** QCD

However

$$\sum_q \sigma(e^+ e^- \rightarrow q\bar{q}) \neq \sigma(e^+ e^- \rightarrow \text{hadrons})$$

The (small) difference is due to hadronisation corrections, and is of non-perturbative origin

We cannot calculate it in pQCD, but in some cases we can get an idea of its behaviour from the incompleteness of pQCD itself
Suppose we keep calculating to higher and higher orders:

\[ \alpha_s^{n+1} \beta_0^nf \]

This is big trouble: the series is not convergent, but only asymptotic

Evidence: try summing

\[ R = \sum_{n=0}^{\infty} \alpha^n n! \]

\((\alpha = 0.1)\)

Asymptotic value of the sum:

\[ R^{\text{asymp}} \equiv \sum_{n=0}^{n_{\text{min}}} R_n \]

\(n_{\text{min}} \approx 1/\alpha\)
The renormalons signal the \textit{incompleteness} of perturbative QCD

One can only \textit{define} what the sum of a perturbative series is (like truncation at the minimal term)

The rest is a \textit{genuine ambiguity}, to be eventually lifted by \textit{non-perturbative corrections}:

\[ R^{\text{true}} = R^{pQCD} + R^{NP} \]

In QCD these non-perturbative corrections take the form of power suppressed terms:

\[ R^{NP} \sim \exp \left( -\frac{p}{\beta_0 \alpha_s} \right) = \exp \left( -p \ln \frac{Q^2}{\Lambda^2} \right) = \left( \frac{\Lambda^2}{Q^2} \right)^p \]

The value of \( p \) depends on the process, and can sometimes be predicted by studying the perturbative series: \( pQCD - \text{NP physics bridge} \)
Cancellation of singularities

**Block-Nordsieck theorem**
IR singularities cancel in sum over soft unobserved photons in final state
(formulated for massive fermions $\Rightarrow$ no collinear divergences)

**Kinoshita-Lee-Nauenberg theorem**
IR and collinear divergences cancel in sum over degenerate initial and final states

These theorems suggest that the observable must be crafted in a proper way for the cancellation to take place
pQCD calculations: hadrons

Turn hadron production in e+e- collisions around: Drell-Yan.

Still easy in Parton Model: just a convolution of probabilities

\[
\frac{d\sigma_{NN\rightarrow \mu\bar{\mu}+X}(Q,p_1,p_2)}{dQ^2d\ldots} \sim \int d\xi_1 d\xi_2 \sum_{a=q\bar{q}} \frac{d\sigma_{a\bar{a}\rightarrow \mu\bar{\mu}}^{EW,\text{Born}}(Q,\xi_1p_1,\xi_2p_2)}{dQ^2d\ldots} \\
\times (\text{probability to find parton } a(\xi_1) \text{ in } N) \\
\times (\text{probability to find parton } \bar{a}(\xi_2) \text{ in } N)
\]

This isn’t anymore an **inclusive process** as far as hadrons are concerned: I find them in the initial state, **I can’t ‘sum over all of them’**

Still, the picture holds at tree level (**parton model**) The parton distribution functions can be roughly equated to those extracted from DIS
The non-inclusiveness of a general strong interaction process is a threat to calculability.

What do we do if we can’t count on Bloch-Nordsieck and Kinoshita-Lee-Nauenberg?

- Infrared and collinear safe observables
  - less inclusive but still calculable in pQCD

- Factorisation
  - trade divergences for universal measurable quantities
A generic (not fully inclusive) observable $O$ is \textbf{infrared and collinear safe if}

\[ O(X; p_1, \ldots, p_n, p_{n+1} \to 0) \to O(X; p_1, \ldots, p_n) \]
\[ O(X; p_1, \ldots, p_n \parallel p_{n+1}) \to O(X; p_1, \ldots, p_n + p_{n+1}) \]

Infrared and collinear safety demands that, in the limit of a collinear splitting, or the emission of an infinitely soft particle, the observable remain \textbf{unchanged}.
Cancellation of divergences in total cross section (KLN)

\[ \sigma_{\text{tot}} = \int_n |M_n^B|^2 d\Phi_n + \int_n |M_n^V|^2 d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 d\Phi_{n+1} \]

A generic observable

\[
\frac{dO}{dX} = \int_n |M_n^B|^2 O(X; p_1, \ldots, p_n) d\Phi_n \\
+ \int_n |M_n^V|^2 O(X; p_1, \ldots, p_n) d\Phi_n + \int_{n+1} |M_{n+1}^R|^2 O(X; p_1, \ldots, p_n, p_{n+1}) d\Phi_{n+1}
\]

In order to ensure the same cancellation existing in \( \sigma_{\text{tot}} \), the definition of the observable must not affect the soft/collinear limit of the real emission term, because it is there that the real/virtual cancellation takes place.
In pQCD (i.e. with gluon emissions), life becomes more complicated

Non fully inclusive process (hadrons in initial state): non cancellation of collinear singularities in pQCD

Same procedure used for renormalising the coupling: reabsorb the divergence into bare non-perturbative quantities, the parton probabilities (collinear factorisation)

**The factorisation theorem**

\[ \sigma^{\text{phys}} = F^{\text{bare}} F^{\text{bare}} \sigma^{\text{divergent}}(\varepsilon) = F(\mu)F(\mu)\hat{\sigma}(\mu) \]

- infrared regulator
- Parton Distribution Function
- factorisation scale
- short-distance cross section

and (schematically)

\[ F(\mu) = F^{\text{bare}} \left( 1 + \alpha_s P \log \frac{\mu^2}{\mu_0^2} \right) \]

This factor universal
A prototype of QCD calculations: many finite terms but, more importantly, a few characteristic large logarithms.

In many circumstances and kinematical situations the logs are much more important than the finite terms: hence in pQCD resummations of these terms are often phenomenologically more relevant than a full higher order calculation.
Factorisation

\[ \sigma^{\text{phys}} = F(\mu) \hat{\sigma}(\mu) \]

Evolution

\[
\frac{d}{d \ln \mu^2} \ln \sigma^{\text{phys}} = 0 \quad \Rightarrow \quad \frac{d \ln \hat{\sigma}(\mu)}{\ln \mu^2} = -\frac{d \ln F(\mu)}{\ln \mu^2} = -\alpha_s \frac{P}{\ln \mu^2}
\]

Resummation

DGLAP evolution equations for PDF's

Solution of evolution equations resums higher order terms

Responsible for scaling violations

(for instance in DIS structure functions)
DGLAP equations

\[
\frac{d f_q(x, t)}{d t} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d z}{z} \left[ P_{qq}(z) f_q\left(\frac{x}{z}, t\right) + P_{qg}(z) f_g\left(\frac{x}{z}, t\right) \right]
\]

\[
\frac{d f_g(x, t)}{d t} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d z}{z} \left[ P_{gq}(z) \sum_{i=q,\bar{q}} f_i\left(\frac{x}{z}, t\right) + P_{gg}(z) f_g\left(\frac{x}{z}, t\right) \right]
\]

The Altarelli-Parisi kernels control the evolution of the Parton Distribution Functions
Altarelli-Parisi kernels

\[ P_{gg} \to 2C_A \left\{ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right\} + \delta(1-x) \left[ \frac{11C_A - 2n_f}{6} \right] \]

\[ P_{qg}(z) \to \left( \frac{1+z^2}{1-z} \right)_+ \equiv \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dy \left( \frac{1+y^2}{1-y} \right) \]

\[ P_{gg} = \frac{1}{2} \left[ z^2 + (1-z)^2 \right] \]

\[ P_{gq}(z) = C_F \frac{1+(1-z)^2}{z} \]

DGLAP evolution of PDFs

Evolution (i.e. higher momentum scale) produces more partons at small momentum fraction (because they lose energy by radiating)

As for the coupling, one can’t predict PDF’s values in pQCD, but only their evolution
• universal character of soft/collinear emission
• both real and virtual diagrams usually contribute to an observable (and are both needed to cancel divergences)
• not everything is calculable. Restrict to IRC-safe observables and/or employ factorisation
QCD, Jets and Monte Carlo techniques

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Lecture 1 - Basics of QCD
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Ingredients and tools

- PDFs
- Hard scattering and shower
- Final state tools
Tools for the hard scattering

Can be divided in

- **Integrators**
  - evaluate the (differential) cross section by integrating the calculation over the phase space, yielding (partly) inclusive quantities
  - Produce weighted events (the weight being the value of the cross section)
  - Calculations exist at LO, NLO, NNLO

- **Generators**
  - generate fully exclusive configurations
  - Events are unweighted (i.e. produced with the frequency nature would produce them)
  - Easy at LO, get complicated when dealing with higher orders
(Higher order) calculations

What goes into them?
Nomenclature

\[ P = \begin{array}{c}
\text{loop correction}
\end{array} \]

\[ P_{11} = \begin{array}{c}
\text{additional emissions}
\end{array} \]

\[ P + 1e = \begin{array}{c}
\text{some final state}
\end{array} \]
N. B.

\[ P + 1 e \neq P + 1 \text{ jet} \]

e = \text{emission}

or

e = \text{ee} e
$P + 1\, e = \begin{array}{c} \\
\end{array}$

Contributes to:

$P + 1\, \text{jet}$

$P + X$

or

(if e is integrated over)
Process P exact at LO, nothing else

(NB. At the matrix element squared level)

Additional QCD loops

Additional $\alpha_s$ powers

Absent

PS approx

Exact

$P$

$P_{3L}$

$P_{2L}$

$P_{IL}$

$P + 1e_{1L}$

$P + 1e_{2L}$

$P + 2e_{1L}$

$P + 2e_{2L}$

$P + 3e$

$P_{+1e}$

$P_{+2e}$

$P_{+3e}$

$\text{Exact}$

$\text{PS approx}$

$\text{Absent}$
Process $P+1j$ exact at LO, nothing else

**Diagram:**
- **P**
- **P+1e**
- **P+2e**
- **P+3e**

**Legend:**
- Absent
- PS approx
- Exact

**Additional QCD loops**
- $3L$
- $2L$
- $1L$

**Additional $\alpha_s$ powers**
- $P$
- $P+1e$
- $P+2e$
- $P+3e$

**Process:**
- Process $P+1j$ exact at LO, nothing else

**Notation:**
- $P$
- $P+1e$
- $P+2e$
- $P+3e$

**Diagonal Lines:**
- Diagonal lines represent additional QCD loops.
- The number of loops increases as we move from left to right:
  - 1 loop
  - 2 loops
  - 3 loops

**Color Coding:**
- Absent: Blank
- PS approx: Grey
- Exact: Green
Process $P$ exact at NLO, $P+1j$ exact at LO, nothing else
Process $P$ exact at NLO, $P+1j$ exact at LO, nothing else

Additional $\alpha_s$ powers

Additional QCD loops

Additional QCD emissions

The "NLO triangle" (for $P+X$)

Cancellation of divergences

Absent

PS approx

Exact
Process $P$ and $P + 1j$ exact at NLO, $P + 2j$ at LO

Additional $\alpha_s$ powers

Additional QCD loops

Additional QCD emissions

The "NLO triangle" (for $P + 1j$)
Process $P$ exact at NNLO, $P+1j$ exact at NLO, $P+2j$ at LO

Additional $\alpha_s$ powers

Additional QCD loops

Additional QCD emissions

The "NLO triangle" (for $P+1j$)

The "NNLO triangle" (for $P+X$)
Process $P$ exact at NNLO, $P+1j$ exact at NLO, $P+2j$ at LO

Interferences and squares down the “equal final state” lines

Additional QCD loops

Additional $\alpha_s$ powers

The “NLO triangle” (for $P+1j$)

The “NNLO triangle” (for $P+X$)
Born

$$d\sigma^{Born} = B(\Phi_B) d\Phi_B$$

NLO

$$d\sigma^{NLO} = \left[ B(\Phi_B) + V(\Phi_B) \right] d\Phi_B + R(\Phi_R) d\Phi_R$$

Problem:

$V(\Phi_B)$ and $\int R d\Phi_R$ are divergent
An observable $O$ is **infrared and collinear safe** if

$$O(\Phi_R(\Phi_B, \Phi_{rad})) \to O(\Phi_B)$$

Soft or collinear limit

One can then write, with $C \to R$ in the soft/coll limit,

$$\langle O \rangle = \int \left[ B(\Phi_B) + V(\Phi_B) + \int C(\Phi_R) d\Phi_{rad} \right] O(\Phi_B) d\Phi_B$$

$$+ \left[ R(\Phi_R) O(\Phi_R) - C(\Phi_R) O(\Phi_B) \right] d\Phi_R$$

This integration performed analytically

Separately finite

This (or a similar) cancellation will always be implicit in all subsequent equations
Exploit factorisation property of soft and collinear radiation

\[ d\sigma_{n+1}(\Phi_{n+1}) = \mathcal{P}(\Phi_{\text{rad}}) \, d\sigma_n(\Phi_n) \, d\Phi_{\text{rad}} \]

Factorisation

Emission probability

Iterate emissions to generate higher orders (in the soft/collinear approximation)
Based on the **iterative emission of radiation** described in the **soft-collinear limit**

\[ d\sigma^{(MC)}(\Phi_R)d\Phi_R = B(\Phi_B)d\Phi_B P(\Phi_{rad})d\Phi_{rad} \]

**Pros:** soft-collinear radiation is resummed to all orders in pQCD

**Cons:** hard large-angle radiation is missing

Overall accuracy will be leading log (LL) for the radiation, and leading order (i.e. Born) for the integrated cross sections
A key ingredient of a parton shower Monte Carlo:

**Sudakov form factor**

\[ \Delta(t_1,t_2) \]

- Probability of no emission between the scales \( t_1 \) and \( t_2 \)

**Example:**
- Decay probability per unit time of a nucleus = \( c_N \)
- Sudakov form factor \( \Delta(t_0,t) = \exp(-c_N(t-t_0)) \)

Probability that nucleus does not decay between \( t_0 \) and \( t \)
Sudakov form factor: derivation

Decay probability per unit time = \( \frac{dP}{dt} = c_N \)

Probability of **no** decay between \( t_0 \) and \( t = \Delta(t_0, t) \) \[\text{[with } \Delta(t_0, t_0) = 1\]

\( \Rightarrow \) Probability of decay between \( t_0 \) and \( t = 1 - \Delta(t_0, t) \)

[\text{unitarity: either you decay or you don’t}]

Decay probability per unit time **at time** \( t \) can be written in two ways:

1. \( P_{\text{dec}}(t) = \frac{d}{dt} \left( 1 - \Delta(t_0, t) \right) = -\frac{d\Delta(t_0, t)}{dt} \)

2. \( P_{\text{dec}}(t) = \Delta(t_0, t) \frac{dP}{dt} \)

No decay until \( t \), probability per unit time to decay at \( t \)
Sudakov form factor: derivation

Equating the two expressions for $P_{\text{dec}}(t)$ we get

$$
-d\Delta(t_0, t) \frac{d\Delta(t_0, t)}{dt} = \Delta(t_0, t) \frac{dP}{dt}
$$

We can solve the differential equation using $dP/dt = c_N$ and we get

$$
\Delta(t_0, t) = \exp(-c_N(t-t_0))
$$

If the decay probability depends on $t$ (and possibly other variables, call them $z$) this generalises to

$$
\Delta(t_0, t) = \exp \left( - \int_{t_0}^{t} dt' \int dz \ c_N(t', z) \right)
$$
Sudakov form factor in QCD

**Emission probability**

\[ \mathcal{P}(\Phi_{\text{rad}}) \, d\Phi_{\text{rad}} \approx \frac{\alpha_S(q)}{\pi} \frac{dq}{q} \, P(z, \phi) \, dz \frac{d\phi}{2\pi} \]

**Sudakov form factor** = probability of **no emission** from large scale \( q_1 \) to smaller scale \( q_2 \)

\[ \Delta_S(q_1, q_2) = \exp \left[ - \int_{q_2}^{q_1} \frac{\alpha_S(q)}{\pi} \frac{dq}{q} \int_{z_0}^{1} P(z) \, dz \right] \]
Conventions for Sudakov form factor

\[ \Delta_S(q_1, q_2) = \exp \left[ - \int_{q_2}^{q_1} \frac{\alpha_S(q)}{\pi} \frac{dq}{q} \int_{z_0}^{1} P(z) \, dz \right] \]

\[ \Delta(p_T) = \exp \left[ - \int_{p_T}^{Q} \frac{d\sigma^{(MC)}}{dy} \frac{dp_T'}{dp_T} \right] \]

\[ \Delta_R(p_T) = \exp \left[ - \int_{p_T}^{\frac{R}{B}} \Theta(k_T(\Phi_R) - p_T) \, d\Phi_{rad} \right] \]

\[ \Delta_R(p_T) = \exp \left[ - \int_{p_T}^{\frac{R}{B}} \frac{R}{B} \, d\Phi_{rad} \right] \]
PS example: Higgs plus radiation

Leading order.
No radiation, Higgs $p_T = 0$

With emission of radiation
Higgs $p_T \neq 0$

Description of hardest emission in PS MC (either event is generated)

\[
\frac{d\sigma^{(MC)}}{dy \, dp_T} = \frac{d\sigma^{(B)}}{dy} \delta(p_T) \Delta(Q_0) + \Delta(p_T) \frac{d\sigma^{(MC)}}{dy \, dp_T}
\]

\[
\Delta(p_T) = \exp \left[ - \int_{p_T}^Q \frac{d\sigma^{(MC)}}{dy \, dp_T} - \frac{d\sigma^{(B)}}{dy} dp_T' \right]
\]

Sudakov form factor

x-sect for no emission
prob. of no emission
(down to the PS cutoff)

prob. of no emission down to $p_T$

x-sect for emission at $p_T$,
as described by the MC
Gavin Salam has made public a ‘toy shower’ that generates the Higgs transverse momentm via successive emissions controlled by the Sudakov form factor

\[ \Delta(p_T) = \exp \left[ - \frac{2 \alpha_s C_A}{\pi} \ln^2 \frac{p_{T,\text{max}}^2}{p_T^2} \right] \]

You can get the code at https://github.com/gavinsalam/zuoz2016-toy-shower

NB. In order to get more realistic results you need at least at the code in v2
It holds

\[ \int_0^Q \left[ \delta(p_T) \Delta(Q_0) + \frac{\Delta(p_T)}{\frac{d\sigma^{(MC)}}{dydp_T}} \right] dp_T = \Delta(Q_0) + \int_{Q_0}^Q \frac{d\Delta(p_T)}{dp_T} dp_T = \Delta(Q) = 1 \]

so that

\[ \int_0^Q dp_T \frac{d\sigma^{(MC)}}{dydp_T} = \frac{d\sigma^{(B)}}{dy} \int_0^Q \left[ \delta(p_T) \Delta(Q_0) + \frac{\Delta(p_T)}{\frac{d\sigma^{(MC)}}{dydp_T}} \right] dp_T = \frac{d\sigma^{(B)}}{dy} \]

A parton shower MC correctly reproduces the Born cross section for integrated quantities
Writing the real cross section as described by the Monte Carlo (i.e. with the parton shower) simply as $R^{MC}$, we can rewrite

$$d\sigma^{MC} = B d\Phi_B \left[ \Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

with

$$\Delta_{MC}(p_T) = \exp \left[ - \int_{p_T} \frac{R^{MC}}{B} d\Phi_{rad} \right]$$

as our Master Formula for a Parton Shower Monte Carlo.

Thanks to the shower unitarity, it holds

$$\Rightarrow \int d\sigma^{MC} = \int B d\Phi_B = \sigma^{LO}$$
Matrix Element corrections

In a PS Monte Carlo

\[ R^{(MC)}(\Phi_R) = B(\Phi_B)\mathcal{P}(\Phi_{rad}) \]

soft-collinear approximation

Replace the MC description of radiation with the **correct** one:

\[ \mathcal{P}(\Phi_{rad}) \rightarrow \frac{R}{B} \]

The Sudakov becomes

\[ \Delta(p_T) = \exp \left[ - \int_{p_T}^{Q} \frac{d\sigma^{(MC)}}{dy \, dp_T'} dp_T' \right] \]

\[ \longrightarrow \Delta_R(p_T) = \exp \left[ - \int \frac{R}{B} \Theta(k_T(\Phi_R) - p_T) d\Phi_{rad} \right] \]

and the x-sect formula for the hardest emission

\[ d\sigma^{MEC} = B d\Phi_B \left[ \Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right] \]
Matrix Element corrections

Hard radiation: full real corrections dominate

Soft radiation: Sudakov dominates (and eliminates the divergence of NLO)

\[ p_T^H \text{ (GeV)} \]

\[ \sigma / dy dp_T^H \]

\[ M_H = 120 \text{ GeV} \]
We wish to go beyond a Parton Shower (+MEC) Monte Carlo, so that

- we can successfully interface matrix elements for multi-parton production with a parton shower

- we can successfully interface a parton shower with a NLO calculation

It’s a quest for exactness of ever more complex processes
Process P exact at LO, the rest PS approximation

Additional QCD loops

Additional $\alpha_s$ powers

Additional QCD emissions

- P 3L
- P 2L
- P 1L
- P+1e 2L
- P+1e 1L
- P+2e 1L
- P+2e
- P+3e

Absent
PS approx
Exact
Process $P$ and $P+1j$ exact at LO, the rest PS approximation. 

[PS+MEC or PS from ME for $P+1e$]

Additional $\alpha_s$ powers

Additional QCD loops

Additional QCD emissions

1. P
2. P + 1e
3. P + 2e
4. P + 3e

Absent
PS approx
Exact
Process $P, P+1j, P+2j, \ldots$ exact at LO, the rest PS approx.

**[PS+Matrix Element (CKKW, MLM, \ldots)]**

Additional QCD loops

Additional $\alpha_s$ powers

Additional QCD emissions

- **Absent**
- **PS approx**
- **Exact**
Process P exact at NLO, the rest PS approximation [PS+NLO (MC@NLO, POWHEG,...)]

Additional QCD loops

Additional $\alpha_s$ powers

Additional QCD emissions

- **P**
  - **3L**
  - **P+1e**
    - **2L**
    - **P+2e**
      - **1L**
      - **P+3e**
  - **P+1e**
    - **1L**
    - **P+2e**
  - **P+2e**
    - **1L**
    - **P+3e**

- **Absent**
- **PS approx**
- **Exact**
Process $P$ exact at NLO, $P+1j$, $P+2j$, ... at LO, the rest $PS$ approx

$[PS+NLO+ME (MENLOPS,...)]$ [Hamilton, Nason '10]
Process $P, P+1j, P+2j, \ldots$ exact at NLO, the rest PS

$[PS+NLO+\text{ME}_{\text{NLO}} (\text{MEPS@NLO}, \ldots)]$
Existing ‘MonteCarlo at NLO’:

- **MC@NLO** [Frixione and Webber, 2002]
- **POWHEG** [Nason, 2004]

NB. MC@NLO is a code, POWHEG is a method

Evolving into (semi)automated forms:

- **The POWHEG BOX** [powhegbox.mib.infn.it 2010]
- **aMC@NLO** [amcatnlo.cern.ch 2011]
Matrix-element corrected shower Monte Carlos still have leading order accuracy for the total rates

$$d\sigma^{MEC} = Bd\Phi_B \left[ \Delta_R(Q_0) + \Delta_R(p_T)\frac{R}{B}d\Phi_{rad} \right] \quad \text{and} \quad \Delta_R(Q_0) + \int \Delta_R(p_T)\frac{R}{B}d\Phi_{rad} = 1$$

$$\Rightarrow \int d\sigma^{MEC} = \int Bd\Phi_B = \sigma^{LO}$$

We want to do better, and **merge** PS and NLO, so that

$$\int d\sigma^{PS+NLO} = \int (B + V)d\Phi_B + \int Rd\Phi_R = \sigma^{NLO}$$
Idea: remove from the NLO the terms that are already generated by the parton shower (NB. MC-specific)

\[ d\sigma^{MC@NLO} = \bar{B}_{MC} d\Phi_B \left[ \Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right] + [R - R^{MC}] d\Phi_R \]

\[ B_{MC} = B + \left[ V + \int R^{MC} d\Phi_{rad} \right] \]

It is easy to see that, as desired,

\[ \int d\sigma^{MC@NLO} = \int d\sigma^{NLO} \]
Idea: generated hardest radiation first, then pass event to MC for generation of subsequent, softer radiation

\[ d\sigma^{POWHEG} = \tilde{B} d\Phi_B \left[ \Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right] \]

\[ \tilde{B} = B + \left[ V + \int R d\Phi_{rad} \right] \]

NLO x-sect

MC shower

It is easy to see that, as desired,

\[ \int d\sigma^{POWHEG} = \int d\sigma^{NLO} \]
Large $p_T$ enhancement in *POWHEG*

The ‘naive’ formulation for POWHEG is

$$d\sigma^{POWHEG} = \tilde{B} d\Phi_B \left[ \Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]$$

In this form $\tilde{B} d\Phi_B$ provides the NLO K-factor (order $1 + \mathcal{O}(\alpha_s)$), but also associates it to large $p_T$ radiation, where the calculation is already $\mathcal{O}(\alpha_s)$ (but only LO accuracy).

This generates an effective (but not necessarily correct) $\mathcal{O}(\alpha_s^2)$ term (i.e. NNLO for the total cross section)

OK because beyond nominal accuracy, but one may feel uncomfortable with such large numerical factors
The ‘problem’ with the naive POWHEG comes from the hard radiation being enhanced by spurious higher orders. In order to suppress this effect, we split

\[ R = R^S + R^F \]

\[ R^S \equiv \frac{h^2}{h^2 + p_T^2} R \]
\[ R^F \equiv \frac{p_T^2}{h^2 + p_T^2} R \]

\[ \Delta_S(p_T) = \exp \left[ - \int_{p_T} \frac{R^S}{B} d\Phi_{rad} \right] \]

\[ \bar{B}^S = B + \left[ V + \int R^S d\Phi_{rad} \right] \]

\[ d\sigma_{POWHEG} = \bar{B}^S d\Phi_B \left[ \Delta_S(Q_0) + \Delta_S(p_T) \frac{R^S}{B} d\Phi_{rad} \right] + R^F d\Phi_R \]
In the $h \to \infty$ limit the exact NLO result is recovered.
Comparisons

\[d\sigma^{MC} = B d\Phi_B \left[ \Delta(Q_0) + \Delta(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right]\]

\[d\sigma^{MEC} = B d\Phi_B \left[ \Delta_R(Q_0) + \Delta_R(p_T) \frac{R}{B} d\Phi_{rad} \right]\]

\[d\sigma^{NLO} = [B + V] d\Phi_B + Rd\Phi_R\]

\[d\sigma^{MC@NLO} = \bar{B}_{MC} d\Phi_B \left[ \Delta_{MC}(Q_0) + \Delta_{MC}(p_T) \frac{R^{MC}}{B} d\Phi_{rad} \right] + [R - R^{MC}] d\Phi_R\]

\[d\sigma^{POWHEG} = \bar{B}^S d\Phi_B \left[ \Delta_S(Q_0) + \Delta_S(p_T) \frac{R^S}{B} d\Phi_{rad} \right] + R^F d\Phi_R\]

POWHEG approaches MC@NLO if \( R^S \rightarrow R^{MC} \)
Monte Carlos in QCD are complicated. I only scratched the surface here and gave almost no details. If interested, check lectures of real MC people (Sjostrand, Skands, Nason, Maltoni, Frixione, Krauss, Richardson, Webber,.....)

Monte Carlos exploit property of universality of soft/collinear radiation to resum its effects to all orders (within some approximations)

Effects of multi-parton, hard, large-angle radiation can be included via exact calculations and proper (and delicate) mergings

The result is a detailed description of the final state, covering as much phase space as possible. Accurate descriptions of data are usually achieved