QUANTUM FIELD THEORY

Problem Set

A) Evaluate the first and second functional derivatives with respect to $\varphi(x)$ of the following functionals.

$$\begin{aligned} F_1 &= \int d^4x\, \varphi^3 \\ F_2 &= \int d^4x\, \varphi^2 \partial_\mu \varphi \partial^\mu \varphi \end{aligned} \tag{1}$$

B) In class I showed that the generating functional of multiparticle propagators is given by

$$\begin{aligned} Z[J] &\equiv \langle 0|\, T\, \exp\left(\int J\phi\right)\, |0\rangle \\ &= \exp\left[\frac{1}{2}\int_{x,y} J(x)G(x,y)J(y)\right] \end{aligned} \tag{2}$$

By taking the fourth-order functional derivative of this with respect to J's at four different points, and setting J equal to zero at the end, obtain

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = G(x_1,x_2)G(x_3,x_4)+G(x_1,x_3)G(x_2,x_4)+G(x_1,x_4)G(x_2,x_3) \tag{3}$$

How do you interpret the fact that individual terms are summed up?

C) The polarization vectors $e_i^{(\lambda)}$, $\lambda = 1, 2$, for the photon, along with $e_i^{(3)} = k_i/|\vec{k}|$ for an orthonormal triad of vectors. Use the completeness relation for them to calculate the propagator $D_{ij}(x,y) = \langle 0|T\, A_i^T(x)\, A_j^T(y)|0\rangle$.

Problem 1

a) The general formula for the Chern-Simons form in $2n+1$ dimensions is given by

$$\omega_{2n+1}(A) = \frac{i^{n+1}}{(2\pi)^{n+1}2^n\, n!}\int_0^1 ds\, \epsilon^{\mu_1\cdots\mu_{2n+1}}\, \mathrm{Tr}(A_{\mu_1}F^{(s)}_{\mu_2\mu_3}\, F^{(s)}_{\mu_4\mu_5}\cdots F^{(s)}_{\mu_{2n}\mu_{2n+1}}) \tag{4}$$

where $F^{(s)}_{\mu\nu} = s\, F_{\mu\nu} + (s^2 - s)[A_\mu, A_\nu]$. Obtain $\omega_5(A)$ by direct calculation from this formula.

b) The five-dimensional Chern-Simons form can be written as $\omega_5(A) = K_5^6$, where

$$K_5^\mu(A) = -\frac{i}{48\pi^3}\epsilon^{\mu\mu_1\cdots\mu_5}\mathrm{Tr}\left(A_{\mu_1}\,\partial_{\mu_2}A_{\mu_3}\,\partial_{\mu_4}A_{\mu_5} + \frac{3}{2}A_{\mu_1}A_{\mu_2}A_{\mu_3}\,\partial_{\mu_4}A_{\mu_5} + \frac{3}{5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right) \tag{5}$$

c) Verify by directly taking the derivative that

$$\partial_\mu K_5^\mu = -\frac{i}{384\pi^3}\epsilon^{\mu_1\mu_2\cdots\mu_6}\mathrm{Tr}(F_{\mu_1\mu_2}F_{\mu_3\mu_4}F_{\mu_5\mu_6}) \tag{6}$$

Verify further that $\omega_5(A^g) - \omega_5(A)$ gives the nonabelian anomaly for a chiral Dirac fermion, for infinitesimal transformations, $g \approx 1 + \theta$.

Problem 2

The winding number for the map $S^3 \to G$ for a Lie group G is given by

$$Q[g] = \frac{1}{24\pi^2} \int d^3x\, \epsilon^{\mu\nu\alpha}\, \mathrm{Tr}(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\alpha g) \tag{7}$$

a) Verify that $Q[g_1 g_2] = Q[g_1] + Q[g_2]$.

b) Consider the configuration

$$g(x) = \cos\theta(r) + i\tau \cdot \hat{x}\, \sin\theta(r) \tag{8}$$

where $\tau_1,\, \tau_2,\, tau_3$ are the Pauli matrices. Here we take $G = SU(2)$. Calculate $Q[g]$ for this case and show that

$$Q[g] = \frac{1}{\pi}\left(\theta(0) - \theta(\infty)\right) \tag{9}$$

c) Why should $\sin\theta(r)$ vanish at $r = 0$?

Problem 3

For the standard model, the Wess-Zumino action in terms of the pseudoscalar meson fields is given by

$$\begin{aligned}
\Gamma_{WZ} \;=\; & -\frac{iN_c}{240\pi^2}\int \mathrm{Tr}\left(dUU^{-1}\right)^5 \\
& +\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(A_L dA_L + dA_L A_L + A_L^3\right) dUU^{-1} \\
& +\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(A_R dA_R + dA_R A_R + A_R^3\right) U^{-1}dU \\
& -\frac{iN_c}{96\pi^2}\int \mathrm{Tr}\left[\left(A_L dUU^{-1}\right)^2 - \left(A_R U^{-1}dU\right)^2\right] \\
& -\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left[A_L\left(dUU^{-1}\right)^3 + A_R\left(U^{-1}dU\right)^3\right] \\
& -\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(dA_L dU A_R U^{-1} - dA_R dU^{-1} A_L U\right) \\
& -\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(A_R U^{-1} A_L U (U^{-1}dU)^2 - A_L U A_R U^{-1}(dUU^{-1})^2\right) \\
& +\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left((dA_R A_R + A_R dA_R)U^{-1}A_L U - (dA_L A_L + A_L dA_L)UA_R U^{-1}\right) \\
& +\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(A_L U A_R U^{-1} A_L dUU^{-1} + A_R U^{-1} A_L U A_R U^{-1} dU\right) \\
& -\frac{iN_c}{48\pi^2}\int \mathrm{Tr}\left(A_R^3 U^{-1} A_L U - A_L^3 U A_R U^{-1} + \frac{1}{2} UA_R U^{-1} A_L U A_R U^{-1} A_L\right).
\end{aligned} \tag{10}$$

a) By restricting the gauge fields to be just the electromagnetic fields, given by

$$A_L = A_R = -ieA\,Q = -ieA\begin{pmatrix} \frac{2}{3} & 0 & \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{11}$$

show that Γ_{WZ} reduces to

$$\begin{aligned} \Gamma_{WZ} &= -\frac{eN_c}{48\pi^2}\int_{\mathcal{M}} A\,\mathrm{Tr}[Q(dUU^{-1})^3 + Q(U^{-1}dU)^3] \\ &\quad -i\frac{e^2N_c}{24\pi^2}\int_{\mathcal{M}} AdA\,\mathrm{Tr}[Q^2(dUU^{-1}+U^{-1}dU)] \\ &\quad +i\frac{e^2N_c}{48\pi^2}\int_{\mathcal{M}} AdA\,\mathrm{Tr}[QUQdU^{-1} - QdUAQU^{-1}] \end{aligned} \tag{12}$$

b) Further expanding the fields U in terms of the mesons to linear order, show that we recover the effective vertex for $\pi^0 \to 2\,\gamma$ decay.

Problem 4

The Wess-Zumino-Witten action for a 2-dimensional theory is given by

$$\mathcal{S}_{WZW} = \frac{k}{8\pi}\int_{\mathcal{M}^2} d^2x\sqrt{g}\,g^{ab}\,\mathrm{Tr}(\partial_a M\partial_b M^{-1}) \;+\; k\,\Gamma[M] \tag{13}$$

where M takes values in a group G, k is an integer and

$$\Gamma[M] = \frac{i}{12\pi}\int_{\mathcal{M}^3} d^3x\;\epsilon^{\mu\nu\alpha}\mathrm{Tr}(M^{-1}\partial_\mu MM^{-1}\partial_\nu MM^{-1}\partial_\alpha M) \tag{14}$$

with $\mathcal{M}^2 = \partial\mathcal{M}^3$.

a) Show that the Euclidean version of this theory obeys the Polyakov-Wiegmann identity

$$\mathcal{S}_{WZW}[M\,h] = \mathcal{S}_{WZW}[M] \;+\; \mathcal{S}_{WZW}[h] \;-\; \frac{k}{\pi}\int_{\mathcal{M}^2}\mathrm{Tr}(M^{-1}\partial_{\bar z}M\,\partial_z h\,h^{-1}) \tag{15}$$

where $z = x_1 - ix_2$, $\bar z = x_1 + ix_2$.

b) In $1+1$ dimensions (i.e. after a Wick rotation), this theory has a current operator

$$J_z = -\frac{k}{\pi}\,\partial_z M\,M^{-1} \tag{16}$$

For the case of $G = SU(2)$, we can do a mode expansion of this as

$$J^a(z) = \sum_{n=-\infty}^{\infty}\frac{J_n^a}{z^{n+1}} \tag{17}$$

The components J_n^a, $a = 1,2,3$, $n = 0,\pm1,\pm2,\cdots$ can be shown to obey the so-called Kac-Moody algebra

$$[J_m^a, J_n^b] = i\epsilon^{abc}J^c_{m+n} \;+\; k\,m\,\delta^{ab}\delta_{m+n,0} \tag{18}$$

m, n are integers. From these rules, notice that $J_{-n},\ n > 0$ behave a bit like creation operators and $J_n,\ n > 0$ behave like annihilation operators. So we define a vacuum state which obeys $J^a_n|0\rangle = 0$, $n \geq 0$. Calculate the two-point function $\langle 0|J^a(z)J^b(w)|0\rangle$.

Problem 5

The quantization of the two-dimensional sphere as a phase space leads to the states

$$f_k(z) = \left[\frac{n!}{k!\,(n-k)!}\right]^{\frac{1}{2}} z^k \tag{19}$$

with the inner product

$$\langle 1|2\rangle = i(n+1)\int \frac{dz \wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}\, f_1{}^* f_2 \tag{20}$$

Consider the operators

$$\begin{aligned} J_+ &= z^2 \frac{\partial}{\partial z} - n\, z \\ J_- &= -\frac{\partial}{\partial z}, \qquad\qquad J_3 = z\frac{\partial}{\partial z} - (n/2) \end{aligned}$$

a) Show that these obey the correct commutation rules

b) Show that J_+ and J_- are adjoints of each other using the given inner product.

c) Show that the the standard representation of angular momentum is obtained, and identify the j-value of this set of states.

Problem 6

a) Obtain the three-dimensional Chern-Simons form from the general integral formula for ω_{2n+1}.

b) From ω_3, obtain the expression for the nonabelian anomaly for two spacetime dimensions.

c) On a three-manifold with no boundary, consider the Chern-Simons theory defined by

$$S = -\frac{k}{4\pi}\int \epsilon^{\mu\nu\alpha}\mathrm{Tr}\left(A_\mu\,\partial_\nu A_\alpha + \frac{2}{3}A_\mu\,A_\nu\,A_\alpha\right) \tag{21}$$

Show that gauge invariance of e^{iS} requires that k should be an integer.

d) Also consider a set of gauge transformations, say for $SU(2)$, in two dimensions defined by $g(x,\lambda)$, $0 \leq \lambda \leq 1$, $g(x,0) = g(x,1) = 1$. This is a loop of gauge transformations from the identity back to the identity. We also define a two parameter family of gauge potentials in two dimensions ($i = 1,2$) by

$$A_i(x,\lambda,\sigma) = (g\,A_i(x)\,g^{-1} - \partial_i g g^{-1})\sigma + (1-\sigma)A_i \tag{22}$$

where $0 \leq \sigma \leq 1$. For the boundary values of the square $0 \leq \lambda, \sigma \leq 1$, the field is either A_i or its gauge transform, so that the boundary of the square can be viewed as a point as far gauge invariant quantities are concerned. For simplicity, we take $A_i = 0$. Integrate the canonical structure over the this square, i.e., evaluate

$$-i\frac{k}{2\pi}\int d\sigma d\lambda \, \mathrm{Tr}\left[\frac{\partial A_{\bar{z}}}{\partial\sigma}\frac{\partial A_z}{\partial\lambda} - \frac{\partial A_{\bar{z}}}{\partial\lambda}\frac{\partial A_z}{\partial\sigma}\right]. \tag{23}$$

Show that this is always 2π times an integer if k is an integer. (The integral of the canonical structure over any closed two-surface in the phase space should be 2π times an integer for consistent quantization. This is the generalization of the famous Dirac quantization condition for monopoles.)