Quantum Field Theory

Anomalies and Related Matters

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Refresher on Field Quantization: Electrodynamics

- We start with the action for the electromagnetic field,

$$\mathcal{S} = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu = \int \tfrac{1}{2}(E^2 - B^2) - A_0 J_0 + A_i J_i$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F_{0i} = E_i, \qquad F_{ij} = \epsilon_{ijk} B_k$$

- The equations of motion are the Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$ which can be written out as

$$\partial_i E_i = J_0, \qquad \partial_0 E_i + \epsilon_{ijk}\partial_j B_k = J_i$$

- Notice that these also imply the conservation of the current, $\partial_\mu J^\mu = 0$.

- There are two issues which arise in quantizing this theory:
  - There is a redundancy of variables; A_μ and $A_\mu + \partial_\mu \theta$, for some function θ, give the same $F_{\mu\nu}$.
  - The equation $\partial_i E_i = 0$ (Gauss law) cannot be obtained as a Heisenberg equation of motion (not of the form $(\partial C/\partial t) = \text{something}$).

Refresher on Field Quantization: Electrodynamics (*cont'd.*)

- Elimination of redundancy: We can choose $A_0 = 0$. If it is not zero, we use $A'_\mu = A_\mu + \partial_\mu \theta$, choose θ such that $A'_0 = 0$. Then use A'_i (which we call A_i).

- Split A_i as

$$A_i = A_i^T + \partial_i f, \qquad \partial_i A_i^T = 0$$

- Dealing the Gauss law: We write the electric field as $E_i = \partial_0 A_i^T + \partial_i(\partial_0 f)$. Then the Gauss law equation becomes

$$\nabla^2 (\partial_0 f) = J_0$$

- The solution is in terms of the Coulomb Green's function G_C,

$$\partial_0 f = \int d^3y \, G_C(\vec{x} - \vec{y}) J_0(x^0, \vec{y})$$

$$G_C(\vec{x} - \vec{y}) = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}, \qquad \nabla^2 G_C(\vec{x} - \vec{y}) = \delta^{(3)}(x - y)$$

- f is not an independent degree of freedom, its dynamics is entirely determined by J_0, i.e., by matter fields.

Refresher on Field Quantization: Electrodynamics (*cont'd.*)

- The interaction term is just $A_i J_i$ and this is simplified as

$$\begin{aligned}\int d^4x\, A_i J_i &= \int d^4x (A_i^T J_i + \partial_i f J_i) = \int d^4x (A_i^T J_i - f \partial_i J_i) \\ &= \int d^4x (A_i^T J_i - f \partial_0 J_0) = \int d^4x (A_i^T J_i + \partial_0 f J_0) \\ &= \int d^4x\, A_i^T J_i + \int dx^0 d^3x d^3y\, J_0(x^0, \vec{x}) G_C(\vec{x} - \vec{y}) J_0(x^0, \vec{y})\end{aligned}$$

- The magnetic field only depends on A_i^T, so the action becomes

$$\begin{aligned}\mathcal{S} &= \int d^4x \frac{1}{2}\left[\partial_0 A_j^T \partial_0 A_j^T - \partial_i A_j^T \partial_i A_j^T\right] \\ &\quad + \frac{1}{2}\int d^4x d^4y\, J_0(x) G_C(\vec{x} - \vec{y}) \delta(x^0 - y^0) J_0(y) + \int d^4x\, A_i^T J_i\end{aligned}$$

- The theory reduces to that of two massless fields corresponding to the two independent directions in A_i^T, with some interaction terms.

- This way of quantizing, with $A_0 = 0$ and $\nabla \cdot A = 0$, corresponds to the radiation gauge.

Refresher on Field Quantization: Electrodynamics (*cont'd.*)

- The quantum operator for A_i^T has the expansion

$$A_i^T(x) = \sum_{k\lambda} a_{k\lambda} e_i^{(\lambda)} u_k(x) + a_{k\lambda}^\dagger e_i^{(\lambda)} u_k^*(x), \qquad u_k(x) = \frac{1}{\sqrt{2\omega_k V}} e^{-ikx}$$

- One choice of polarization vectors, consistent with $\nabla \cdot A = 0$, is

$$e^{(1)} = \frac{1}{\sqrt{k_1^2 + k_2^2}}(k_2, -k_1, 0), \qquad e^{(2)} = \frac{\vec{k}}{|\vec{k}|} \times e^{(1)}$$

- The Hamiltonian is

$$H = \sum_{k\lambda} \omega_k a_{k\lambda}^\dagger a_{k\lambda}$$

- A small aside on zero-point energy:

$$\delta_{ij} \langle 0|H|0\rangle = \langle 0| \left(K_i\, P_j - P_j\, K_i\right) |0\rangle = 0 \implies \text{No zero} - \text{point energy}$$

- The propagator can be obtained as

$$D_{ij}(x,y) = \langle 0|T\, A_i^T(x) A_j^T(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}\cdot\vec{k}}\right) \frac{i}{k^2 + i\epsilon} e^{-ik(x-y)}$$

  This is not manifestly covariant as it stands.

- The S-matrix functional can be written down as

$$\mathcal{F}(A) = \exp\left[\frac{1}{2}\int D_{ij}(x,y) \frac{\delta}{\delta A_i^T(x)} \frac{\delta}{\delta A_j^T(y)}\right] e^{iS_{int}}$$

- The first term which involves a photon propagator is quadratic in the currents,

$$\begin{aligned} \mathcal{F}^{(2)} \quad &= \quad i \int d^4x d^4y\, \frac{1}{2} J_0(x) G_C(\vec{x} - \vec{y}) \delta(x^0 - y^0) J_0(y) \\ &\quad - \frac{1}{2} \int d^4x d^4y\, J_i(x) D_{ij}(x,y) J_j(y) \end{aligned}$$

- The term involving the $k_i k_j / \vec{k}\cdot\vec{k}$ part of the propagator is

$$\begin{aligned}\int_{x,y} J_i(x)J_j(y)\int_k \frac{k_i k_j}{\vec{k}\cdot\vec{k}}\frac{e^{-ik(x-y)}}{k^2+i\epsilon} &= \int_{x,y}\partial_i J_i(x)\partial_j J_j(y)\int_k \frac{1}{\vec{k}\cdot\vec{k}}\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\\ &= \int_{x,y}\partial_0 J_0(x)\partial_0 J_0(y)\int_k \frac{1}{\vec{k}\cdot\vec{k}}\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\\ &= \int_{x,y} J_0(x)J_0(y)\int_k \frac{k_0^2}{\vec{k}\cdot\vec{k}}\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\end{aligned}$$

  We used the conservation of the current.

- $G_C(\vec{x}-\vec{y})$ is the Fourier transform of $-(1/\vec{k}\cdot\vec{k})$, so we can combine as

$$\begin{aligned}\mathcal{F}^{(2)} &= \frac{i}{2}\int_{x,y}\left[J_0(x)J_0(y)\int_k \frac{e^{-ik(x-y)}}{k^2+i\epsilon} - J_i(x)J_i(y)\int_k \frac{e^{-ik(x-y)}}{k^2+i\epsilon}\right]\\ &= \frac{1}{2}\int d^4x d^4y\, J^\mu(x)D_{\mu\nu}(x,y)J^\nu(y)\\ D_{\mu\nu}(x,y) &= \eta_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}e^{-ik(x-y)}\end{aligned}$$

Refresher on Field Quantization: Electrodynamics (*cont'd.*)

- Invariance under $A_\mu \to A_\mu + \partial_\mu \theta$ (and the associated conservation law $\partial_\mu J^\mu = 0$) are crucial to
  - Eliminate redundancy
  - Correctly implement all equations of motion
  - Obtain a Lorentz covariant answer

- Thus the action for quantum electrodynamics would involve the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$,

$$\mathcal{S} = \int d^4x\, \bar{\psi}(i\gamma \cdot D - m)\psi = \bar{\psi}(i\gamma \cdot \partial - m)\psi + e\bar{\psi}\gamma^\mu \psi A_\mu$$

- This has the gauge invariance

$$\mathcal{S}(\psi', \bar{\psi}', A') = \mathcal{S}(\psi, \bar{\psi}, A), \qquad A'_\mu = A_\mu + \partial_\mu \theta, \qquad \psi' = e^{ie\theta}\,\psi$$

- The interaction part of the Lagrangian has the form of $A^\mu J_\mu,\ J^\mu = e\bar{\psi}\gamma^\mu \psi.$

- The S-matrix functional for quantum electrodynamics (QED) is

$$\mathcal{F} = \hat{W}\, e^{ie \int A_\mu \bar{\psi} \gamma^\mu \psi}$$

$$\hat{W} = \exp\left[- \int \left(\frac{1}{2} D_{\mu\nu}(x,y) \frac{\delta}{\delta A_\mu(x)} \frac{\delta}{\delta A_\nu(y)} + \frac{\delta}{\delta \psi_r(x)} S_{rs}(x,y) \frac{\delta}{\delta \bar{\psi}_s(y)} \right) \right]$$

- We formulate a functional integral version. For a scalar field, propagators and amplitudes can be calculated from

$$Z[J] = \mathcal{N} \int [d\varphi]\, e^{-\mathcal{S}(\varphi) + i \int J\varphi}$$

- For a gauge theory, the same result holds except that we must integrate over physical (dynamical, nonredundant) degrees of freedom.

- Physical fields are A_μ modulo the identification $A_\mu + \partial_\mu \theta$. We need $[dA]_{phys}$ for the functional integral Z.

- By definition we have

$$[dA] = [dA]_{phys}\,[d\theta]$$

- We can then write

$$Z = \int [dA]_{phys}\, e^{-\mathcal{S}(A)} = \int [dA]_{phys}\,[d\theta]\,\delta[\theta]\,e^{-\mathcal{S}(A)}$$
$$= \int [dA]\,\delta[\theta]\,e^{-\mathcal{S}(A)}$$

- Consider separating A_μ into a (4-dim) transverse part A^T_μ and $\partial_\mu\theta$ as $A_\mu = A^T_\mu + \partial_\mu\theta$. then

$$\partial_\mu A^\mu = \Box\theta \implies \delta[\partial_\mu A^\mu] = \delta[\Box\theta] = \frac{1}{\det(-\Box)}\,\delta[\theta]$$

- So we finally have a manifestly covariant form

$$Z = \int [dA]\,\det(-\Box)\,\delta[\partial_\mu A^\mu]\,e^{-\mathcal{S}(A)}$$

What are Anomalies?

- We consider quantum field theory defined in terms of a functional integral

$$Z = \int [d\,\text{physical fields}]\; e^{-\mathcal{S}(\text{fields})}$$

- We are interested in quantum anomalies which arise because there is no regularization of this integral which preserves all the symmetries of the classical action.

- We will consider a general action of the form

$$\begin{aligned}\mathcal{S} &= \int \left[\frac{1}{4e_L^2}F_L^2 + \frac{1}{4e_R^2}F_R^2 + \bar{\psi}_L\gamma\cdot(\partial + L)\psi_L + \bar{\psi}_R\gamma\cdot(\partial + R)\psi_R\right] \\ &= \int \left[\frac{1}{4e_L^2}F_L^2 + \frac{1}{4e_R^2}F_R^2 + \bar{\psi}\gamma\cdot(\partial + V + \gamma^5 A)\psi\right]\end{aligned}$$

where $\psi_L = \frac{1}{2}(1+\gamma^5)\psi,\ \psi_R = \frac{1}{2}(1-\gamma^5)\psi,\ V = \frac{1}{2}(L+R),\ A = \frac{1}{2}(L-R).$

- These correspond to a general symmetry group $U(N)_L \times U(N)_R$; thus $L_\mu = -iT^A L_\mu^A, R_\mu = -iT^A R_\mu^A, T^A.$

What are Anomalies? (*cont'd.*)

- The field strength tensors have the usual form

$$F^A_{L\mu\nu} = \partial_\mu L^A_\nu - \partial_\nu L^A_\mu + f^{ABC} L^B_\mu L^C_\nu \qquad F^A_{R\mu\nu} = \partial_\mu R^A_\nu - \partial_\nu R^A_\mu + f^{ABC} R^B_\mu R^C_\nu$$

$$\begin{aligned} F^A_{V\mu\nu} &= \partial_\mu V^A_\nu - \partial_\nu V^A_\mu + f^{ABC} V^B_\mu V^C_\nu + f^{ABC} A^B_\mu A^C_\nu \\ F^A_{A\mu\nu} &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f^{ABC} (V^B_\mu A^C_\nu - V^B_\nu A^C_\mu) \end{aligned}$$

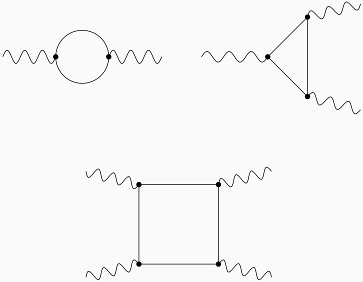
- Although a bit cumbersome, we can regularize using

$$\mathcal{S}_{Reg} = -\sum_{L,R} \int \mathrm{Tr} \left(F_{\mu\nu} \frac{(-D^2)}{\Lambda^2} F^{\mu\nu} \right), \qquad G \sim \frac{\Lambda^2}{p^4 + \Lambda^2 p^2} \sim \frac{\Lambda^2}{p^4}$$

- This takes care of gauge boson loops, but fermion one-loop diagrams are not regularized by this $\Longrightarrow$ Fermion loops can give anomalies.

What are Anomalies? (*cont'd.*)

- The potential diagrams for anomalies are



- Under a charge conjugation

$$\int d^4x\,\bar{\psi}\gamma\cdot(\partial + V + \gamma^5 A)\psi = \int d^4x\,\bar{\psi}^c\gamma\cdot(\partial -$$

$\Longrightarrow VVA$, AAA are the diagrams to worry about.

- Instead of evaluating diagrams, we use a functional integral method due to Fujikawa.

- First consider the fermion functional integral with only vector fields

$$Z = \int [d\psi d\bar{\psi}]\, e^{-\mathcal{S}(\psi,\bar{\psi})} \qquad \mathcal{S}(\psi,\bar{\psi}) = \int d^4x\, \bar{\psi}\gamma\cdot(\partial + V)\psi$$

- The classical action has the chiral $U(1)$ symmetry

$$\psi \to e^{-i\gamma_5\theta}\,\psi, \qquad \bar{\psi} \to \bar{\psi}\, e^{-i\gamma_5\theta}$$

for constant (spacetime-independent) θ.

- We make a change of variables in the functional integral with $\theta(x)$,

$$\begin{aligned} Z &= \int [d\psi' d\bar{\psi}']\, e^{-\mathcal{S}(\psi',\bar{\psi}')} \\ &= \int [d\psi d\bar{\psi}]\ \det(e^{2i\gamma_5\theta})\, e^{-\mathcal{S}(\psi,\bar{\psi})}\ \exp\left[-\int d^4x\, \theta(x)\partial_\mu J_{\mu 5}\right] \\ &= \int [d\psi d\bar{\psi}]\, e^{-\mathcal{S}(\psi,\bar{\psi})} \exp\left[2\,i\,\mathrm{Tr}(\gamma_5\theta) - \int d^4x\, \theta\partial_\mu J_{\mu 5}\right] \\ J_{\mu 5} &= i\bar{\psi}\gamma_\mu\gamma_5\psi \end{aligned}$$

Anomalies: Calculation (*cont'd.*)

- Since Z is unaltered by the change of variables, we get

$$\int [d\psi d\bar{\psi}]\, e^{-\mathcal{S}(\psi,\bar{\psi})} \left[\int d^4x\, \theta\partial_\mu J_{\mu 5} \;-\; 2i\mathrm{Tr}(\gamma_5\theta)\right] = 0$$

This is the basic Ward-Takahashi identity.

- The trace involves a functional trace as well, and can be evaluated by regularization as

$$\mathrm{Tr}(\gamma_5\theta) = \lim_{M\to\infty} \int d^4x\, \mathrm{Tr}\langle x|\gamma_5\, e^{(\gamma\cdot D)^2/M^2}|x\rangle$$

- Using $(\gamma\cdot D)^2 = D^2 + \frac{1}{2}\gamma_\mu\gamma_\nu\, F^{\mu\nu}$, we get

$$\begin{aligned}
\mathrm{Tr}(\gamma_5\theta) &= \int d^4x\, \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma_5 \epsilon^{-p^2/M^2}\, \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \frac{1}{2!\, 2^2\, M^4} F_{\mu\nu}F_{\alpha\beta}\right) \\
&= \frac{1}{32\pi^2} \int d^4x\, \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(\theta\, F_{\mu\nu}F_{\alpha\beta}) \\
&= \frac{1}{16\pi^2} \int d^4x\, \theta\, \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})
\end{aligned}$$

- Thus the WT identity becomes

$$\int [d\psi d\bar{\psi}]\, e^{-\mathcal{S}(\psi,\bar{\psi})} \int d^4x\, \theta \left[\partial_\mu J_{\mu 5} - \frac{i}{8\pi^2}\, \mathrm{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) \right] = 0$$

This shows the breaking of conservation of axial current by the quantum effects.

- For a full nonabelian case, the regularization has to be done a bit more carefully.

We can use the ζ-function regularization:

$$\mathrm{Tr}(\gamma_5 \theta) = \int d^4x\, \mathrm{Tr}(\gamma_5 \theta(x)\, \delta^{(4)}(x-y)) \Big]_{y\to x} = \int d^4x \lim_{s\to 0,\, y\to x} \mathrm{Tr}(\theta\, \zeta(s,x,y))$$

where the ζ-function is defined by

$$\zeta(s,x,y) = \sum_n \frac{\phi_n(x)\, \phi_n^\dagger(y)}{\lambda_n^{2s}}, \qquad \gamma \cdot D\, \phi_n = i\lambda_n\, \phi_n$$

Anomalies: Calculation (*cont'd.*)

- The ζ-function has an expression in terms of the so-called heat kernel,

$$\zeta(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1}\, h(\tau,x,y) \qquad h(\tau,x,y) = \langle x| e^{\tau(\gamma\cdot D)^2} |y\rangle$$

- The heat kernel has a short-distance expansion in the form,

$$h(\tau,x,y) = \frac{1}{16\pi^2\tau^2} e^{-(x-y)^2/4\tau} \sum_{n=0} \tau^n\, a_n(x,y)$$

with $\zeta(0,x,x) = a_2/(16\pi^2)$.

- Calculating a_2 and taking the trace

$$\begin{aligned}\mathrm{Tr}(\gamma_5\theta) \quad = \quad -\frac{1}{8\pi^2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta}\, \mathrm{Str}\Bigg[\theta\Bigg(&\frac{1}{4}F_{V\mu\nu}F_{V\alpha\beta} + \frac{1}{12}F_{A\mu\nu}F_{A\alpha\beta} \\ &-\frac{2}{3}(A_\mu A_\nu F_{V\alpha\beta} + A_\mu F_{V\nu\alpha}A_\beta + F_{V\mu\nu}A_\alpha A_\beta) \\ &+\frac{8}{3}A_\mu A_\nu A_\alpha A_\beta\Bigg)\Bigg]\end{aligned}$$

Anomalies: Properties

- The above expression gives the full nonabelian anomaly. It is in terms of F_V and F_A, sometimes referred to as the Bardeen form of the anomaly.

- We can express the anomaly as a nonzero change of the effective action under the symmetry transformation as

$$\delta_\xi \Gamma \equiv G(\xi)$$

- If we only have left-handed gauge fields and left-chiral fermions, the Bardeen expression reduces to

$$\delta_\xi \Gamma = G(\xi) = -\frac{1}{24\pi^2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta}\mathrm{Str}\left[\partial_\mu \xi \left(L_\nu \partial_\alpha L_\beta + \frac{1}{2} L_\nu L_\alpha L_\beta\right)\right]$$

- Under an infinitesimal gauge transformation with parameters ξ^A,

$$L_\mu^A \to L_\mu^A + (D_\mu \xi)^A = L_\mu^A + \partial_\mu \xi^A + f^{ABC} L_\mu^B \xi^C$$

Anomalies: Properties (*cont'd.*)

- This corresponds to the functional transformation

$$\delta_\xi = \int d^4x \, (D^\mu \xi)^A(x) \frac{\delta}{\delta L_\mu^A(x)}$$

- These obey the identity

$$\delta_\xi \, \delta_{\xi'} \; - \; \delta_{\xi'} \, \delta_\xi \; - \; \delta_{\xi \times \xi'} = 0$$

which is just the expression of the group composition law.

- This implies that $G(\xi)$ should obey the integrability or (Wess-Zumino) consistency conditions

$$\delta_\xi G(\xi') - \delta_{\xi'} G(\xi) - G(\xi \times \xi') = 0$$

- The expression we have, namely,

$$G(\xi) = -\frac{1}{24\pi^2} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \mathrm{Str} \left[\partial_\mu \xi \left(L_\nu \partial_\alpha L_\beta + \frac{1}{2} L_\nu L_\alpha L_\beta \right) \right]$$

satisfies these conditions.

Anomalies: Properties (*cont'd.*)

- Can we get rid of the anomaly by redefining Γ?

  A true anomaly is one for which

$$\delta_\xi G(\xi') - \delta_{\xi'} G(\xi) - G(\xi \times \xi') = 0, \qquad G(\xi) \neq \delta_\xi W$$

  The anomaly we found cannot be eliminated. Its form can be modified to some extent by adding counterterms.

- Can we live with an anomaly?
  - If there is an anomaly in a gauge symmetry, the theory loses unitarity; so we must eliminate it by choice of representations for matter fields.
  - If there is an anomaly in a global (non-gauge) symmetry, there is no inconsistency; but there are physical consequences.

Anomalies: Properties (*cont'd.*)

- To see how to cancel out anomalies, we need the group structure. Since $L_\mu = -iT^A L_\mu^A$, $\xi = -iT^A \xi^A$, we get
$$\delta_\xi \Gamma = -\frac{i}{24\pi^2}\, d^{ABC} \int d^4x\, \epsilon^{\mu\nu\alpha\beta} \left[\partial_\mu \xi^A \left(L_\nu^B \partial_\alpha L_\beta^C + \frac{1}{4} f^{CRS} L_\nu^B L_\alpha^R L_\beta^S\right)\right]$$
where $d^{ABC} = \mathrm{Str}(T^A T^B T^C)$. d^{ABC} is the symmetric rank 3 invariant of the algebra of the generators of the transformation.

- This is zero for all groups and all representations except for the $U(1)$ and $SU(N)$ groups with $N \geq 3$.

- The anomaly has opposite signs for the left and right handed fields, since $R_\mu = V_\mu - A_\mu$ as opposed to $L_\mu = V_\mu + A_\mu$ and we have an odd number of A's in the diagrams.

- Anomaly is in the imaginary part of the action. Γ is usually real in our Euclidean calculation, but with anomaly, $\delta_\xi \Gamma$ is imaginary.

Physics Implication I:

Anomaly constrains the gauge groups and representations for a consistent theory

- b^3-type terms

$$t^a = \frac{1}{2}\tau^a \;\Rightarrow\; d^{abc} = \mathrm{Str}\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\frac{\tau^c}{2}\right) = \frac{1}{8}\mathrm{Tr}(\tau^a\delta^{bc}) = 0$$

- $b^2\ c$-type terms

  In this case we need $d^{Yab} = \frac{1}{4}\mathrm{Str}(Y\tau^a\tau^b) = \frac{1}{4}\delta^{ab}\mathrm{Tr}(Y)$

$$\mathrm{Tr}(Y) = \underbrace{(-1)}_{\nu_L} + \underbrace{(-1)}_{e_L} + \left(\underbrace{\frac{1}{3}}_{u_L} + \underbrace{\frac{1}{3}}_{d_L}\right)\times 3 = 0$$

- c^3-type terms
  The c^3 anomaly is given by $\mathrm{Tr}(Y^3)$ and for this, both left and right fermions can contribute. We get

$$\mathrm{Tr}(Y^3) = \Bigg[\underbrace{(-1)}_{\nu_L} + \underbrace{(-1)}_{e_L} + \Bigg(\underbrace{\frac{1}{27}}_{u_L} + \underbrace{\frac{1}{27}}_{d_L}\Bigg) \times 3\Bigg] - \Bigg[\underbrace{-8}_{e_R} + \Bigg(\underbrace{\frac{64}{27}}_{u_R} - \underbrace{\frac{8}{27}}_{d_R}\Bigg) \times 3\Bigg]$$

$$= \left(-\frac{16}{9}\right)_L - \left(-\frac{16}{9}\right)_R = 0$$

  In this case, the cancellation involves quarks and leptons and both chiralities.

- Among the global symmetries with anomalies are the baryon and lepton numbers.

Physics from Anomalies (*cont'd.*)

- Lepton number is defined by the transformation

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \to e^{i\alpha} \begin{pmatrix} \nu \\ e \end{pmatrix}_L , \qquad e_R \to e^{i\alpha} e_R, \qquad u \to u, \qquad d \to d$$

Leptons $\nu,\ e$ have lepton number equal to 1, quarks have no lepton number.

- Baryon number corresponds to the transformation

$$\nu \to \nu, \quad e \to e, \qquad \begin{pmatrix} u \\ d \end{pmatrix}_L \to e^{i\beta/3} \begin{pmatrix} u \\ d \end{pmatrix}_L , \qquad (u_R, d_R) \to e^{i\beta/3} (u_R, d_R)$$

- These are both anomalous symmetries with

$$\begin{aligned} \delta_{\alpha,\beta}\Gamma &= -i \int d^4x \, (\alpha(x) + \beta(x)) \left(c_2[b] - 2\, c_2[c] \right) \\ c_2[b] &= \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr} \, F_{\mu\nu}(b) F_{\alpha\beta}(b) = -\frac{1}{64\pi^2} \epsilon^{\mu\nu\alpha\beta} G^a_{\mu\nu} G^a_{\alpha\beta} \\ c_2[c] &= -\frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} f_{\mu\nu} f_{\alpha\beta} \end{aligned}$$

Physics from Anomalies (*cont'd.*)

- The field strengths in the previous expression are

$$G^a_{\mu\nu} = \partial_\mu b^a_\nu - \partial_\nu b^a_\mu + \epsilon^{abc} b^b_\mu b^c_\nu, \qquad f_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu$$

- The axial $U(1)$ transformation is another global symmetry with anomalies in QCD. This corresponds to

$$Q'_L = e^{i\lambda} Q_L, \qquad Q'_R = e^{-i\lambda} Q_R, \qquad Q' = e^{i\lambda\gamma^5}\, Q$$

- The anomaly is given by

$$\begin{aligned} \delta_\lambda \Gamma &= -i\, 2N_f \int d^4x\, \lambda\, \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\left(F_{\mu\nu} F_{\alpha\beta}\right) = i\, 2N_f \int d^4x\, \lambda\, \rho[A] \\ \rho[A] &= -\frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\left(F_{\mu\nu} F_{\alpha\beta}\right) \end{aligned}$$

Physics Implication II:

Anomaly in global symmetries have observable consequences, e.g. $\pi^0 \to 2\,\gamma$

- Consider the transformation

$$u \to \exp(i\gamma_5\varphi)\,u, \qquad d \to \exp(-i\gamma_5\varphi)\,d, \qquad \begin{pmatrix} u \\ d \end{pmatrix} \to e^{i\gamma_5\tau_3\varphi} \begin{pmatrix} u \\ d \end{pmatrix}$$

- In terms of the Goldstone fields (meson fields) U, we have

$$U \to U' = g_L\,U\,g_R^\dagger, \qquad U \sim e^{i\pi^0\tau_3/f_\pi} \implies \pi^0 \to \pi^0 + 2\,f_\pi\varphi$$

- The up and down quarks have electrical charges $\frac{2}{3}e$ and $-\frac{1}{3}e$, respectively, and there are three colors of each.

$$\delta_\varphi\Gamma = -i\frac{e^2}{8\pi^2}\int d^4x\,\varphi\,F_{\mu\nu}\tilde{F}_{\mu\nu}\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]\times 3 = -i\frac{\alpha}{2\pi}\int d^4x\,\frac{\delta\pi^0}{2f_\pi}F_{\mu\nu}\tilde{F}_{\mu\nu}$$

$$\Gamma = -i\frac{\alpha}{4\pi f_\pi}\int d^4x\,\pi^0 F_{\mu\nu}\tilde{F}_{\mu\nu} = -i\frac{\alpha}{\pi f_\pi}\int d^4x\,\vec{E}\cdot\vec{B}\,\pi^0$$

Physics Implication III:

Anomaly can solve the axial $U(1)$ problem in QCD related to the mass of the η'

- Even though pseudoscalar mesons are only pseudo-Goldstone bosons, the mass η' (~ 958 MeV) is abnormally high and violates the bound $m_{\eta'} \leq \sqrt{3}\, m_\pi$.
- The $U_A(1)$ axial anomaly can be represented in terms of the meson fields by

$$S_{\text{eff}} = \frac{i}{2}\left(\log\det U - \log\det U^\dagger\right)\left(\frac{1}{16\pi^2}\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}\right) = \frac{\sqrt{2N_{\text{f}}}}{f_\pi}\,\eta'\,\rho[A,x]$$

- If the two-point function for ρ has the expansion

$$\langle\rho(x)\rho(y)\rangle = m_0^4\,\delta^{(4)}(x-y) + \mathcal{O}(\partial), \qquad m_0 \neq 0$$

then this effective action gives a mass for the η',

$$S_{\eta'\,\text{mass}} = \frac{1}{2}\,\left[\frac{2\,N_{\text{f}}\,m_0^4}{f_\pi^2}\right]$$

$U_A(1)$ problem: Why do we need instantons?

- However, this needs instantons.

$$\begin{aligned} \partial_\mu J_A^\mu &= 2\,N_\mathrm{f}\,\frac{1}{32\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,F_{\mu\nu}F_{\alpha\beta} = -2N_\mathrm{f}\,\partial_\mu K^\mu \\ K^\mu &= -\frac{1}{8\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(A_\nu\partial_\alpha A_\beta + \frac{2}{3}A_\mu A_\alpha A_\beta\right) \end{aligned}$$

- There is a conserved current $J_A^\mu + 2\,N_\mathrm{f}\,K^\mu$, but K^μ is not gauge-invariant.

- Is $\int d^3x\,K^0 = \int d^3x\,\omega_3(A)$ gauge-invariant ?

$$\begin{aligned} \int\left[\omega_3(A^g) - \omega_3(A)\right] = &-\frac{1}{8\pi^2}\int \epsilon^{ijk}\partial_i\mathrm{Tr}(g^{-1}\partial_j g\,A_k) \\ &-\frac{1}{24\pi^2}\int \epsilon^{ijk}\mathrm{Tr}(g^{-1}\partial_i g\,g^{-1}\partial_j g\,g^{-1}\partial_k g) \end{aligned}$$

- The last term is nonzero if we have instantons.

- $K^0 \equiv \omega_3(A)$ is the Chern-Simons 3-form.

Differential forms

- We combine the gauge fields with dx^μ to write it as 1-form,

$$A = (-iT^a)A^a_\mu \, dx^\mu = A_\mu \, dx^\mu$$

- Advantage: Change in components $A'_\mu = A_\nu(\partial x^\nu / \partial x'^\mu)$ under coordinate transformation is cancelled by the transformation of $dx'^\mu = dx^\alpha(\partial x'^\mu / \partial x^\alpha)$. The 1-form is coordinate invariant.

- When we take derivatives, we must antisymmetrize indices to keep this property,

$$dA = \frac{\partial}{\partial x^\mu} A_\nu \, dx^\mu \wedge dx^\nu = \frac{1}{2}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) dx^\mu \wedge dx^\nu$$

- The field strength tensor for the nonabelian gauge field is

$$\begin{aligned} F &= dA + A \wedge A = \frac{1}{2}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + A_\mu A_\nu - A_\nu A_\mu\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \, dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \end{aligned}$$

Differential forms (*cont'd.*)

- If we are in four dimensions, we can write

$$F\,F = F \wedge F = \frac{1}{4} F_{\mu\nu} F_{\alpha\beta}\, dx^\mu dx^\nu dx^\alpha dx^\beta = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\, d^4x$$

- Some other important properties:
  - For the product of a p-form α and a q-form β,

$$\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha$$

  - Further, since antisymmetrized derivatives vanish,

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu} \Phi\, dx^\mu \wedge dx^\nu = 0 \implies d^2 = 0$$

  - Using this, we find that F should obey the Bianchi identity

$$dF = F\,A - A\,F$$

Differential forms (*cont'd.*)

- Gravitational fields are treated in a similar way, with

$$A \to \Omega = \text{spin connection}, \qquad F \to \mathcal{R} = \text{Riemann curvature}$$

- Σ_{ab} generate Lorentz transformations, so we have

$$\Omega = (-i\Sigma_{ab})\, \Omega_\mu^{ab}\, dx^\mu$$

- The curvature is given by

$$\begin{aligned} \mathcal{R} &= d\Omega + \Omega\,\Omega = \frac{1}{2}\left(\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu]\right) dx^\mu\, dx^\nu \\ &= (-i\Sigma_{ab})\, \frac{1}{2}\, \mathcal{R}_{\mu\nu}^{ab}\, dx^\mu\, dx^\nu \end{aligned}$$

- Because the forms do not involve metric and are invariant under coordinate transformations, many topological properties are expressed as integrals of combinations of forms known as characteristic classes.

Characteristic Classes

- Chern classes are defined by

$$c(F) = \det\left(1 + i\frac{F}{2\pi}\right) = 1 \,+\, c_1(F) \,+\, c_2(F) \,+\cdots$$

c_1 is callled the first Chern class; c_2 is the second Chern class and so on.

- Explicitly

$$\begin{aligned} c_1(F) &= \frac{i}{2\pi}\,\mathrm{Tr}F \\ c_2(F) &= \frac{1}{8\pi^2}\left[\mathrm{Tr}(F\wedge F) - (\mathrm{Tr}F)\wedge(\mathrm{Tr}F)\right] \end{aligned}$$

- Chern character $Ch(F)$ is another characteristic class defined by

$$\begin{aligned} Ch(F) &= \mathrm{Tr}\exp\left(i\frac{F}{2\pi}\right) = 1 \,+\, Ch_1(F) \,+\, Ch_2(F) \,+\cdots \\ Ch_1(F) &= \frac{i}{2\pi}\mathrm{Tr}(F), \qquad Ch_2(F) = -\frac{1}{8\pi^2}\mathrm{Tr}(F\,F),\cdots \end{aligned}$$

Characteristic Classes (*cont'd.*)

- The $\hat{A}$-genus is anothe characteristic class defined in terms of the Riemann curvature two-form $\mathcal{R}$ by

$$\begin{aligned}
\hat{A}(\mathcal{R}) &= \prod_i \frac{x_i/2}{\sinh(x_i/2)} = \prod_i \left(1 - \frac{1}{24}x_i^2 + \cdots\right) \\
&= \left(1 - \frac{1}{24}\sum_i x_i^2 + \cdots\right) \\
&= 1 + \frac{1}{24}\frac{1}{8\pi^2}\mathrm{Tr}(\mathcal{R} \wedge \mathcal{R}) + \cdots
\end{aligned}$$

- The x_i's are defined by

$$\frac{\mathcal{R}}{2\pi} = \begin{bmatrix} 0 & x_1 & 0 & 0 & . & . \\ -x_1 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & x_2 & . & . \\ 0 & 0 & -x_2 & 0 & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

The Index Theorem

- For us, characteristic classes are important because they are related to the index theorem and to anomalies.

- Let $\mathcal{M} = 2n$-dimensional spin manifold. We have the Dirac algebra

$$\gamma_\mu \, \gamma_\nu \; + \; \gamma_\nu \, \gamma_\mu \; = \; 2 \, \delta_{\mu\nu} \mathbf{1}$$

  These matrices can be represented explicitly as $(2^n \times 2^n)$-matrices.

- The chirality matrix is given by

$$\gamma_{2n+1} = i^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$$

- We can define the chiral projections of a Dirac spinor by

$$\psi_\pm = \frac{1}{2}(1 \pm \gamma_{2n+1}) \, \psi, \qquad \gamma_{2n+1} \, \psi_\pm = \pm \, \psi_\pm$$

- For any gauge field or gravitational background, let

$$\begin{aligned} n_+ \quad &= \quad \text{Number of zero modes of } \gamma \cdot D \text{ of positive chirality} \\ n_- \quad &= \quad \text{Number of zero modes of } \gamma \cdot D \text{ of negative chirality} \end{aligned}$$

The Index Theorem (*cont'd.*)

- The Atiyah-Singer index theorem gives the result

$$n_+ - n_- = \int_{\mathcal{M}} \hat{A}(\mathcal{R}) \wedge Ch(F)$$

  This is essentially the trace of γ_{2n+1}. The rule is to expand the right side and pick the term with $2n$ dx's.

- The axial $U(1)$ transformation is given by

$$\psi \to \exp(-i\gamma_{2n+1}\theta)\, \psi$$

- Since the anomaly for this transformation involves the trace of γ_{2n+1}, we can write it generally as

$$2\,\mathrm{Tr}(\gamma_{2n+1}\theta) = 2 \int_{\mathcal{M}} \theta\, \hat{A}(\mathcal{R}) \wedge Ch(F)$$

Index density $\Longleftrightarrow$ Anomaly for the axial $U(1)$ symmetry for a Dirac spinor

The Index Theorem & the Nonabelian Anomaly

- One can get nonabelian anomaly also from the index density.

$$\left.\begin{array}{r}\text{Index density in } 2n+2 \\ \text{dimensions}\end{array}\right\} = d\left\{\begin{array}{c}\text{Chern} - \text{Simons form in } 2n+1 \\ \text{dimensions}\end{array}\right\}$$

$$\left.\begin{array}{r}\text{Gauge, Lorentz variation of} \\ C.S._{2n+1}\end{array}\right\} = d\left[\omega^1_{2n}\right]$$

- The nonabelian anomaly for a chiral Dirac fermion in $2n$ dimensions is

$$\delta\Gamma = \int_{\mathcal{M}} \omega^1_{2n}$$

For 4 dim. we start with the index density in 6 dimensions,

$$\mathcal{I}_6 = -\frac{i}{48\pi^3}\mathrm{Tr}F^3 + \frac{i}{384\pi^3}\mathrm{Tr}F\,\mathrm{Tr}\mathcal{R}^2$$

The Index Theorem & the Nonabelian Anomaly (*cont'd.*)

- Consider F^3 term first.

$$-\frac{i}{48\pi^3}\mathrm{Tr}F^3 = d\,\omega_5$$

$$\omega_5(A) = -\frac{i}{48\pi^3}\mathrm{Tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right)$$

- The gauge transformation is $A \to A^g = g\,A\,g^{-1} - dg\,g^{-1}$.

- The CS form changes as

$$\omega_5(A^g) - \omega_5(A) = d\alpha_4 \;+\; \frac{i}{480\pi^3}\mathrm{Tr}(dg\,g^{-1})^5 \qquad (\text{WZ term})$$

$$\begin{aligned}\alpha_4 &= -\frac{i}{48\pi^3}\,\mathrm{Tr}\Big[g^{-1}dg\;\left(\frac{1}{2}AdA + \frac{1}{2}dAA + \frac{1}{2}A^3\right) \\ &\qquad + \frac{1}{4}(g^{-1}dg\;A\,g^{-1}dg\;A) \;-\; \frac{1}{2}(g^{-1}dg)^3A\Big] \\ &\approx \frac{i}{48\pi^3}\mathrm{Tr}\left[d\theta\;\left(AdA + \tfrac{1}{2}A^3\right)\right] + \text{total derivative}\end{aligned}$$

- This agrees with the anomaly calculated earlier.

- How do we integrate the last term in $\omega_5(A^g) - \omega_5(A)$? We need to extend the matrix g into a fifth dimension, $g \to U$.

- The version of anomaly for finite transformation is

$$\det(\gamma \cdot D^g) = \det(\gamma \cdot D)\ \exp\left(-2\pi i \int_{\mathcal{D}} [\omega_5(A^U) - \omega_5(A)]\right)$$

$$\Delta\Gamma = 2\pi i \int_{\mathcal{D}} [\omega_5(A^U) - \omega_5(A)]$$

- The 5-form term in $\omega_5(A^U) - \omega_5(A)$ is

$$\Omega^{(5)} = \frac{i}{480\pi^3}\mathrm{Tr}(dU\, U^{-1})^5 = \frac{i}{480\pi^3}\mathrm{Tr}(dUU^{-1}\, d(dUU^{-1})\, d(dUU^{-1}))$$
$$= \frac{i}{480\pi^3}\mathrm{Tr}\left[\partial_{\mu_1}UU^{-1}\,\partial_{\mu_2}(\partial_{\mu_3}UU^{-1})\,\partial_{\mu_4}(\partial_{\mu_5}UU^{-1})\right] dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_5}$$

The Index Theorem & the Nonabelian Anomaly (*cont'd.*)

- The curl (or d) of the integrand in $\Omega^{(5)}$ vanishes, but it cannot be written as a total derivative.

- Consider two different extensions U_1, U_2. The difference in the finite anomaly is given by

$$\Delta\Gamma(U_1) - \Delta\Gamma(U_2) = \oint_{S^5} \Omega^{(5)}(U) \tag{1}$$

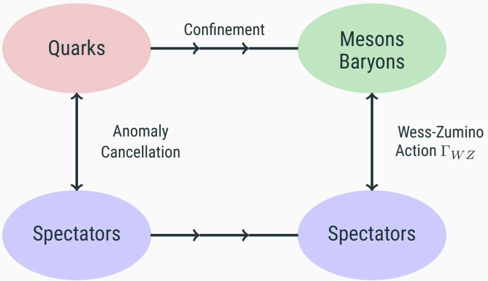
  where $U = U_1$ for the upper hemisphere and $U = U_2$ for the lower hemisphere. On the equator (which is spacetime $\mathcal{M}$) $U_1 = U_2 = g$, so there is no difficulty of continuity of the functions on S^5.

- The integral in (1) gives the winding number of the map $U:\ S^5 \to G$ considered as an element of $\Pi_5(G)$.

- This is an integer and so the ambiguity of different extensions will not affect equation $e^{-\Gamma}$ or the transformation law for $\det \gamma \cdot D$.

Flavor Anomalies of QCD

- Should we reproduce flavor anomalies of QCD in the effective action for low energies?



- We must match anomalies between different phases of a theory because
  - Anomalies are topological in nature, not affected by energy scales.
  - They can also be obtained in low energy physics from unitarity and cross sections

Flavor Anomalies of QCD (*cont'd.*)

Physics Implication IV:

The Wess-Zumino term can be used to represent flavor anomalies of QCD

- QCD has a chiral flavor symmetry for the u, d, s quarks, if we neglect weak interaction effects (including quark masses).
- This approximate symmetry $SU_L(3) \times SU_R(3)$ is spontaneously broken by strong forces to the diagonal $SU_V(3)$.
- The corresponding Goldstone bosons (identified with the pseudoscalar mesons) can be represented by the group element $U \in SU(3),\ U = \exp\left(i \frac{\sqrt{2}\,M}{f_\pi}\right)$,

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\,\eta \end{pmatrix}$$

Flavor Anomalies of QCD (*cont'd.*)

- The effective low energy ($\lesssim 1$ GeV) action is given by

$$S_{\text{eff}} = \frac{f_\pi^2}{4} \int d^4x \, \text{Tr} \left(\partial_\mu U^\dagger \, \partial^\mu U \right) \; + \; \Gamma_{WZ}$$

$$\Gamma_{WZ} = -i \frac{N}{240\pi^2} \int_D (\text{Tr}(dU \, U^{-1})^5 + 2\pi N [\alpha_4(U^{-1}, A_L) - \alpha_4(U, A_R)] \; + \Gamma_{count}$$

- This contains the $\pi^0 \to 2\,\gamma$ we discussed, and many other processes such as $K^+K^- \to \pi^+\pi^-\pi^0$.

- But more importantly, it leads to a picture of baryons as solitons made of mesons.

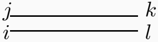
The Wess-Zumino term can change and spin and statistics for solitons.

Baryons as Solitons

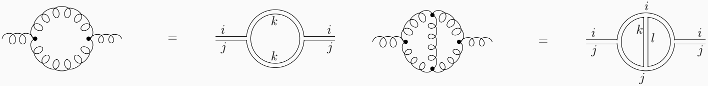
- The Feynman diagrams generated by an $SU(N)$ gauge theory can be classified by the power of N, by taking the coupling constant $g \sim \frac{1}{\sqrt{N}}$; color traces generate a factor of N.

- For this we use a double line representation

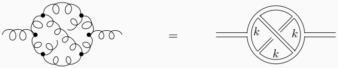
$$\langle A_{\mu ij}(x) A_{\nu kl}(y) \rangle \quad = $$



- These are of order N^0



- This is of order N^{-2}



Baryons as Solitons (*cont'd.*)

- So we can write the effective action for a gauge theory as

$$\Gamma = N^2\,\Gamma_0 + N^0\,\Gamma_1 + N^{-2}\,\Gamma_2 + \cdots = \sum_h N^{2-2h}\,\Gamma_h$$

  $1/N$ plays the role of a "coupling constant".

- The large N term seems to capture many nonperturbative features of the theory.

- In this expansion, baryon masses $\sim N$, since there are N quarks in it, allowing $N(N-1)/2$ pair-interactions which go like $g^2 \sim \frac{1}{N}$.

- This is typical of nonperturbative particle-like solutions, or solitons.

- But the low energy limit is known, it is the theory of mesons based on U.

- So this leads to the idea:

At low energies, baryons can be viewed as solitons made of the meson fields

Baryons as Solitons (*cont'd.*)

- This idea is due to Skyrme, hence the name skyrmions for these solitons.

  But there are difficulties:

  - Are there stable solitons we can make of meson fields?
  - Baryons are fermions. How can we make a fermion from composites of bosons?
  - Baryons have spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, etc. How can we get half-integral spin from composing integral spins?

- The effective low energy action for meson fields is given by

$$\begin{aligned}\mathcal{S} &= \frac{1}{4} f_\pi^2 \int d^4x \, \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32\epsilon^2} \mathrm{Tr}([\partial_\mu U U^{-1}, \partial_\nu U U^{-1}]^2) \\ & \qquad - \frac{iN}{240\pi^2} \int_{\mathcal{D}} (\mathrm{Tr}(dU \, U^{-1})^5\end{aligned}$$

- The energy of a configuration $U(\vec{x})$ is given by

$$\mathcal{E} = \int d^3x \left[\frac{1}{4} f_\pi^2 \, \mathrm{Tr}(\partial_i U \partial_i U^\dagger) - \frac{1}{32\epsilon^2} \mathrm{Tr}([\partial_i U U^{-1}, \partial_j U U^{-1}]^2) \right]$$

Baryons as Solitons (*cont'd.*)

- Here $U(\vec{x})$ is a map: $\mathbb{R}^3 \to SU(3)$, with $U \to 1$ at spatial infinity. These are equivalent to maps $S^3 \to SU(3)$.

- Consider a 3-sphere with the standard embedding coordinates y_μ with $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$.

- We can find $\vec{x}$ such that (stereographic map)

$$y_4 = \frac{R^2 - |\vec{x}|^2}{R^2 + |\vec{x}|^2}, \qquad y_i = \frac{2Rx_i}{R^2 + |\vec{x}|^2}$$

- If we take a map $U(y)$ from S^3 to $SU(3)$, we can substitute these values and get a map $\mathbb{R}^3 \to SU(3)$, with spatial infinity corresponding to the south pole of S^3.

- These maps can be classified, all members within a class being continuously deformable to each other. These are called homotopy classes, in this case labeled by an integer, the winding number.

Baryons as Solitons (*cont'd.*)

- The winding number is given by

$$Q[U] = -\frac{1}{24\pi^2}\int d^3x\, \epsilon^{ijk}\mathrm{Tr}(U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U)$$

- It is easy to check that

$$Q[U_1 U_2] = Q[U_1] + Q[U_2]$$

- This proves many things.
  - If $U_2 \approx 1 + \Theta$, $Q[UU_2] = Q[U] \implies Q[U]$ is invariant under small continuous deformations.
  - $Q[U]$ does not depend on the metric of space, being the integral of a differential form. These two properties $\implies Q[U]$ is a topological invariant.
  - If $U^{(1)}$ has $Q = 1$, $(U^{(1)})^2$ has $Q = 2$, $(U^{(1)})^n$ has $Q = n$.

- The space of U's has disconnected sectors, each connected piece labeled by an integer.

- Minimize the energy for, say, $Q = 1$; that is a static soliton.

Baryons as Solitons (*cont'd.*)

- An example for $SU(2)$ is

$$U(x) = U_S(x) \equiv \exp\left(i\theta(r)\tau \cdot \hat{x}\right) = \cos\theta(r) + i\tau \cdot \hat{x}\sin\theta(r)$$

- This has

$$Q = \frac{1}{\pi}\left[\theta(0) - \theta(\infty)\right], \qquad \mathcal{E} = a f_\pi^2 R + \frac{b}{\epsilon^2 R}$$

- In $SU(3)$ we write

$$U_S(x) = \begin{pmatrix} \exp(i\theta(r)\tau \cdot \hat{x}) & 0 \\ 0 & 1 \end{pmatrix}$$

- This has "rotation symmetry",

$$U_S(\mathbf{R}(\alpha)x) = G\, U_S(x)\, G^\dagger, \qquad G = \begin{pmatrix} \exp(i\tau \cdot \alpha) & 0 \\ 0 & 1 \end{pmatrix}$$

- We can use

$$U(x,t) = A(t)\, U_S\, A^\dagger(t), \qquad A'(t) = V\, A(t), \qquad A'(t) = A(t)\, G$$

Baryons as Solitons (*cont'd.*)

- Using this ansatz in the action

$$\mathcal{S} = \int dt \left[-\frac{\alpha}{2}\{\mathrm{Tr}(t_i A^\dagger \partial_t A)\}^2 - \frac{\beta}{2}\{\mathrm{Tr}(t_k A^\dagger \partial_t A)\}^2 \; - i\frac{QN}{\sqrt{3}}\mathrm{Tr}(t_8 A^\dagger \partial_t A) \right]$$

- We have the property

$$\mathcal{S}(Ae^{it_8\lambda}) = \mathcal{S} \, + \, \frac{NQ}{2\sqrt{3}}\lambda \implies \Psi(Ae^{it_8\lambda}) = \Psi(A) \, \exp\left(i\frac{NQ}{2\sqrt{3}}\,\lambda \right)$$

- The wave functions can be generally written as

$$\Psi(A) = C_R \, \mathcal{D}^R(A)_{I,I_3,Y;I',I'_3,Y'} = C_R \, \langle I, I_3, Y | \, \hat{A} \, | I', I'_3, Y' \rangle$$

- We must choose $Y' = 1$ for $Q = 1$. Lowest dim. reps are **8** and **10**.
  - **8** $\implies I' = \frac{1}{2} \equiv J$; gives $SU(3)$ octet of spin-$\frac{1}{2}$ baryons
  - **10** $\implies I' = \frac{3}{2} \equiv J$; gives $SU(3)$ decuplet of spin-$\frac{3}{2}$ baryons.
  - Further, Q = baryon number.

Coadjoint Orbits and Fluids

- The contribution of the WZ term for skyrmions is part of a more general set of actions called coadjoint orbit actions

$$S = i \sum_{\alpha} w_s \int dt \, \mathrm{Tr}(h_s \, g^{-1} \dot{g})$$

g = some matrix, an element of some group G

h_s = diagonal generators of the Lie algebra, $\mathrm{Tr}(h_s \, h_{s'}) = \delta_{ss'}$

w_s = a set of numbers

Theorem

Quantization of this action gives a Hilbert space corresponding to one unitary irreducible representation of G with the highest weight $(w_1, w_2, \cdots, w_r)$.

- Useful for point-particles with nonabelian charges.

Coadjoint Orbits and Fluids (*cont'd.*)

- We have the correspondence

$$\text{Point-particle with mass and spin} \longleftrightarrow \text{UIR of Poincaré group}$$

$$\text{Point-particles with color charge} \longleftrightarrow \text{Extra UIR of color group}$$

- For example for a particle with $SU(2)$ color charge, the action is

$$\begin{aligned}\mathcal{S} &= \int dt \left[\frac{1}{2} m \dot{x}^2 - A_i^a Q^a \dot{x}_i - i\,\frac{n}{2}\,\mathrm{Tr}(\sigma_3\, g^{-1} \dot{g})\right] \\ &= \int dt \left[\frac{1}{2} m \dot{x}^2 - i\,\frac{n}{2}\,\mathrm{Tr}(\sigma_3\, g^{-1} D_0\, g)\right]\end{aligned}$$

$$Q^a = \tfrac{n}{4}\mathrm{Tr}(\sigma_3\, g^{-1} \sigma^a g) \qquad D_0 = \partial_0 + A_i^a \dot{x}_i (-i\sigma^a/2).$$

- We will consider a fluid version soon, but first consider quantizing the action for g.

Coadjoint Orbits and Fluids (*cont'd.*)

- Start with the action

$$\mathcal{S} = i\frac{n}{2}\int dt\ \operatorname{Tr}(\sigma_3\, g^{-1}\dot{g})$$

g = (2×2)-matrix $\in SU(2)$, $g = \exp(i\,(\sigma_i/2)\theta_i)$.

- Under $g \to g\,h$, $h = \exp(i\sigma_3\varphi/2)$,

$$\mathcal{S} \to \mathcal{S} - \frac{n}{2}\int dt\,\dot{\varphi} = \mathcal{S} - \frac{n\varphi}{2}$$

- The dynamics is actually restricted to $SU(2)/U(1)$ which is a two-sphere S^2.

- Parametrizing g as

$$g = \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}\begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \implies \mathcal{S} = i\frac{n}{2}\int dt\,\frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{1+z\bar{z}}$$

Coadjoint Orbits and Fluids (*cont'd.*)

- Strategy for quantizing: Take wave functions as functions of g and impose restrictions.

$$\Psi = \sum_j \sum_{a,b} C^{(j)}_{ab}\, \mathcal{D}^{(j)}_{ab}(g) = \sum_j \sum_{a,b} C^{(j)}_{ab}\, \langle a| e^{i \hat{J}_i \theta_i}\, |b\rangle$$

where $\hat{J}_i$ = angular momentum or $SU(2)$ generator in an arbitrary representation.

- From behavior of $\mathcal{S}$ under $g \to g\,h$, $h = \exp(i\sigma_3 \varphi/2)$.

$$\Psi\left(g\, e^{i \hat{J}_3 \varphi}\right) = \Psi(g)\ \exp\left(-i \frac{n}{2} \varphi\right)$$

$\implies |b\rangle = |j, -\frac{n}{2}\rangle$.

- The action has only one power of $\dot{z}$ or $\dot{\bar{z}}$. $\implies z, \bar{z}$ are phase space variables. Ψ can depend only on half of the phase space directions.

- Define right action on g by

$$R_i\, g = g\, \frac{\sigma_a}{2}$$

- The combinations $R_\pm = R_1 \pm iR_2$ are complex and conjugate to each other, these are the two derivatives on phase space.

- So we can set R_Ψ to zero to ensure dependence on only "half" of phase space coordinates,

$$R_-\, \Psi = R_- \sum_j \sum_{a,b} C_{ab}^{(j)} \langle a| e^{i\hat{J}_i\theta_i} |b\rangle = \sum_j \sum_{a,b} C_{ab}^{(j)} \langle a| e^{i\hat{J}_i\theta_i}\, \hat{J}_- |b\rangle = 0$$

- This means that $|b\rangle$ must also be the lowest weight state, so $|b\rangle = |\frac{n}{2}, -\frac{n}{2}\rangle$.

- There is only one representation,

$$\Psi = \sum_a C_{a,-\frac{n}{2}}^{(\frac{n}{2})}\, \mathcal{D}_{a,-\frac{n}{2}}^{(\frac{n}{2})}(g)$$

Coadjoint Orbits and Fluids (*cont'd.*)

- The action for many particles, labeled by λ, is

$$\mathcal{S} = -i \int dt \sum_{\lambda} w_{\lambda} \, \mathrm{Tr}\left(\sigma_3 g_{\lambda}^{-1} \dot{g}_{\lambda}\right)$$

- Take a limit to a continuous index λ by $\lambda \to \mathbf{x}$, $\sum_{\lambda} \to \int d^3\mathbf{x}/v$, $w_{\lambda}/v \to \rho_3(\mathbf{x})$

$$\mathcal{S} = -i \int d^4x \, \rho_3 \, \mathrm{Tr}(\sigma_3 g^{-1} \dot{g})$$

- Taking this as the crucial and leading term,

$$\mathcal{S} = -i \int d^4x \, j_3^{\mu} \, \mathrm{Tr}\left(\sigma_3 g^{-1} D_{\mu} g\right) - \int F(n_3) + S_{YM}$$

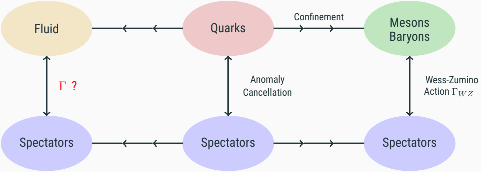
  This describes dynamics of nonabelian charge transport in a fluid, $n_3^2 = j_3^{\mu} j_{3\mu}$.

- With mass transport included, we get nonabelian magnetohydrodynamics given by

$$\mathcal{S} = \int d^4x \left[j^{\mu} \left(\partial_{\mu}\theta + \alpha \, \partial_{\mu}\beta\right) - i \sum_{s} j_s^{\mu} \mathrm{Tr}\left(h_s \, g^{-1} D_{\mu} g\right) - F(n, n_s, \cdots)\right]$$

Anomalies and Fluids

- Should we have an anomaly term?



- Yes, and we can use Γ_{WZ} with a reinterpretation. The action for the fluid phase of the standard model is

$$\begin{aligned}\mathcal{S} \quad = \quad & \int \Big[j^\mu \, (\partial_\mu \theta + \alpha \, \partial_\mu \beta) - i \, j_3^\mu \mathrm{Tr} \left(t_3 g_L^{-1} D_\mu g_L \right) - i \, j_8^\mu \mathrm{Tr} \left(t_8 g_L^{-1} D_\mu g_L \right) \\ & + j_0^\mu \partial_\mu \theta_B - i \, k_3^\mu \mathrm{Tr} \left(t_3 g_R^{-1} D_\mu g_R \right) - i \, k_8^\mu \mathrm{Tr} \left(t_8 g_R^{-1} D_\mu g_R \right) \\ & - F(n, n_3, n_8, m_3, m_8, \cdots) \Big] + \Gamma_{WZ}(A_L, A_R, g_L g_R^\dagger) + S_{YM}(A)\end{aligned}$$

Physics Implication V:

The Wess-Zumino term can lead to the chiral magnetic effect

- Focus just on the electromagnetic field and axial angle θ,

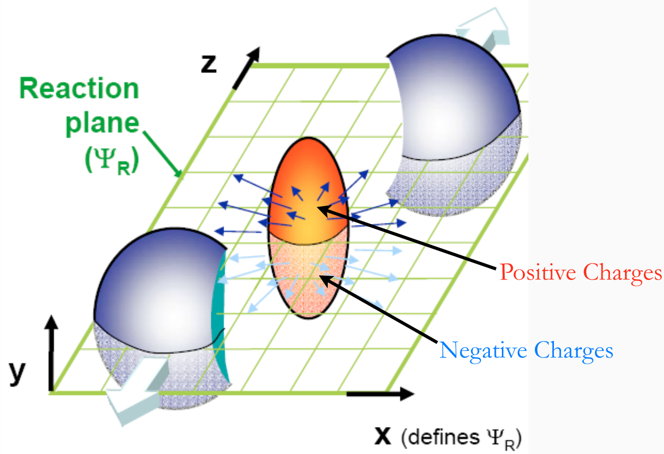
$$\Gamma_{WZ} = -\frac{e^2}{4\pi}\epsilon^{\mu\nu\alpha\beta}A_\mu\partial_\nu A_\alpha\,\partial_\beta\theta \implies J^\mu = -\frac{e^2}{2\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\nu A_\alpha\,\partial_\beta\theta$$

- In a statistical distribution, $\dot{\theta} \to \mu$, the chemical potential, so

$$J^i = \frac{e^2}{2\pi}\,B^i\dot{\theta} = \frac{e^2}{4\pi}(\mu_L - \mu_R)\,B^i$$

- With axial asymmetry a current is created in the direction of the magnetic field.

Chiral Magnetic Effect (*cont'd.*)



Thank You