Hydrodynamics of quantum information from random circuits

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Outline

1. Motivation: Scrambling and chaos

2. Operator hydrodynamics in random unitary circuits
   - Behavior of out-of-time-ordered correlators
   - Entanglement growth

3. Coupling to a conserved charge
Thermalization: information of initial state is lost
Thermalization: information of initial state is lost locally

For all initial states $\Psi, \Phi$ and subsystem $A$

\[
|\psi\rangle \rightarrow \rho^{(1)}(t) = |\psi(t)\rangle\langle\psi(t)|
\]
\[
|\phi\rangle \rightarrow \rho^{(2)}(t) = |\phi(t)\rangle\langle\phi(t)|
\]
\[
\text{tr} |\rho_A^{(1)} - \rho_A^{(2)}| \approx 0
\]

“Scrambling” of information Requires signaling between subsystems

Lashkari et. al. JHEP (2013)
We can quantify scrambling via operator spreading

Spin $\frac{1}{2}$ chain:

Operators grow and get scrambled (look random within lightcone)

Pauli strings: $\sigma^{\vec{\mu}} = \sigma_1^{\mu_1} \sigma_2^{\mu_2} \ldots \sigma_L^{\mu_L}$ \quad $\mu_i = 0, 1, 2, 3$

- e. g. $Z_1 X_2 \mathbb{1}_3 Z_4 \ldots$

\[
Z_j(t) = \sum_{\vec{\nu}} c_{\vec{\nu}}(t) \sigma^{\vec{\nu}} \quad \sum_{\vec{\nu}} |c_{\vec{\nu}}(t)|^2 = 1
\]

How to diagnose?
Motivation I: Out-of-time-ordered correlator measures the spreading of quantum information

\[ Z_j(t) = \sum_{\bar{\nu}} c_{\bar{\nu}}(t) \sigma^{\bar{\nu}} \]

Operators grow and get scrambled

How to diagnose?  Out-of-time-ordered correlator (OTOC):

\[ C(s, t) = \frac{1}{2} \left\langle \left| [Z_j(t), Z_s] \right|^2 \right\rangle_{\beta=0} = \sum_{\bar{\nu}} |c_{\bar{\nu}}(t)|^2 \]

\[ [\sigma^{\bar{\nu}}, Z_s] \neq 0 \]
Motivation II: Many-body quantum chaos

Classical chaos: \[ \left( \frac{\partial q(t)}{\partial q(0)} \right)^2 = \langle \{q(t), p(0)\} \rangle^2 \propto e^{2\lambda_L t} \]

\( \lambda \) measures how fast information spreads (Kolmogorov-Sinai entropy)

\[
C(t) \equiv \left\langle \left[ A(t), B \right] \right|^2 = -\text{Re} \langle A(t)BA(t)B \rangle_\beta + \ldots
\]

\( C \propto e^{2\lambda_L t} \) in weakly coupled field theories, SYK model

Larkin, Ovchinnikov JETP 28 (1969); Maldacena et. al. JHEP (2015); Maldacena, Stanford PRD 94 (2016), etc.

What about local lattice systems?

- Exponential growth?
- Universal features?
- Relationship to entanglement growth?

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Operator spreading in 1D has a hidden conservation law

Local operator density (of right endpoints):

\[ \rho_R(s, t) = \sum_{\nu} |c_{\nu}(t)|^2 \delta(\sigma^\nu_s \neq 1 \text{ and } \sigma_{r>s}^\nu = 1) \]

Conserved during time evolution:

\[ \sum_s \rho_R(s, t) = 1 \]

Initial condition: \[ \rho_R(s, 0) = \delta(s - j) \]
Random unitary circuits are a toy model for chaotic systems

Random $q^2 \times q^2$ unitary with uniform (Haar) distribution

Light cone velocity: \( v_{LC} = \frac{\Delta s}{\Delta t} = 1 \)

After averaging: \( \overline{c_{\vec{v}}} = 0 \quad \overline{c_{\vec{v}} c_{\vec{v}'}} = \delta_{\vec{v} \vec{v}'} \left| \overline{c_{\vec{v}}} \right|^2 \)

No local conserved quantities, only constraint is locality + unitarity
Random unitary circuits are a toy model for chaotic systems

Random \(q^2 \times q^2\) unitary with uniform (Haar) distribution

Light cone velocity: \(v_{LC} = \frac{\Delta s}{\Delta t} = 1\)

After averaging: \(\overline{c_{\vec{v}}} = 0\) \(\overline{c_{\vec{v}}c_{\vec{v}'}} = \delta_{\vec{v}\vec{v}'} \left|c_{\vec{v}}\right|^2\)

No local conserved quantities, only constraint is locality + unitarity
Average operator density obeys biased diffusion equation

Density of right endpoints:

\[ \rho_R(s, t) = \sum_{\nu} |c_{\nu}(t)|^2 \delta(\sigma^\nu_s \neq 1) \text{ and } \sigma^\nu_r = 1 \]

Applying 2-site unitary:

\[ \overline{\rho_R}(1, t + 1) = p [\overline{\rho_R}(1, t) + \overline{\rho_R}(2, t)] \]
\[ \overline{\rho_R}(2, t + 1) = (1 - p) [\overline{\rho_R}(1, t) + \overline{\rho_R}(2, t)] \]

After two layers:

\[ p^2 \quad 2p(1 - p) \quad (1 - p)^2 \]
Biased diffusion determines the OTOC

\[ \partial_t \rho_R = v_B \partial_x \rho_R + D_{\rho} \partial_x^2 \rho_R \]

Drift (butterfly) velocity:

\[ v_B = \frac{q^2 - 1}{q^2 + 1} < v_{LC} \]

Diffusion constant:

\[ D_{\rho} = \frac{q}{1 + q^2} \]

OTOC:

\[ \overline{C}(s, t) \approx 1 - \sum_{r \leq s} \overline{\rho_R}(r, t) \]

(+ terms exponentially small in \( s, t \))

All operators are equally probable
Once we reach site \( s \)
Operator spreading is described by biased diffusion.
Out-of-time-order correlator has 3 distinct regimes

\[ C(s, t) = \frac{1}{2} \left\langle \left| [Z_j(t), Z_s] \right|^2 \right\rangle_{\beta=0} \]

\[ \bar{C} \approx \frac{1}{2} \text{erfc} \left( \frac{s - v_B t}{\sqrt{2t(1-v_B^2)}} \right) \]

\[ 1 - \bar{C} \sim \left( \frac{2q}{1 + q^2} \right)^t \]

\[ \bar{C} \sim \left( \frac{q^2}{1 + q^2} \right)^s e^{\frac{\delta}{2} \log \frac{\gamma s}{\delta}} \]

\[ \gamma = e(1 - v_B^2)/2 \]

- Initial exponential increase, exponent depends on s
- Saturates exponentially
Fluctuations decrease algebraically in time

\[ R(s) = \sum_{r \leq s} \rho_{R}(r) \approx 1 - C(s) \]

Time evolving the matrix product operator using TEBD
Diffusive broadening appears also in clean driven spin chain

Kicked Ising model:

$$\hat{U} = e^{-i\frac{T}{2}g}\sum_s X_s e^{-i\frac{T}{2} \sum_s Z_s Z_{s+1}} + hZ_s$$

$$T = 0.8 \quad h = 0.809$$

$$\sigma(t) = \sqrt{\sum_s s^2 \rho_R(s) - \left(\sum_s s \rho_R(s)\right)^2}$$

More recently: static tilted field Ising model; Leviatan et. al. ArXiv 1702.08894
Entanglement grows when an operator leaves the subsystem

Start from ‘ferromagnetic’ product state:

$$\hat{\omega}(t = 0) = |00 \ldots 0\rangle \langle 00 \ldots 0| = \frac{1}{q^L} \sum_{\bar{\mu} \in \text{Z-strings}} \sigma^{\bar{\mu}}$$

$$e^{-S_A^{(2)}(t)} \equiv \text{tr} \hat{\omega}^2_A(t) = \frac{1}{q^{L_A}} + \frac{q^2 - 1}{q^2} \sum_{s_0=1}^{L} \sum_{s=1}^{L_A} \frac{\rho_R(s, t; s_0)}{q^{L_A - s_0}}$$

(Purity / exponentiated 2$^{nd}$ Rényi entropy)

Entanglement: ● only sensitive to operator growth (not scrambling)
● average behavior of many operators
Diffusion leads to slower entanglement growth

If \( t < L_A \), then

\[
e^{-S_A^{(2)}(t)} \approx \left( \frac{2q}{1 + q^2} \right)^t \equiv q^{-v_E t}
\]

Entanglement velocity:

\[
v_E = 1 - \frac{\log 2}{\log q} + \frac{\log (1 + q^{-2})}{\log q} < v_B
\]

\[
S_{avg}^{(2)} \equiv \overline{S^{(2)}} \quad S_{exp}^{(2)} \equiv -\log e^{-S^{(2)}}
\]

Graphs showing the evolution of Rényi entropy \( S^{(2)} \) with discrete time \( t \) and the ratio \( S_{exp}^{(2)} / \log q \) for different values of \( q \).
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Hydrodynamic approach: conserved quantities are essential

Random circuit:  
- Only conserved quantity is $\rho_R$  
- Within lightcone all operators are equally probable  
- OTOC measures probability of having reached site $s$

Most systems have more structure: conserved energy, charge etc.  

Consider modified circuit with conserved charge $Q$

$C^2 \sim |0\rangle, |1\rangle$

Local op. basis: $\mathbb{1}, Z, \sigma^+, \sigma^-$
Charge diffusion leads to slow relaxation for the OTOC

Gate on sites $r, r+1$: \[ \hat{Q}_r(t + 1) = \frac{1}{2}(\hat{Q}_r + \hat{Q}_{r+1}) \]

\[
|c_{Z_0 \rightarrow Z_l}| \sim \frac{1}{\sqrt{t}}
\]

\[
\sum_l |c_{Z_0 \rightarrow Z_l}|^2 > \sum_l |c_{Z_0 \rightarrow Z_l}|^2 \sim \frac{1}{\sqrt{t}}
\]

OTOC: \[ \mathcal{F} \sim 1 - \mathcal{C} \sim \frac{1}{\sqrt{t}} \]

\[
\mathcal{F}_{\mu}^{VW}(t) \equiv \text{Re}\langle V^\dagger(t)W^\dagger V(t)W \rangle_\mu
\]

\[
\langle O \rangle_\mu \equiv \frac{\text{tr}(O e^{-\mu \sum \hat{Q}_l})}{\text{tr}(e^{-\mu \sum \hat{Q}_l})}
\]
The OTOC develops a power law tail behind the front

Coarse-graining:

Physical picture: in each step there is a conversion from “conserved” to “non-conserved” Pauli strings

\[ |c_{Z_0 \rightarrow Z_s(t)}|^2 \]

\[ \rho_R(s, t) = \rho^c_R(s, t) + \rho^{nc}_R(s, t) \]

\[ \partial_t \rho_R = v_B \partial_x \rho^{nc}_R + D \rho \partial^2_x \rho^{nc}_R + D_Q \partial^2_x \rho^c_R \]

See: Khemani et. al. ArXiv 1710.09835
At low filling, the ballistic front can only develop at long times

\[
\mu \gg 1 \quad \Rightarrow \quad \mathcal{F}_\mu^W(t) \equiv \text{Re}\langle V(t)W^\dagger V^\dagger(t)W\rangle_\mu \approx \sum_{N} e^{-N\mu} \mathcal{F}_{(N)}^W(t)
\]

diffusion of \(2N + Q_V + Q_W\) interacting particles

\[
\mu = 3
\]
Conclusions, open questions

- Random circuits provide a toy model for chaotic dynamics

- One dimension → ‘hidden’ conserved density → obeys biased diffusion

- Diffusion → slower entanglement growth

- Conserved charge → hydrodynamic tails

Open questions:
- Other universality classes?
- Many-body localized phase?
- Comparison with field theories?

Details: ArXiv 1705.08910 and 1710.09827
Related work: Nahum et. al: PRX (2017); ArXiv 1705.08975 and 1705.10364
Khemani et. al: ArXiv 1710.09835

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