Computational power of symmetry-protected topological phases

David T. Stephen
*Max Plank Institute for Quantum Optics*

Robert Raussendorf, Dong-Sheng Wang, Cihan Okay
*University of British Columbia*

Tzu-Chieh Wei, Abhishodh Prakash
*Stony Brook University*

Hendrik Poulsen-Nautrup
*University of Innsbruck*
Part 0: Motivation

Exploiting the algebraic structure of quantum phases
“Zoo” of quantum phases

Symmetry breaking
- Crystals, ferromagnets
- Local order parameters
- Group Theory

Topological
- FQHE, Toric code
- Patterns of long range entanglement
- Category Theory

Symmetry-protected
- Topological insulator, Haldane phase
- Patterns of short range, symmetric entanglement
- Group cohomology

- Quantum phases of matter have a rich algebraic classification.
- Can we take advantage of this in application?
Example: Topological QC

- Anyonic excitations can be braided, fused to achieve fault-tolerant quantum computation
- Braiding, fusion rules etc. are described by language of category theory
- Category theory classifies possible topological orders, and identifies which ones allow universal TQC!
Before:

- Intrinsic topological order
- Category theory
- Topological quantum computation

Now:

- Symmetry-protected topological order
- Group cohomology
- Measurement-based quantum computation

Who cares?

QI: Classify MBQC resource states, understand source of quantum computational power + “robust” computation

CM: Apply algebraic structure of quantum phases to other fields, classify phases by their computational capability → new insights
Part 1: Background

Tensor networks as a unifying framework for SPT order and MBQC
Matrix product states

\[ |\psi\rangle = \sum_{i_1, \ldots, i_n} \langle R | A^i_n A^{i_{n-1}} \ldots A^{i_1} | L \rangle | i_1 \ldots i_n \rangle \]

Example: Cluster state

\[ |+\rangle |+\rangle |+\rangle |+\rangle |+\rangle |+\rangle \]

\[ |0\rangle \langle +| \]

\[ |1\rangle \langle -| \]

\[ A^{++} = I \]

\[ A^{+-} = Z \]

\[ A^{-+} = X \]

\[ A^{--} = XZ \]
Classification of SPTO with MPS

On-site symmetry group $G$: $u(g)^{\otimes n} |\psi\rangle = |\psi\rangle$

$A = V_g^\dagger A V_g$

$V_g V_h = \omega(g, h) V_{gh}$

$[\omega] \in H^2(G, U(1))$

SPT phases classified by group cohomology

Example: Cluster state $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$u_{x_1} = X \quad X \quad X \quad X$

$u_{x_2} = X \quad X \quad X \quad X$

$V_e = I \quad V_{x_2} = Z$

$V_{x_1} = X \quad V_{x_1 x_2} = XZ$

1D cluster state has SPT order.
Measurement-based QC

- Measurement on entangled state $\rightarrow$ computation (universal?)
- MBQC defines the **computational power** of a quantum state
- What features of a state make it a useful resource?
MBQC in tensor networks

- Measurement $\rightarrow$ Unitary evolution in virtual space

Tensor networks capture both SPT order and computation
Entanglement “sweet spot”

Which states are useful as MBQC resources? Goldilocks problem:

- Product state - Too trivial!
- Most states - Too entangled!
- SPT states - Just right!

Phases are related by patterns of entanglement → SPT phases are useful?*

Phases are related by patterns of entanglement → SPT phases are useful?*

Gross et. al. 2009
For certain SPT phases, MPS matrices have the form:

\[ A^i = B^i \otimes V_g^i \]

Wire basis:

\[ B = \{ |0\rangle, |1\rangle, \ldots, |d-1\rangle \} \]

Junk subspace contains microscopic details

Logical subspace uniform throughout phase

**Theorem:** For general abelian phases:

\[ A^i = \bigoplus_{\alpha} \left( B^i_{\alpha} \otimes V_{g\alpha}^i \right) P^i \]
Computation in SPT phases

• Encode information in logical subspace: \( |L\rangle = |J\rangle \otimes |\psi\rangle \)

\[
\begin{align*}
&J \\
&\psi
\end{align*}
\]

\( B \) = \{ |0\rangle, |1\rangle, \ldots, |d - 1\rangle \}

• Measure within wire basis: \( |L'\rangle = B_i |J\rangle \otimes V_{g_i} |\psi\rangle \)

➤ Perfect “identity gate” throughout phase!

• Measure outside wire basis: \( |L'\rangle = B_i |J\rangle \otimes V_{g_i} |\psi\rangle + B_j |J\rangle \otimes V_{g_j} |\psi\rangle \)

➤ Microscopic details become entangled with logical state…

Need a way to get non-trivial gates in all phases
Part 2: 1D Results

Extracting the useful entanglement from 1D SPT phases
The solution...

Two simple ingredients allow computation throughout the phase:

1. Non-trivial gates are infinitesimal rotations.
2. Gates are separated by a distance $\gg$ correlation length.

Idea: slowly accumulate unitary rotation while keeping junk subspace separated from logical subspace.
What gates can we do?

Primitive gates are infinitesimal rotations:

$$
T(i, j; d\alpha) = e^{d\alpha \left( \nu_{ij} V_{g_i}^\dagger V_{g_j} - \nu_{ji} V_{g_j}^\dagger V_{g_i} \right)}
$$

$$
\mathcal{L}(G, u, [\omega]) \quad \text{Lie group of executable gates}
$$

$$
\nu_{ij} = \text{Tr} \, B_i B_j^\dagger
$$

Microscopic details

Computational power is uniform within 1D SPT phases
Abelian symmetry and SPT phase

\[ \omega \ud u \]

\[ G \]

Quantum phases

Quantum computation

- Physical measurement bases
- Lie group of executable gates
- Algebra of measurable observables
Determining computational power

Theorem: If $G$ is a finite abelian (sub)group, and $[\omega]$ is maximally non-commutative, then:

$$\mathcal{L}(G, u, [\omega]) \supset SU(p^n)$$

<table>
<thead>
<tr>
<th>Symmetry Group $G$</th>
<th>Group of Gates $\mathcal{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(N), Z_N \times Z_N$</td>
<td>$SU(N)$</td>
</tr>
<tr>
<td>$SO(N), Sp(2N)$</td>
<td>$SU(2)$</td>
</tr>
<tr>
<td>$D_{2n}, A_4, S_4$</td>
<td>$SU(2)$</td>
</tr>
<tr>
<td>$(Z_2)^4$</td>
<td>$SU(4)$ or $SU(2) \times SU(2)$</td>
</tr>
<tr>
<td>$Z_4 \times Z_2$</td>
<td>$SU(2)$ or $U(1)$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Algebraic structure of SPT order can be used to identify universal resources (like TQC)
Part 3: 2D Results

A computationally universal phase of matter
Example: 2D cluster state

This quantum cellular automata structure is enough to prove universality of the 2D cluster state
“Cluster Phase”

- Define the cluster phase as the gapped quantum phase which contains the 2D cluster state and respects the “line-like” symmetries of the cluster state:

- This is a non-conventional SPT phase.
Cluster phase computation

• For every state in the cluster phase:

\[
\begin{align*}
A \cdot X = X & \quad \text{and} \quad \boxed{A \cdot X = X} = \boxed{A \cdot Z = Z}, \\
A \cdot Z = Z & \quad \text{and} \quad \boxed{A \cdot Z = Z} = \boxed{A \cdot X = X}.
\end{align*}
\]

• These symmetries imply cluster QCA exists on virtual subspace:

\[
\begin{align*}
A_{s_1} \cdot Z^{s_2} \cdot Z^{s_3} = Z^{s_1} \cdot H \cdot Z^{s_2} \cdot H \cdot Z^{s_3} \cdot H = B(s)
\end{align*}
\]

Combine with 1D methods \(\rightarrow\) cluster phase is a *universal* computational phase of matter!
Summary

• **Result 1**: 1D SPT phases are ubiquitously useful for quantum computation. The algebraic structure of SPT phases can be used to classify their computational power.
  
  \[\text{[PRL 119, 010504, '17], [PRA 96, 012302, '17]}\]

• **Result 2**: There exists a computationally universal phase of matter, the “cluster phase”. It is a non-conventional 2D SPT phase.
  
  \[\text{[arXiv tomorrow]}\]
What’s Next

• Extend 2D results to other cluster-like phases and uncover any underlying algebraic structure.

• Does conventional SPT order lead to similar results in 2D?

• Understand physics of 2D cluster phase. How to characterize SPT phases with non-global symmetries?

• Extend general idea to other phases of matter (symmetry enriched order, fracton order, etc.).

Can we classify phases of matter by their computational power?
Thank you!