Stationary solutions in theories beyond GR

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Why beyond-GR theories:
See morning talk by Thomas (Theoretical motivation for beyond-GR theories).

1 point to remember:
To test GR we also need solutions from theories other than GR in order to form a testbed.

2 approaches:
• Find solutions in specific modifications to GR and work on a case by case basis.
• Construct generic parametric deviations from known GR solutions, like the Kerr solution.

3 ways to proceed:
• Generate an analytic parameterised solution without approximations
• Employ some approximation scheme (slow rotation, small coupling)
• Numerical solution

The Einstein-Hilbert action in GR is $S = \frac{1}{16\pi} \int d^4x \sqrt{-g} R + S_m(g_{\mu\nu}, \psi)$,
which results to the field equations $R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right)$

Stationary spacetime: symmetry with respect to time translations and rotations (the spacetime admits a timelike, ξ^a, and a spacelike, η^a, killing vector).
The line element for such a spacetime can be written as[1]

$$ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2\theta B^2 e^{-2\nu} (d\varphi - \omega dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2)$$

This line element describes the spacetime of a rotating compact object.
The field equations can be solved either for vacuum spacetimes (BH solutions) or spacetimes that have matter (NS solutions).

• In vacuum we have the well known BH solutions of the Kerr family (which can be extended to include electromagnetic fields as well).

• For NSs the full field equations can only be solved numerically.

[1] E. M. Butterworth and J. R. Ipser, ApJ **204**, 200 (1976).

• The alternative is to employ the slow rotation Hartle-Thorne method.[2]

$$ds^2 = -e^{\bar{\nu}}dt^2 + e^{\lambda}dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right] - 2\varepsilon(\Omega - \omega_1)r^2\sin^2\theta d\varphi dt.$$

where ε is a slow rotation small bookkeeping parameter. The first order correction ω_1 is given by the equation

$$\omega_1'' = \frac{4}{r}\left[\pi r^2(\epsilon + p)e^{\lambda} - 1\right]\omega_1' + 16\pi(\epsilon + p)e^{\lambda}\omega_1.$$

• Finally there is a general algorithm for constructing any vacuum stationary axisymmetric space-time. Such a spacetime can be described by the Weyl-Papapetrou line element[3],

$$ds^2 = -f\,(dt - wd\varphi)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right].$$

By introducing the complex potential $\mathcal{E}(\rho, z) = f(\rho, z) + i\psi(\rho, z)$ [4], the Einstein field equations take the form, $(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}$, where, $f = \xi^a\xi_a$ and ψ is the scalar twist, $\nabla_a\psi = \varepsilon_{abcd}\,\xi^b\nabla^c\xi^d$. By prescribing an Ernst potential $\mathcal{E}$ one can calculate a vacuum GR solution.

[2] Hartle J. B., Astrophys.J. 150, 1005 (1967); Hartle J. B., Thorne K. S., ApJ **153**, 807 (1968)

[3] A. Papapetrou, Ann. Phys., **12**, 309 (1953).

[4] F.J. Ernst, Phys. Rev., **167**, 1175 (1968); Phys. Rev., **168**, 1415 (1968).

A brief (incomplete) list of theories beyond GR.

- Scalar-tensor theories
- f(R) theories
- Quadratic gravity theories
    - Einstein-dilaton-Gauss-Bonnet (EdGB)
    - dynamical Chern-Simons (dCS)
- Lorentz-violating theories
- Massive gravity theories
- Theories with non-dynamical fields

The *Bergmann-Wagoner* action for Scalar-Tensor theories is,

$$S = \int d^4x\sqrt{-\hat{g}}\left(\varphi\hat{R} - \frac{\omega(\varphi)}{\varphi}\hat{\nabla}^{\mu}\varphi\hat{\nabla}_{\mu}\varphi U(\varphi)\right) + S_m(\hat{g}_{\mu\nu},\psi)$$

In the Einstein frame it takes the form,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2\tilde{\nabla}^{\mu}\phi\tilde{\nabla}_{\mu}\phi - V(\phi)\right) + S_m(\hat{g}_{\mu\nu},\psi)$$

where φ is redefined to ϕ, $\hat{g}_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu}$, and $V(\phi) \equiv A^4(\phi)U(\varphi(\phi))$. Then the field equations take the form,

$$\tilde{R}_{ab} = 2\partial_a\phi\partial_b\phi + 8\pi\left(T_{ab} - \frac{1}{2}\tilde{g}_{ab}T\right) + 2V\tilde{g}_{ab}, \quad \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\phi = -4\pi\alpha(\phi)T + \frac{1}{4}\frac{dV}{d\phi}$$

These equations can be solved as in GR in vacuum or in the presence of matter.[5] Since the actual physics is done in the Jordan frame (the particles follow the geodesics of the Jordan metric), one can return to that frame by the conformal transformation $\hat{g}_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu}$.

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[5] D.D. Doneva, S.S. Yazadjiev, N. Stergioulas, K.D. Kokkotas, Phys. Rev. D 88, 084060 (2013)

- In the case of a massless scalar field ($V(\phi)=0$), the vacuum field equations

$$\tilde{R}_{ab}=2\partial_a\phi\partial_b\phi,\quad \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\phi=0$$

can admit an Ernst formulation as in GR,[6] having the metric,

$$ds^2=-f\left(dt-wd\varphi\right)^2+f^{-1}\left[e^{2\gamma}\left(d\rho^2+dz^2\right)+\rho^2d\varphi^2\right]$$

and the field equations, $\quad (Re(\mathcal{E}))\nabla^2\mathcal{E}=\nabla\mathcal{E}\cdot\nabla\mathcal{E},$

with the addition of a Laplace equation for the scalar field, $\nabla^2\phi=0$, and a set of equations for the metric function γ of the Weyl-Papapetrou metric,

$$\frac{\partial\gamma}{\partial\rho}=\left(\frac{\partial\gamma}{\partial\rho}\right)_{GR}+\rho\left[\left(\frac{\partial\phi}{\partial\rho}\right)^2-\left(\frac{\partial\phi}{\partial z}\right)^2\right],\frac{\partial\gamma}{\partial z}=\left(\frac{\partial\gamma}{\partial z}\right)_{GR}+2\rho\left(\frac{\partial\phi}{\partial\rho}\right)\left(\frac{\partial\phi}{\partial z}\right),$$

Any vacuum stationary axisymmetric GR solution can be turned into a scalar-tensor solution with a massless scalar field.[7]

[6] GP, T.P. Sotiriou, Phys. Rev. D91, 044011 (2015)

[7] GP, MNRAS, 466, 4381 (2017)

• In the case of a massive scalar field, one doesn't have the Ernst formulation, but can still do a fully numerical calculation,
• Or employ a HT-like slow rotation approximation, like in GR.[8]
The TOV equations then become (0th-order in the rotation),

$$M' = 4\pi r \left(rA^4 \epsilon_0 + \frac{1}{2}(r-2M)\phi_0'^2 + rV \right),$$

$$\nu' = \frac{2\left(4\pi r^3\left(A^4 p_0 - V\right) + M\right)}{r(r-2M)} + 4\pi r \left(\phi_0'\right)^2,$$

$$p' = -(p_0+\epsilon_0)\frac{1}{2}\nu' - \alpha\phi_0',$$

$$\phi_0'' = \frac{2\phi_0'\left(r\left(2\pi r^2\left(A^4(\epsilon_0-p_0)+2V\right)-1\right)+M\right) + r^2\left(A^3A'(\epsilon_0-3p_0)+V'\right)}{r(r-2M)},$$

and at 1st-order in the rotation,

$$\omega_1'' = \frac{4\left(\pi A^4 r^2(p_0+\epsilon_0)\left(r\omega_1'+4\omega_1\right) + (r-2M)\omega_1'\left(\pi r^2\left(\phi_0'\right)^2 - 1\right)\right)}{r(r-2M)}$$

[8] Pani, Berti, Phys. Rev. D 90, 024025 (2014); Yazadjiev et al., Phys. Rev. D 93, 084038 (2016)

A lot of work has been done in scalar-tensor theories for both BHs and NSs. These results also extend to f(R) theories due to the equivalence between them (although there are subtleties).

- Black Holes:
  - Real scalar field:
    No-hair theorems. Same BH solutions as in GR, i.e., Kerr BHs.
  - Complex scalar field:
    Evade the no-hair theorems for $\Psi(t,\varphi,x^i) = e^{-i\omega t}e^{im\varphi}\phi(x^i)$.
    Stationary BH solutions with scalar clouds when $\omega = m\Omega_H$.
- Neutron Stars:

  Systematic studies in slow rotation and rapid rotation for massless and massive scalar fields. Spontaneously scalarised solutions.

  f(R) theories such as $f(R) = R + aR^2$ studied in their scalar-tensor formulation.

The most general action for quadratic gravity with a scalar field is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - 2\nabla^{\mu}\phi\nabla_{\mu}\phi - V(\phi) + f_1(\phi)R^2 + f_2(\phi)R_{\mu\nu}R^{\mu\nu}\right.$$
$$\left. + f_3(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + f_4(\phi)^{*}RR\right] + S_m(\gamma(\phi)g_{\mu\nu},\psi)$$

Special cases are the EdGB gravity (Gauss-Bonnet scalar) $R^2_{GB} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$,
and dCS gravity (Pontryagin scalar), $^{*}RR \equiv \frac{1}{2}R_{\mu\nu\rho\sigma}\epsilon^{\nu\mu\lambda\kappa}R^{\rho\sigma}_{\lambda\kappa}$.

Quite some work has been done in quadratic gravity for both BHs and NSs.

- Black Holes:
  Solutions have been found in the perturbative slow-rotation limit. The BH solutions are scalarised. In EdGB solutions have been also found for rapidly rotating BHs. EdGB also has spontaneously scalarised BHs.
- Neutron Stars:
  - EdGB: Studies in slow rotation and rapid rotation. Spontaneously scalarised solutions. There exists a critical p_c.
  - dCS: Studies in 1st and 2nd order in rotation. Scalar dipole.

  No-scalar-monopole theorem for both theories at the perturbative level.

It is probably impossible to study all solutions of all the theories beyond GR.

Still, without committing to a specific theory, one could construct parameterised spacetimes that can be used to test GR.

These spacetimes do not necessarily satisfy some specific field equations.

• One such example is the Cardoso, Pani, Rico extension of the Johannsen-Psaltis non-Kerr metric,[9]

$$ds^2 = -fdt^2 + \frac{\Sigma(1+h^r)}{\Delta + a^2\sin^2\theta h^r}dr^2 + \Sigma d\theta^2 - 2a\sin^2\theta\,(H-f)\,d\varphi dt$$
$$+\sin^2\theta\left[\Sigma + a^2\sin^2\theta\,(2H-f)\right]d\varphi^2$$

where $f = \left(1-\frac{2mr}{\Sigma}\right)(1+h^t)$, $H = \sqrt{(1+h^r)(1+h^t)}$, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$, and $h^{r,t} = \sum_{k=0}^{k=\infty}\left(\varepsilon_{2k}^{r,t} + \varepsilon_{2k+1}^{r,t}\frac{mr}{\Sigma}\right)\left(\frac{m^2}{\Sigma}\right)^k$

Asymptotic flatness imposes $\varepsilon_0^{r,t} = 0$, while the mass is $M = m(1-\varepsilon_1^t/2)$.

Caveat: seems to be mapped to known static solutions but not stationary.

There are other more successful approaches in that respect.[10]

[9] Cardoso, Pani, Rico, Phys.Rev. D 89, 064007 (2014);Johannsen, Psaltis, Phys.Rev. D83, 124015 (2011)

[10] Konoplya et al., Phys. Rev. D 93, 064015 (2016): rotating EdGB BHs.

• Another approach: metrics with special properties, i.e., a Carter constant. Such examples are the bumpy Kerr and the Johannsen metric,[11]

$$ds^2 = -\frac{2a\left[(r^2+a^2)A_1(r)A_2(r)-\Delta\right]\tilde{\Sigma}\sin^2\theta d\varphi dt}{\left[(r^2+a^2)A_1(r)-a^2A_2(r)\sin^2\theta\right]^2} + \frac{\tilde{\Sigma}dr^2}{\Delta A_5(r)} + \tilde{\Sigma}d\theta^2$$
$$-\frac{\tilde{\Sigma}\left[\Delta - a^2A_2(r)^2\sin^2\theta\right]dt^2}{\left[(r^2+a^2)A_1(r)-a^2A_2(r)\sin^2\theta\right]^2} + \frac{\tilde{\Sigma}\sin^2\theta\left[(r^2+a^2)A_1(r)^2-a^2\Delta\sin^2\theta\right]d\varphi^2}{\left[(r^2+a^2)A_1(r)-a^2A_2(r)\sin^2\theta\right]^2}$$

where $\tilde{\Sigma} = \Sigma + f(r)$, and A_1, A_2, A_5, and f are expansions in powers of $1/r$. The Johannsen metric can be related to the bumpy Kerr and these two can describe slowly rotating dCS, and some static EdGB BHs.

History note: Johannsen's metric is of the form of Carter's canonical metric[12],

$$ds^2 = \frac{Z}{\Delta_r}dr^2 + \frac{Z}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{Z}\left(P_r d\varphi - Q_r dt\right)^2 + \frac{\Delta_r}{Z}\left(Q_\theta dt - P_\theta d\varphi\right)^2,$$

where $Z = P_r Q_\theta - Q_r P_\theta$, and the P, Q, and Δ are functions of either r or θ.

[11] Vigeland, Yunes, Stein, Phys.Rev. D83,104027 (2011); Johannsen, Phys.Rev. D88, 044002 (2013)

[12] B. Carter, Comm.Math.Phys. 10,280 (1968).

Theory	Solutions	Stability	Geodesics	Quadrupole
Extra scalar field				
Scalar-tensor	≡GR [55–60]	[61–67]	–	–
Multiscalar/Complex scalar	⊃GR [56, 68, 69]	?	?	[68, 69]
Metric $f(R)$	⊃GR [58, 59]	[70, 71]	?	?
Quadratic gravity				
Gauss-Bonnet	NR [72–74]; SR [75, 76]; FR [77]	[78, 79]	SR [75, 80, 81]; FR [77]	[76, 82]
Chern-Simons	SR [83–85]; FR [86]	NR [87–90]; SR [79]	[74, 91]	[85]
Generic	SR [80]	?	[80]	Eq. (3.12)
Horndeski	[92–94]	? [95, 96]	?	?
Lorentz-violating				
Æ-gravity	NR [97–99]	?	[98, 99]	?
Khronometric/ Hořava-Lifshitz	NR, SR [98–101]	? [102]	[98, 99]	?
n-DBI	NR [103, 104]	?	?	?
Massive gravity				
dRGT/Bimetric	⊃GR, NR [105–108]	[109–112]	?	?
Galileon	[113]	?	?	?
Nondynamical fields				
Palatini $f(R)$	≡GR	–	–	–
Eddington-Born-Infeld	≡GR	–	–	–

Testing General Relativity

Table 2. Catalogue of BH properties in several theories of gravity. The column "Solutions" refers to asymptotically-flat, regular solutions. Legend: ST="Scalar-Tensor," ≡GR="Same solutions as in GR," ⊃GR="GR solutions are also solutions of the theory," NR="Non rotating," SR="Slowly rotating," FR="Fast rotating/Generic rotation," ?=unknown or uncertain.

Berti et al. (2015) Class. Quantum Grav. 32 243001