Grassmannian Geometry of Scattering Amplitudes

LECTURE 3

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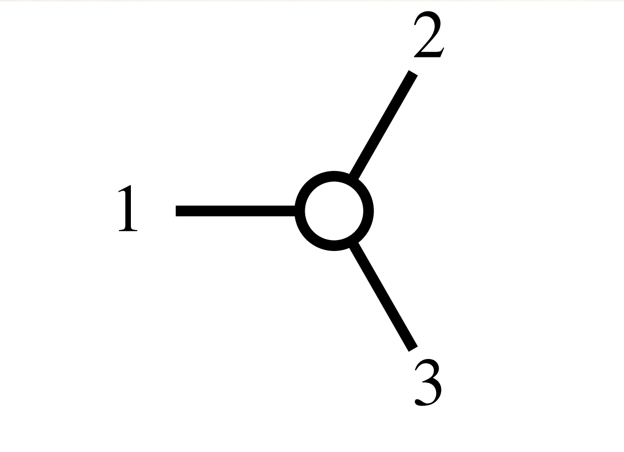
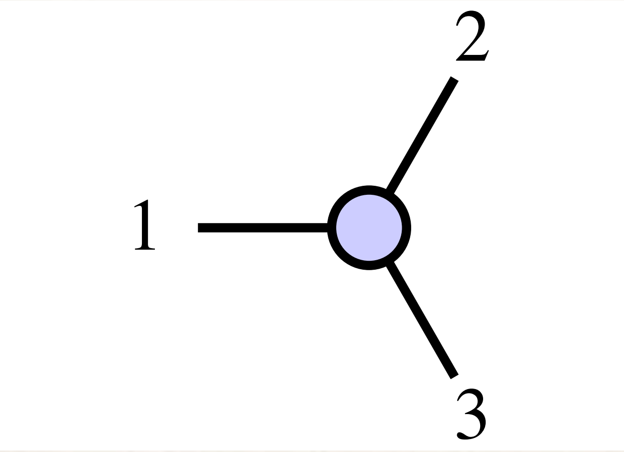
Qspace summer school, Benasque, September 2018

Building blocks: 3pt amplitudes

- In N=4 SYM



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1 + [31]\widetilde{\eta}_2 + [12]\widetilde{\eta}_3)}{[12][23][31]}$$



$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1 + \lambda_2\widetilde{\eta}_2 + \lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Note for Q fermion

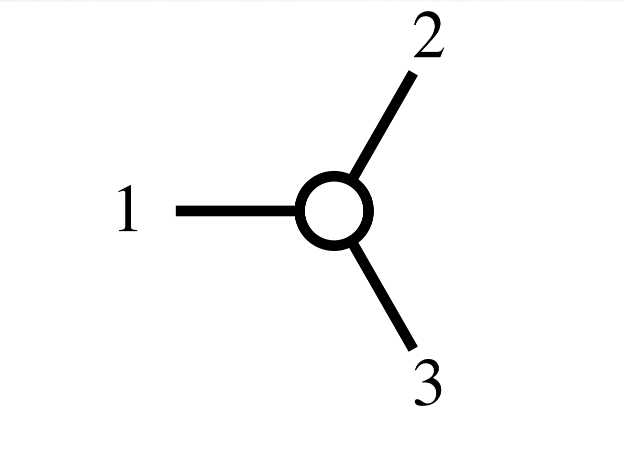
$$\delta^4(Q) = Q^4$$

To extract 1^- helicity

take $\widetilde{\eta}_1^4$

Building blocks: 3pt amplitudes

- In N=4 SYM



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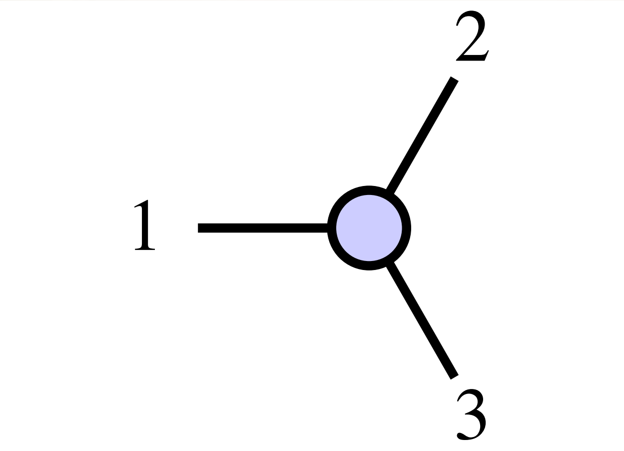

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Note for Q fermion

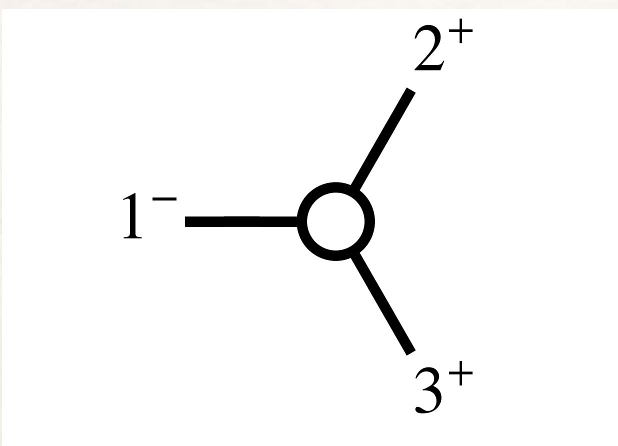
$$\delta^4(Q) = Q^4$$

$$\delta^8(\lambda Q) = (\lambda^{(1)}Q)^4(\lambda^{(2)}Q)^4$$

To extract 1^- helicity take $\tilde{\eta}_1^4$

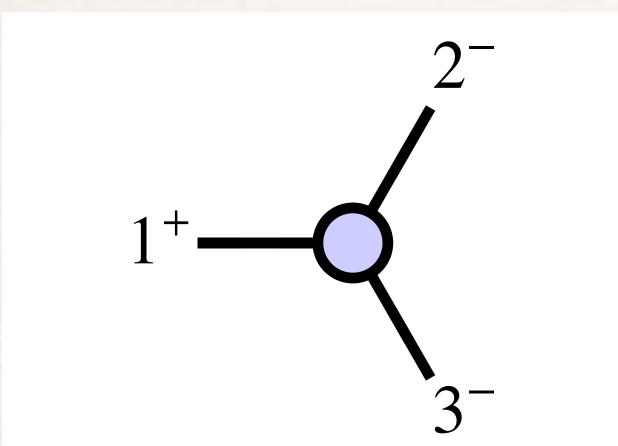
Building blocks: 3pt amplitudes

- In N=4 SYM



$\widetilde{\eta}_1^4$ component → $[23]^4$

$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$

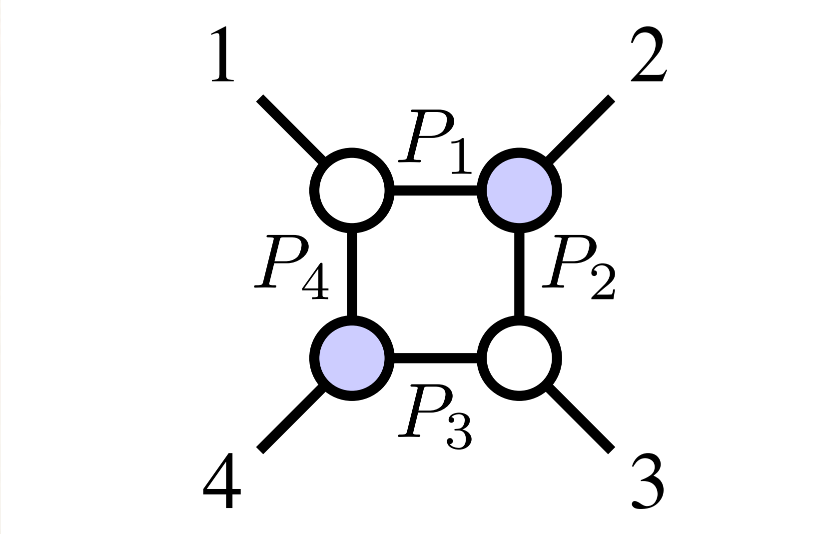


$\widetilde{\eta}_2^4\widetilde{\eta}_3^4$ component → $\langle 23\rangle^4$

$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Gluing three point amplitudes

- Let us build a diagram



Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

Gluing three point amplitudes

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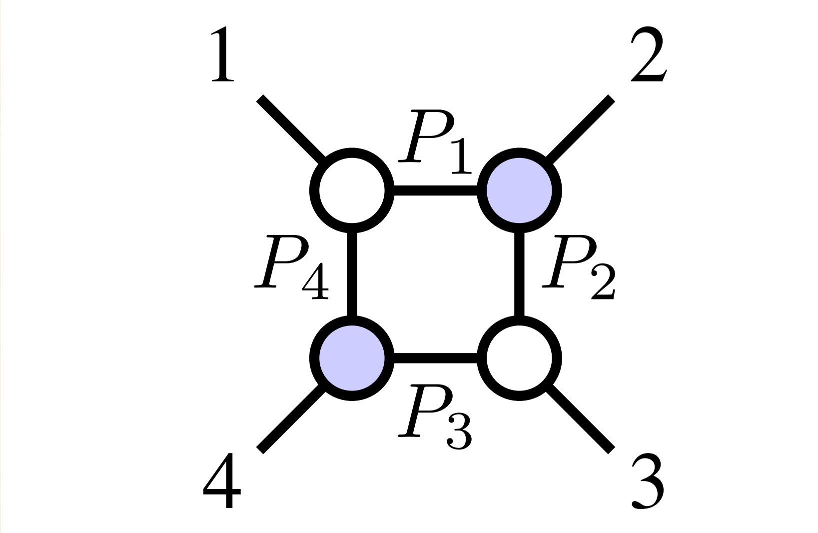

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$\delta^4(\widetilde{\eta}_{P_1}, \widetilde{\eta}_{P_4})$ $\delta^8(\widetilde{\eta}_{P_1}, \widetilde{\eta}_{P_2})$ $\delta^4(\widetilde{\eta}_{P_2}, \widetilde{\eta}_{P_3})$ $\delta^8(\widetilde{\eta}_{P_3}, \widetilde{\eta}_{P_4})$

Gluing three point amplitudes

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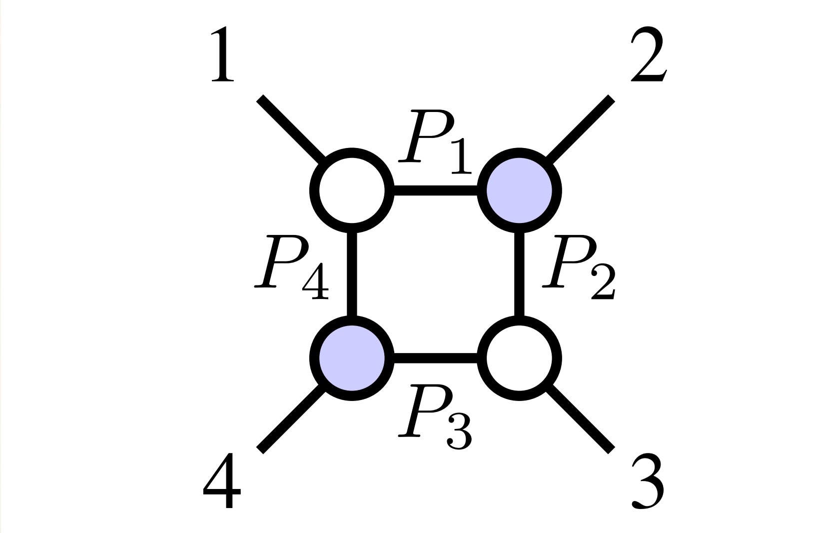

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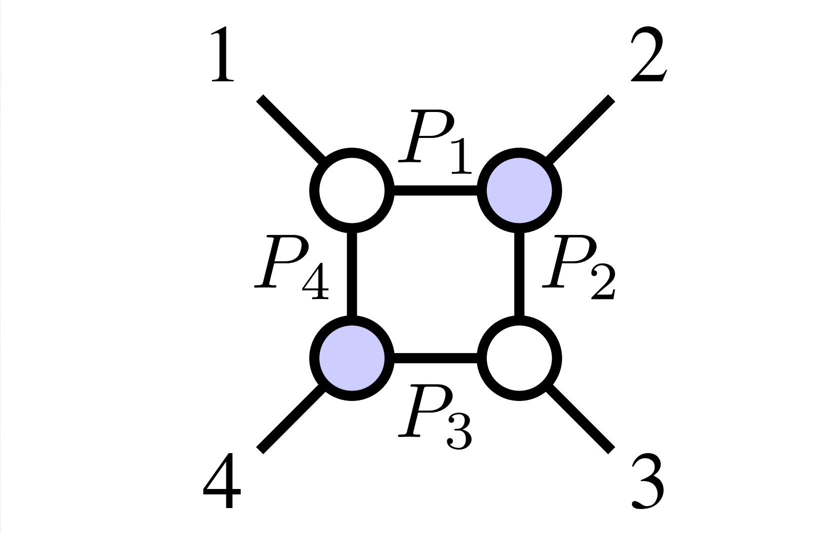
↓

$$\delta^{(24)}(\widetilde{\eta}_{P_1}, \widetilde{\eta}_{P_2}, \widetilde{\eta}_{P_3}, \widetilde{\eta}_{P_4})$$

Gluing three point amplitudes

- Let us build a diagram



Multiply four three point amplitudes

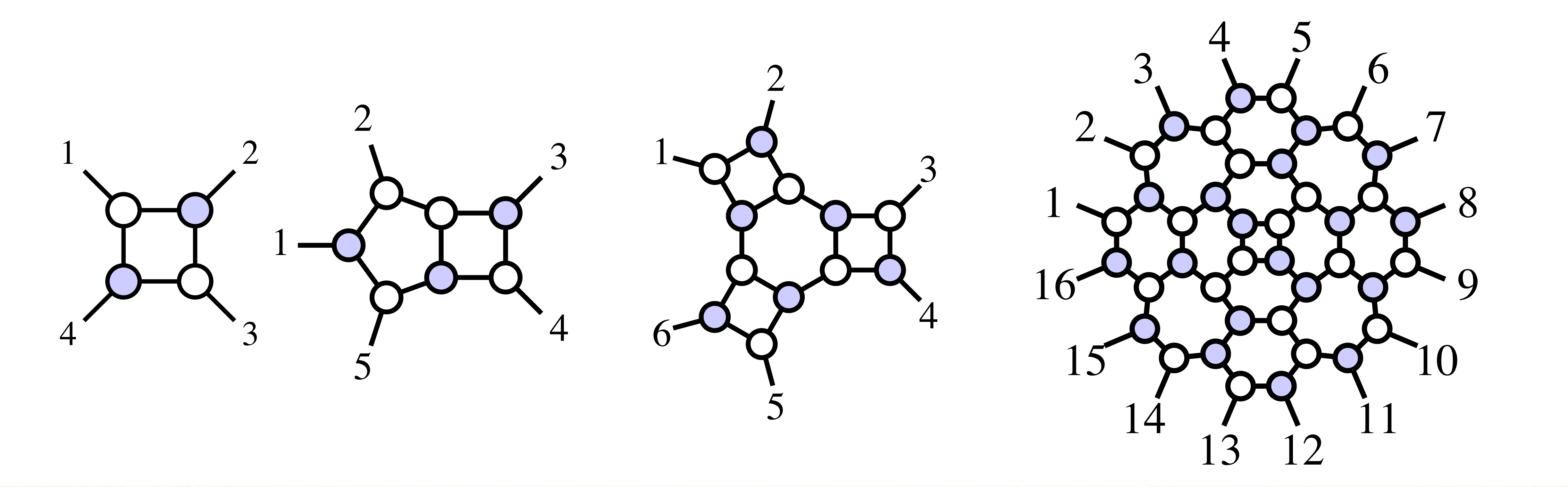
$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

Some work with delta functions

$$\widetilde{\eta}_{P_1}^4 \widetilde{\eta}_{P_2}^4 \widetilde{\eta}_{P_3}^4 \widetilde{\eta}_{P_4}^4 \;\; \delta^8(\lambda_1\widetilde{\eta}_1 + \lambda_2\widetilde{\eta}_2 + \lambda_3\widetilde{\eta}_3 + \lambda_4\widetilde{\eta}_4)$$

On-shell diagrams

- Draw arbitrary graph with three point vertices



On-shell diagrams given by products of 3pt amplitudes

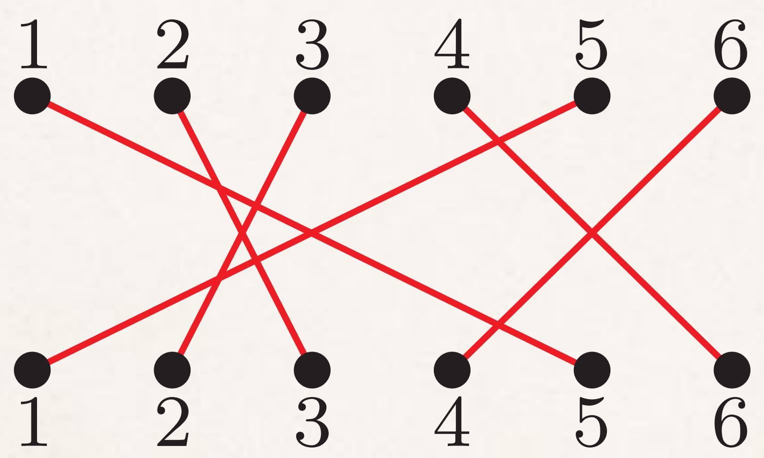
- Parametrized by n, k $\qquad k = 2B + W - P$

Permutations

Permutations

- Graphical way to represent permutations

$$(1,2,\ldots,n) \to (\sigma(1),\sigma(2),\ldots,\sigma(n))$$

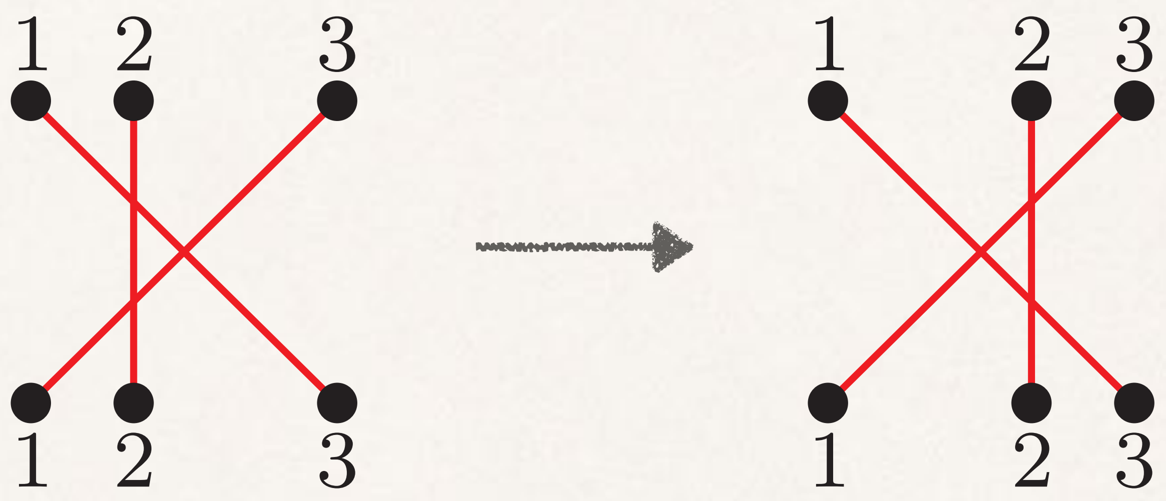


$$(1,2,3,4,5,6) \to (5,3,2,6,1,4)$$

- This picture actually represents a scattering process in 1+1 dimensions

Permutations

- These pictures are not unique: they satisfy Yang-Baxter move: anywhere in the diagram

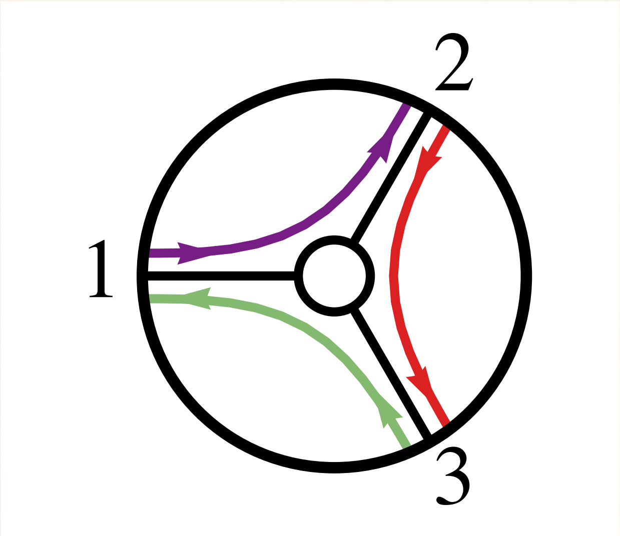


- Unfortunately, this picture can not apply to 3+1 dimensions where the fundamental vertices are 3pt

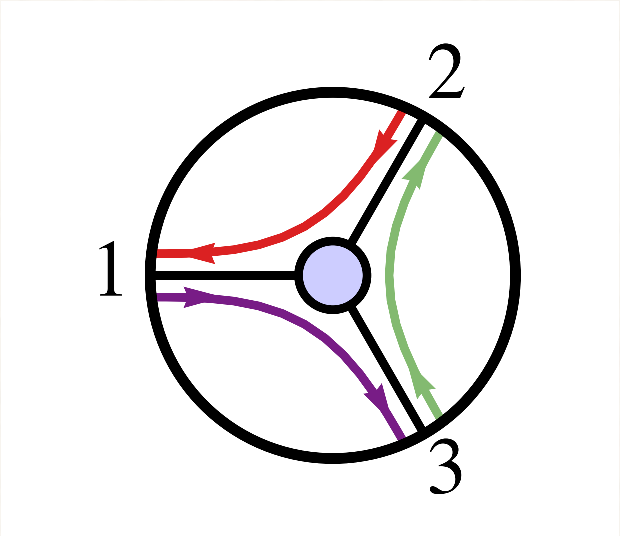
New look at permutations

- Can we represent permutation using 3pt vertices?

Two different non-trivial permutations



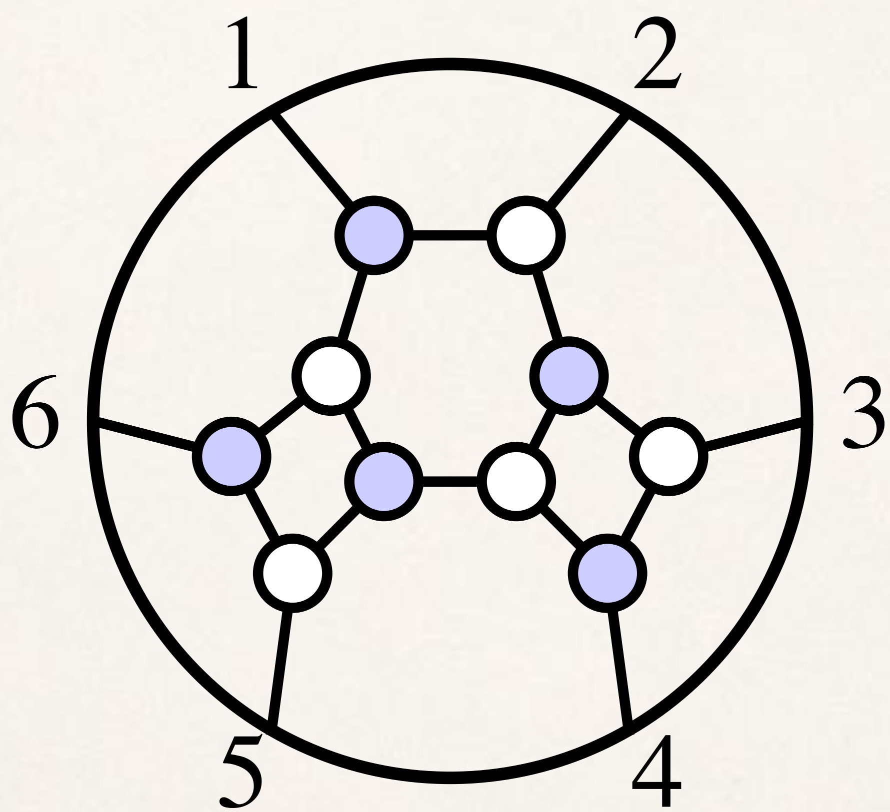
$(1, 2, 3) \to (2, 3, 1)$



$(1, 2, 3) \to (3, 1, 2)$

New look at permutations

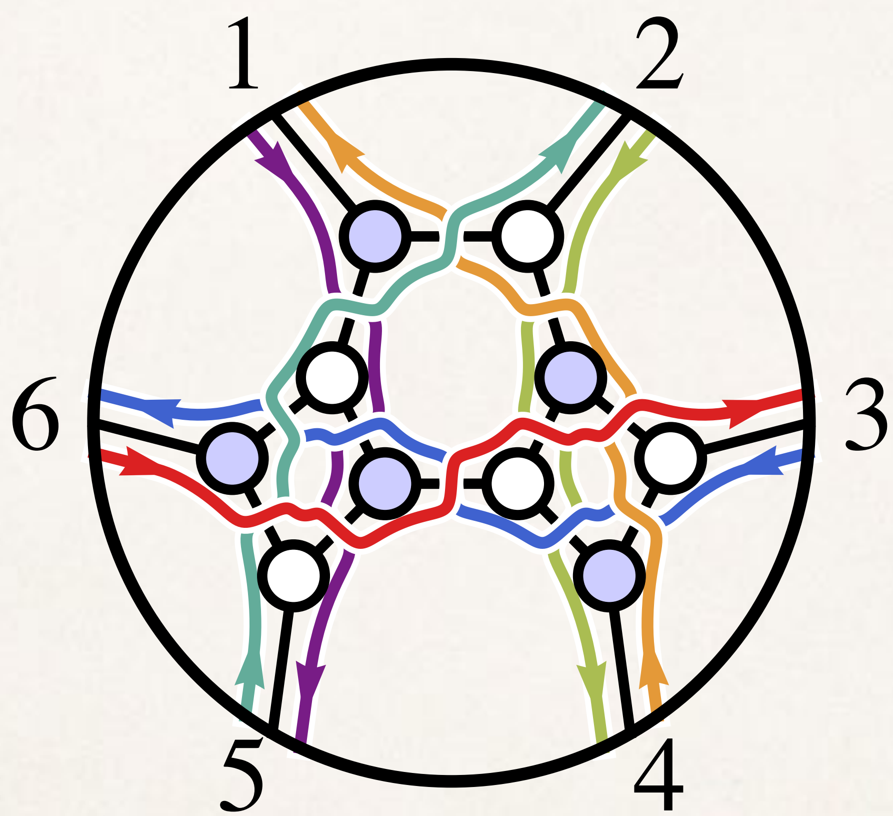
- Glue these vertices into diagrams



For any permutation there is a diagram

New look at permutations

- Glue these vertices into diagrams: plabic graph

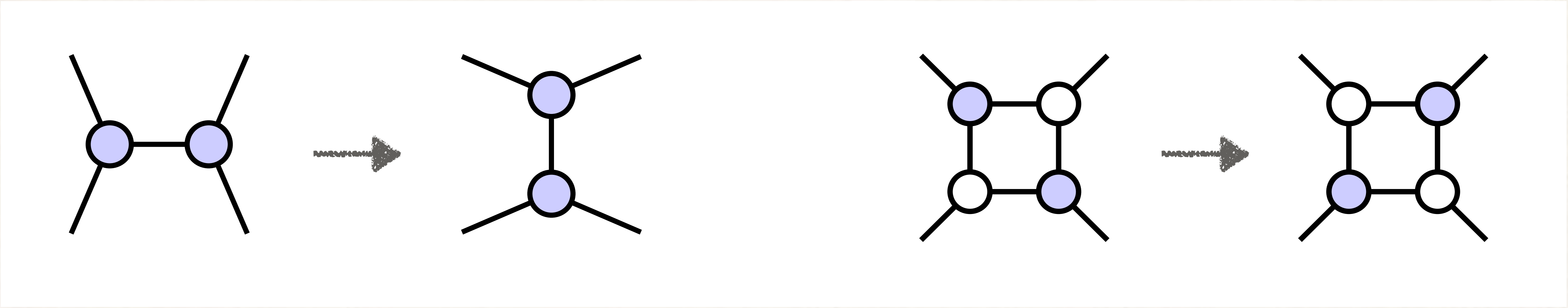


For any permutation there is a diagram

$$(1,2,3,4,5,6) \to (5,4,6,1,2,3)$$

Identity moves

- Are these diagrams unique for a given permutation?
- No! There are identity moves - do not change permutation



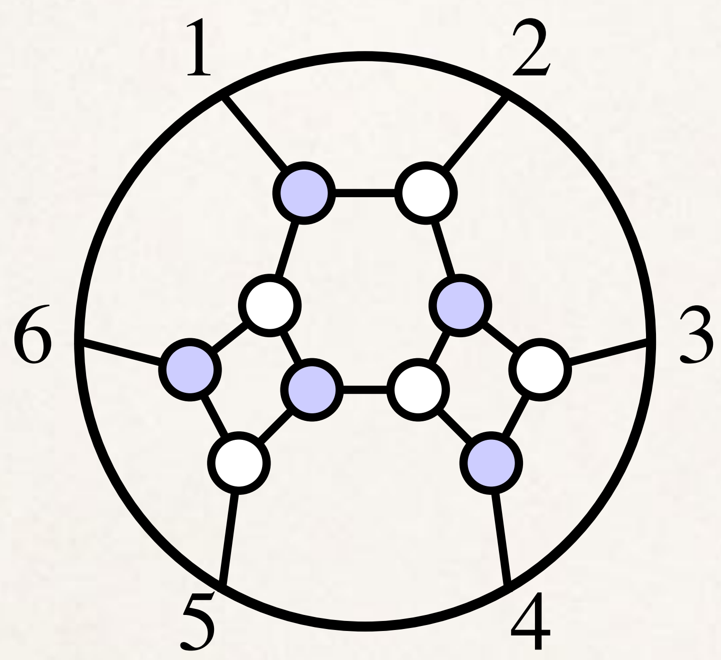
merge-expand

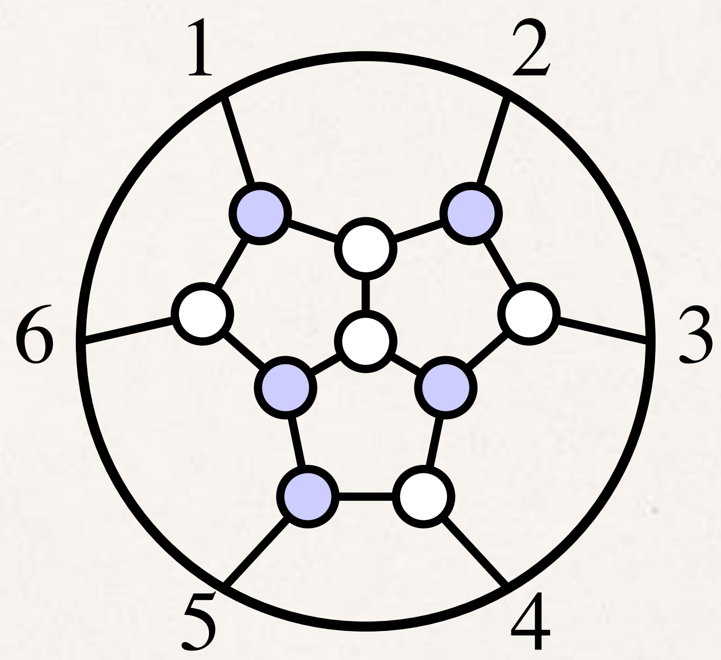
square move

- We already saw it in the context of on-shell diagrams

Identity moves

- Example: related by a sequence of identity moves





Identity moves

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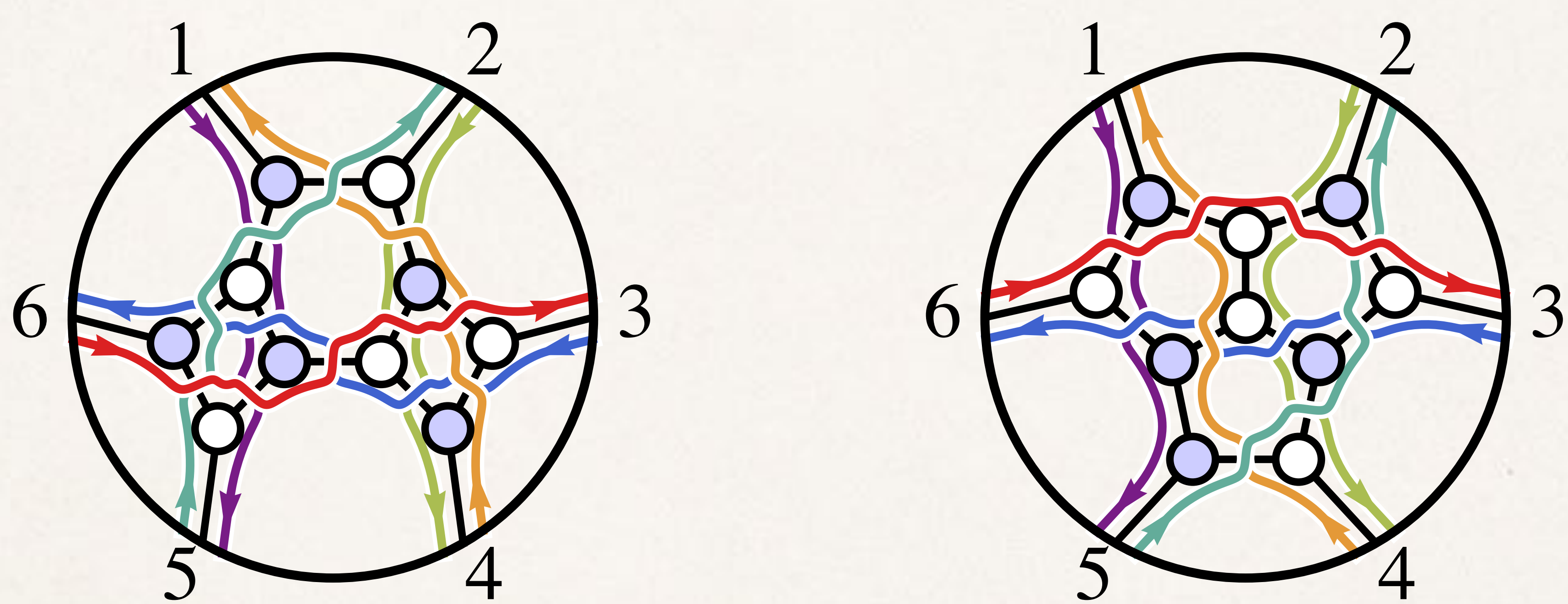
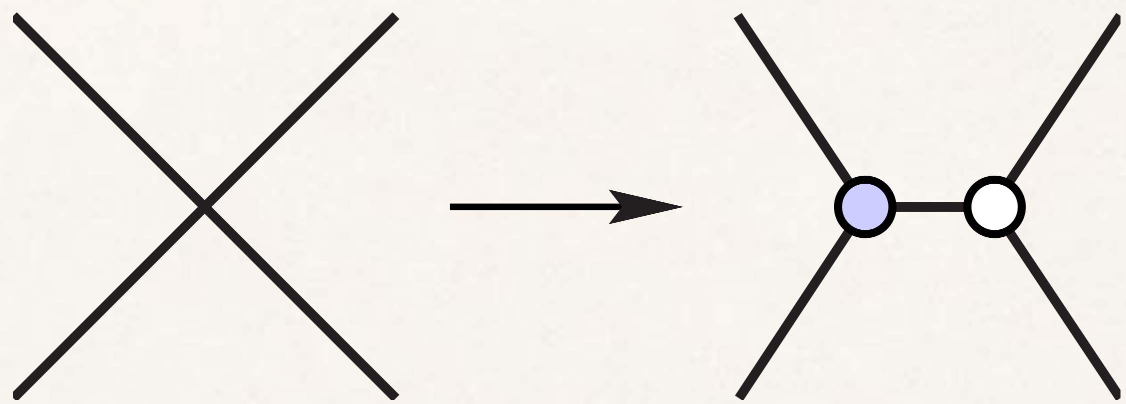


$$(1, 2, 3, 4, 5, 6) \to (5, 4, 6, 1, 2, 3)$$

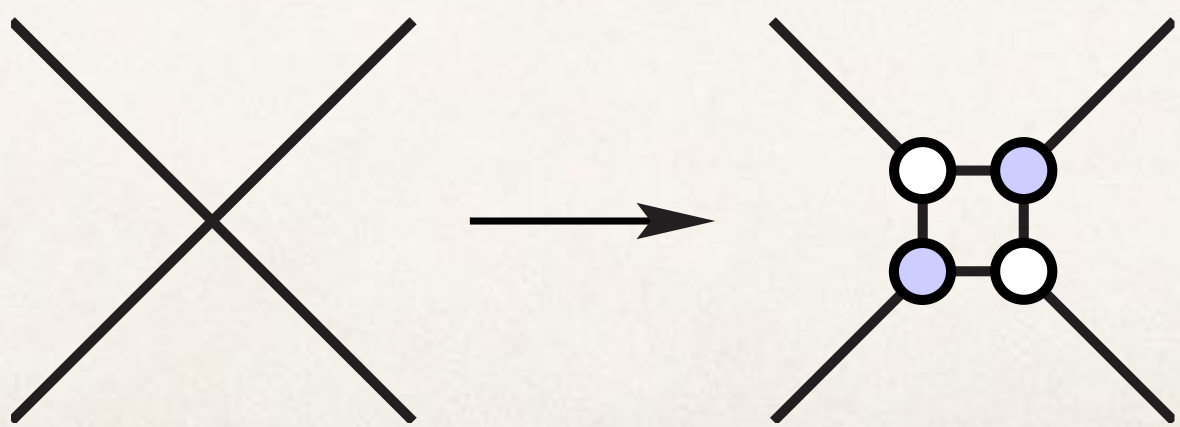
- If permutations are the same the diagrams are related by identity moves

Yang-Baxter moves

- Let us replace

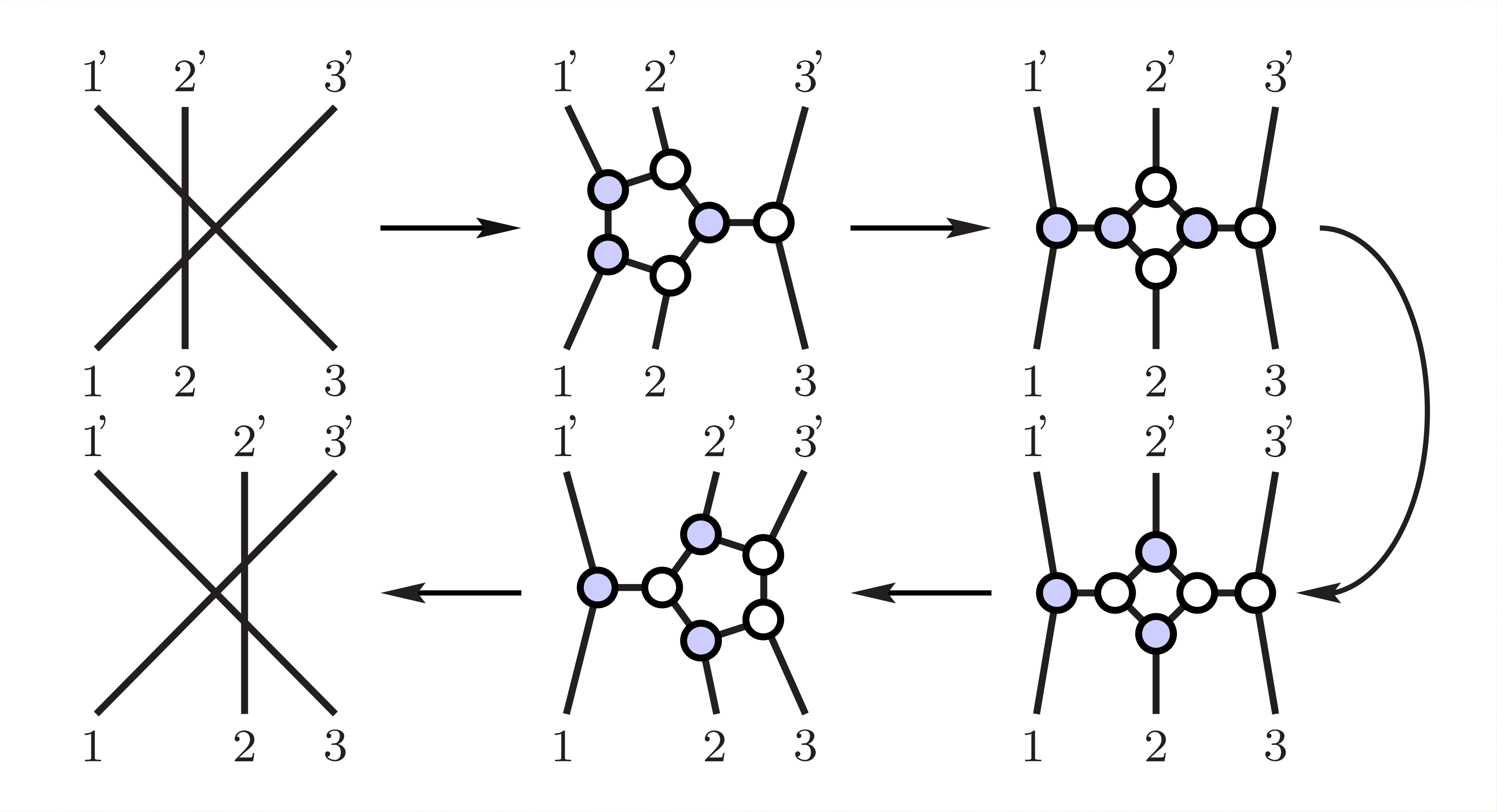


or



Yang-Baxter moves

- And check the Yang-Baxter:



Reduced information

- Reduced diagrams: plabic graphs which represent permutations
- They include diagrams which were relevant for tree-level amplitudes (but so far it is just pictures)
- The information to fully reconstruct the tree-level amplitudes is given by a set of permutations

permutation $\longrightarrow$ reduced on-shell diagram

Positive Grassmannian

Positive matrices

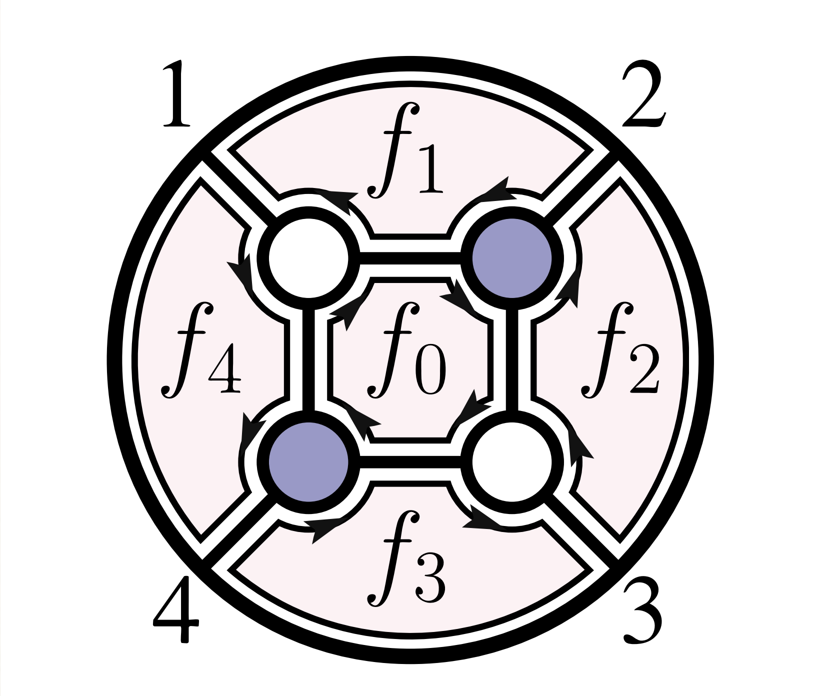
- Same diagrams came up in a very different context
- Build matrices with positive maximal minors

$$k \overset{n}{\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}} \qquad \begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix} \geq 0$$

- Positive Grassmannian: mod out by GL(k)

Face variables

- Draw a graph with two types of three point vertices
- Associate variables with the face of diagram

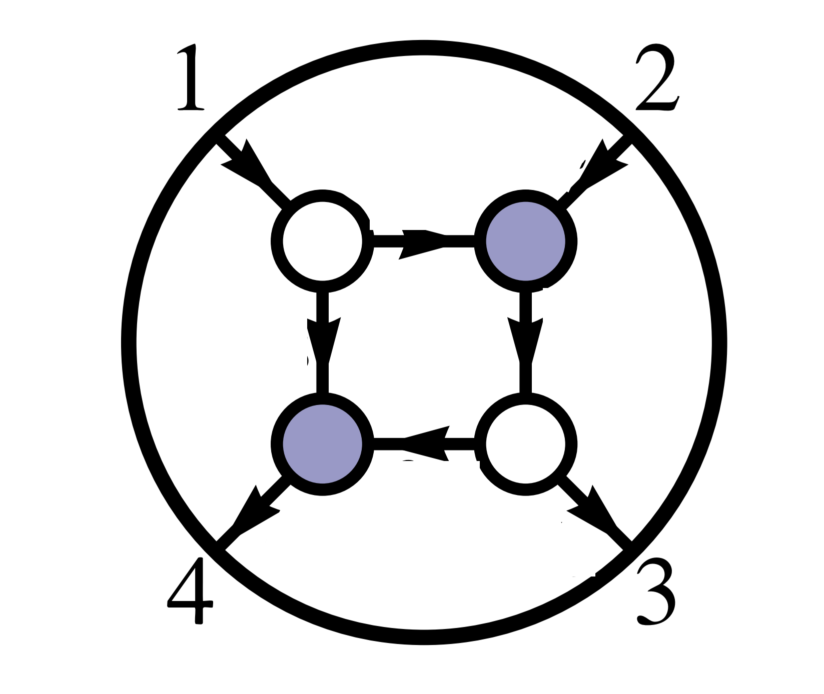


with the property

$$\prod_j f_j = -1$$

Perfect orientation

- Add arrows:



Perfect orientation

White vertex: one in, two out

Black vertex: two in, one out

- Not unique, always exists at least one
- Two (k) incoming, two (n-k) outgoing

Boundary measurement

- Define elements of $(k \times n)$ matrix

$$c_{ab} = -\sum_{\Gamma}\prod_{j}(-f_j)$$

product of all face variables to the right of the path

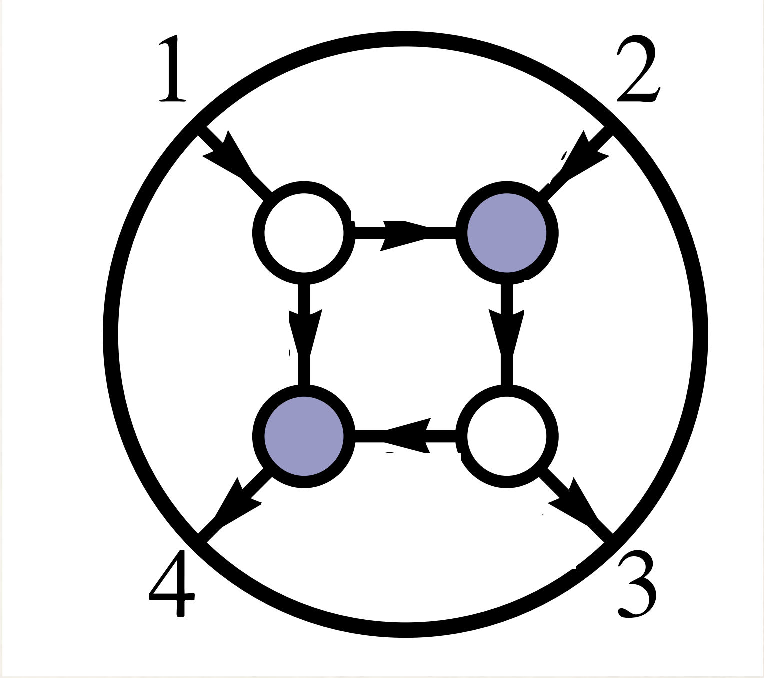
incoming

if b incoming

$c_{aa} = 1$

$c_{ab} = 0$

sum over all allowed paths



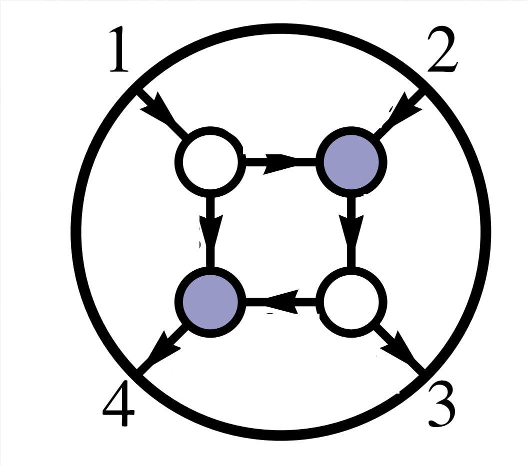
- Example: $c_{11} = c_{22} = 1 \quad c_{12} = c_{21} = 0$

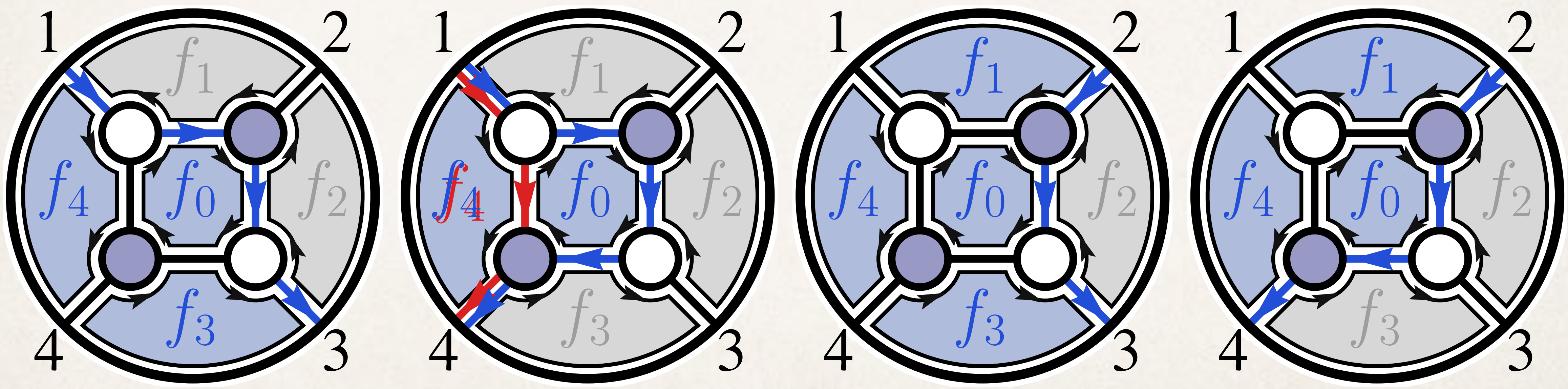
$$c_{13} = *,\ c_{14} = *,\ c_{23} = *,\ c_{24} = *$$

Entries of matrix

Apply on our example

$$c_{ab} = -\sum_{\Gamma} \prod_{j} (-f_j)$$





$-c_{13} = -f_0 f_3 f_4$ $\quad -c_{14} = f_0 f_4 - f_4$ $\quad -c_{23} = f_0 f_1 f_3 f_4$ $\quad -c_{24} = f_0 f_1 f_4$

Positive matrix

- The matrix is

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

f_2 eliminated

- There always exists choice of signs for f_i such that

$$C \in G_+(k,n)$$

- For our case:

$$m_{12} = 1 \qquad m_{23} = -f_0 f_3 f_4$$
$$m_{13} = -f_0 f_1 f_3 f_4 \qquad m_{24} = -f_4(1-f_0)$$
$$m_{14} = -f_0 f_1 f_4 \qquad m_{34} = f_0 f_1 f_3 f_4^2$$

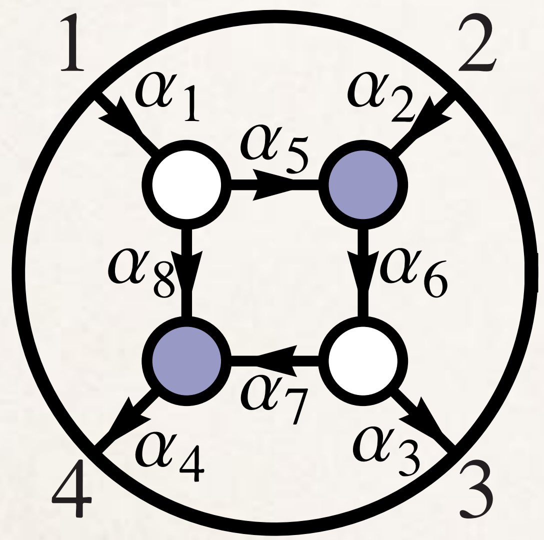
$\rightarrow$

$$f_0 < 0$$
$$f_1 < 0$$
$$f_3 > 0$$
$$f_4 < 0$$

All minors positive

Edge variables

- There is another set of variables which are redundant but have nice interpretation



$$C_{iJ} = -\sum_{\text{paths } i\to J} \prod \alpha_i \quad \text{edges along path}$$

$$c_{11} = 1, \qquad c_{12} = 0, \qquad c_{21} = 0, \qquad c_{22} = 1$$

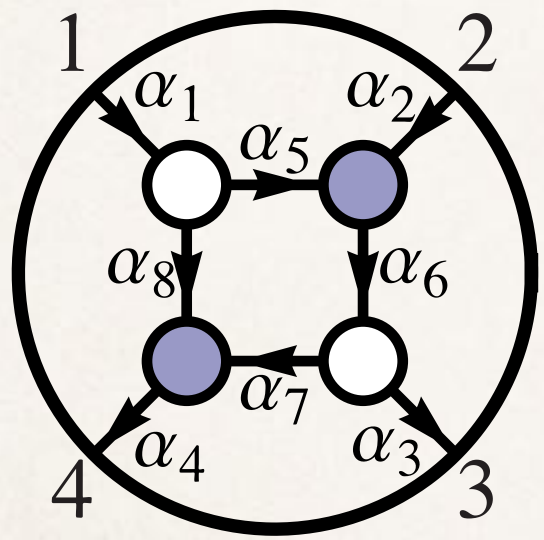
$$c_{13} = -\alpha_1\alpha_5\alpha_6\alpha_3, \qquad c_{14} = -\alpha_1(\alpha_5\alpha_6\alpha_7 + \alpha_8)\alpha_4$$

$$c_{23} = -\alpha_2\alpha_6\alpha_3, \qquad c_{24} = -\alpha_2\alpha_6\alpha_7\alpha_4$$

- We have to fix one α_j in each vertex

Edge variables

- There is another set of variables which are redundant but have nice interpretation



$$C_{iJ} = -\sum_{\text{paths } i\to J} \prod \alpha_i \quad \text{edges along path}$$

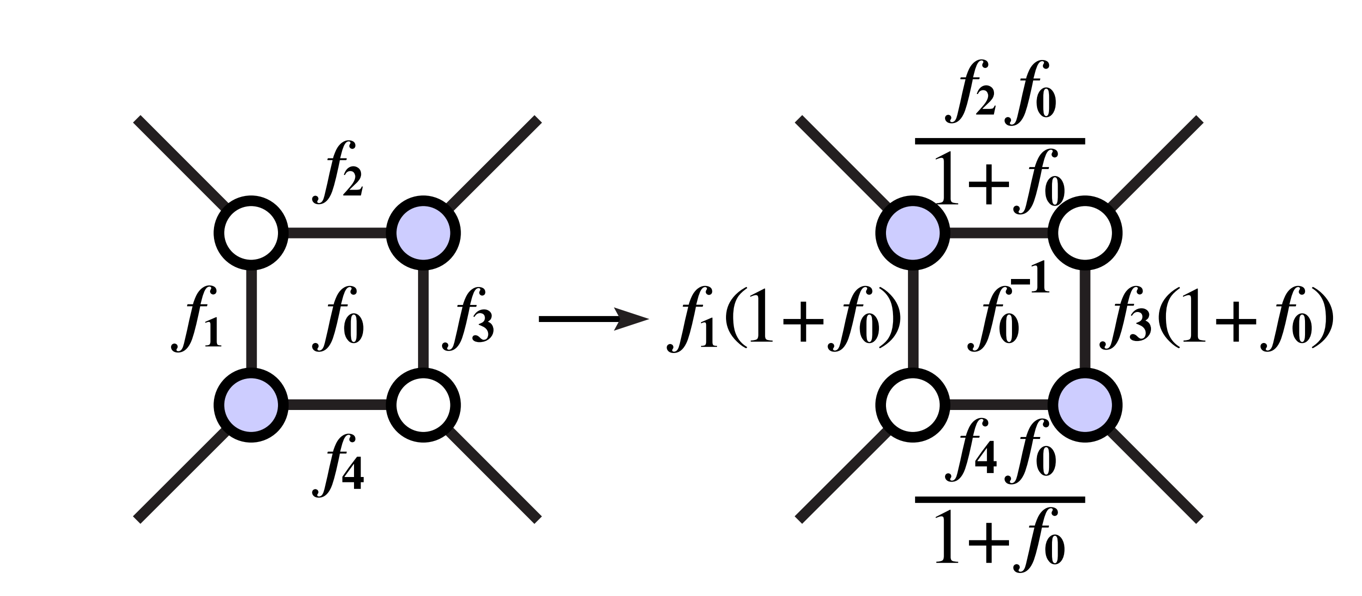
$$C = \begin{pmatrix} 1 & 0 & -\alpha_1\alpha_3\alpha_5\alpha_6 & -\alpha_1\alpha_4\alpha_5\alpha_6\alpha_7 - \alpha_1\alpha_4\alpha_8 \\ 0 & 1 & -\alpha_2\alpha_3\alpha_6 & -\alpha_2\alpha_4\alpha_6\alpha_7 \end{pmatrix}$$

- We have to fix one α_j in each vertex

  Setting α_j to zero means erasing the edge in the vertex

Cluster variables

- Face variables are cluster X-variables
- Identity moves: cluster transformations on face variables - compositions of cluster mutations



They preserve positivity

Cell in the Positive Grassmannian

- Cell in $G_+(k,n)$: specified by a set of non-vanishing Plucker coordinates

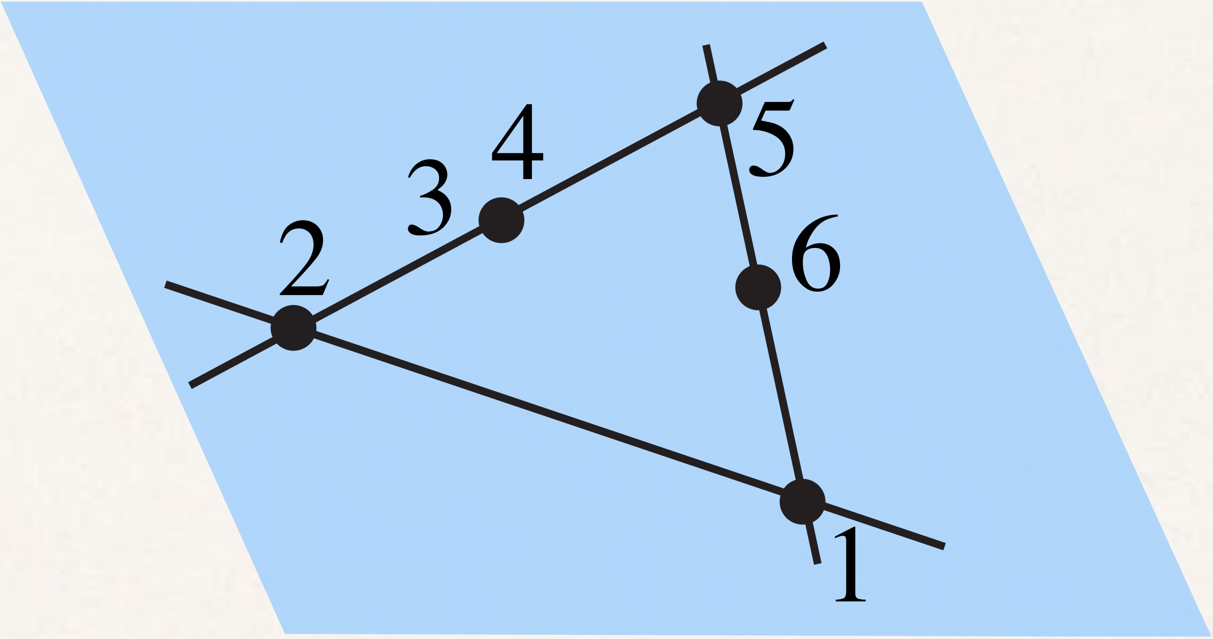
- Corresponds to configuration of points in $\mathbf{P}^{k-1}$

$$C = \begin{pmatrix} * & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}$$

- Positivity = convexity of the configuration

Example of configuration

- G(3,6) example



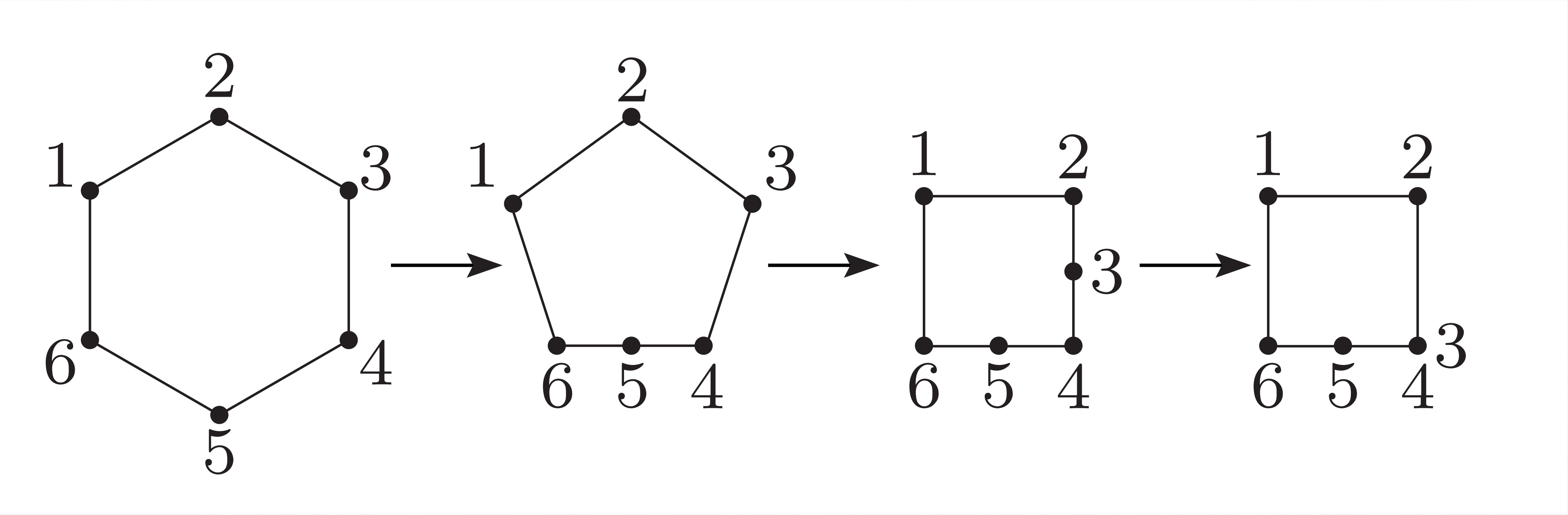
$$c_4 = a_{34}c_3 \qquad c_5 = a_{25}c_2 + a_{35}c_3$$

$$c_6 = a_{16}c_1 + zc_5 = a_{16}c_1 + za_{25}c_5 + za_{35}c_5$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 1 & 0 & 0 & a_{25} & za_{25} \\ 0 & 0 & 1 & a_{34} & a_{35} & za_{35} \end{pmatrix}$$

Example of stratification

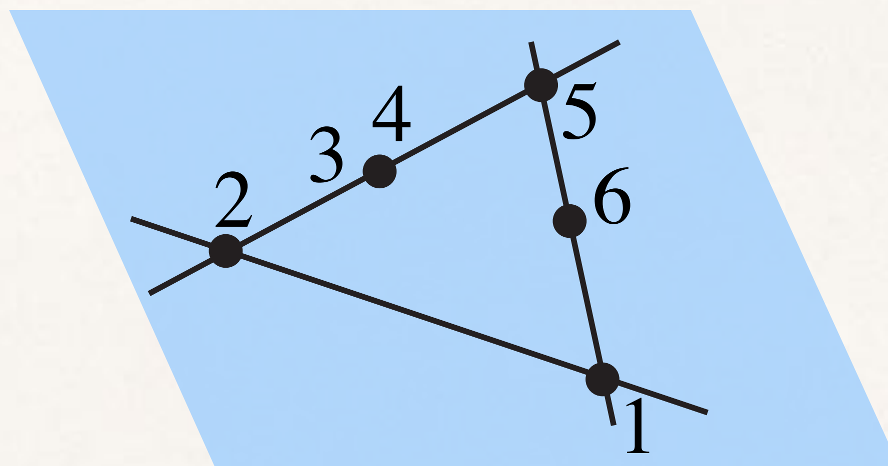
- Boundaries: deformed special configurations



- In the C-matrix send some positive variables to zero
- Positivity: linear relations between consecutive points

Relation to permutations

- The configuration of points gives the link to permutations

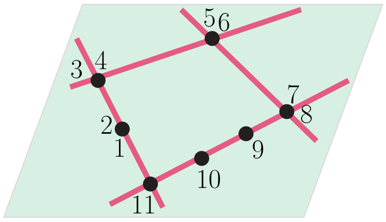


- Point $i \to \sigma(i)$ if $i \in (i+1, i+2, \ldots \sigma(i))$

$$1 \subset (2, 34, 5, 6) \to \sigma(1) = 6, \qquad 2 \subset (34, 5) \to \sigma(2) = 5,$$
$$3 \subset (4) \to \sigma(3) = 4, \qquad 4 \subset (5, 6, 1, 2) \to \sigma(4) = 2,$$
$$5 \subset (6, 1) \to \sigma(5) = 1, \qquad 6 \subset (1, 2, 3) \to \sigma(6) = 3.$$

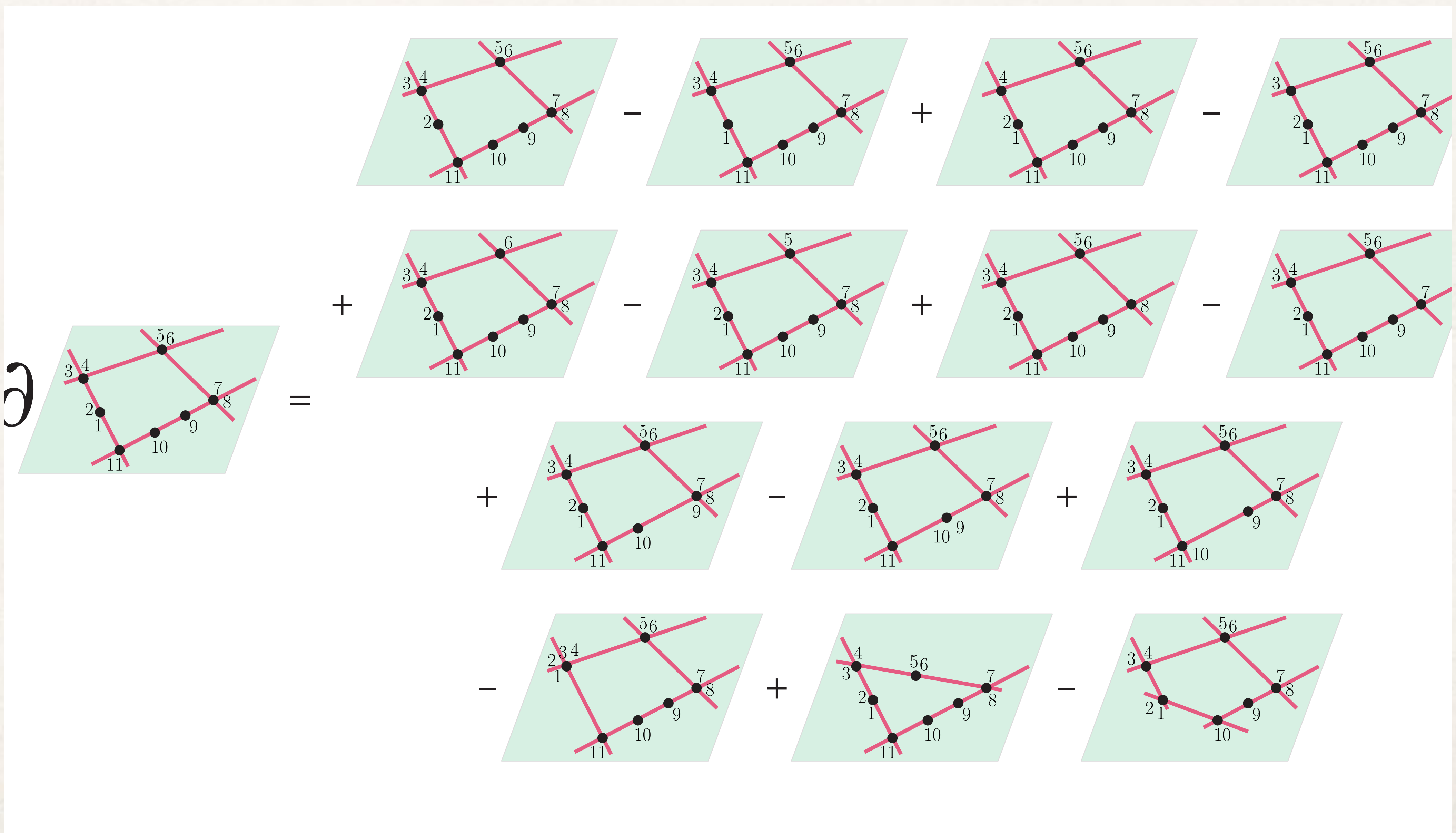
Boundary operator

- There is a notion of the boundary operator and stratification



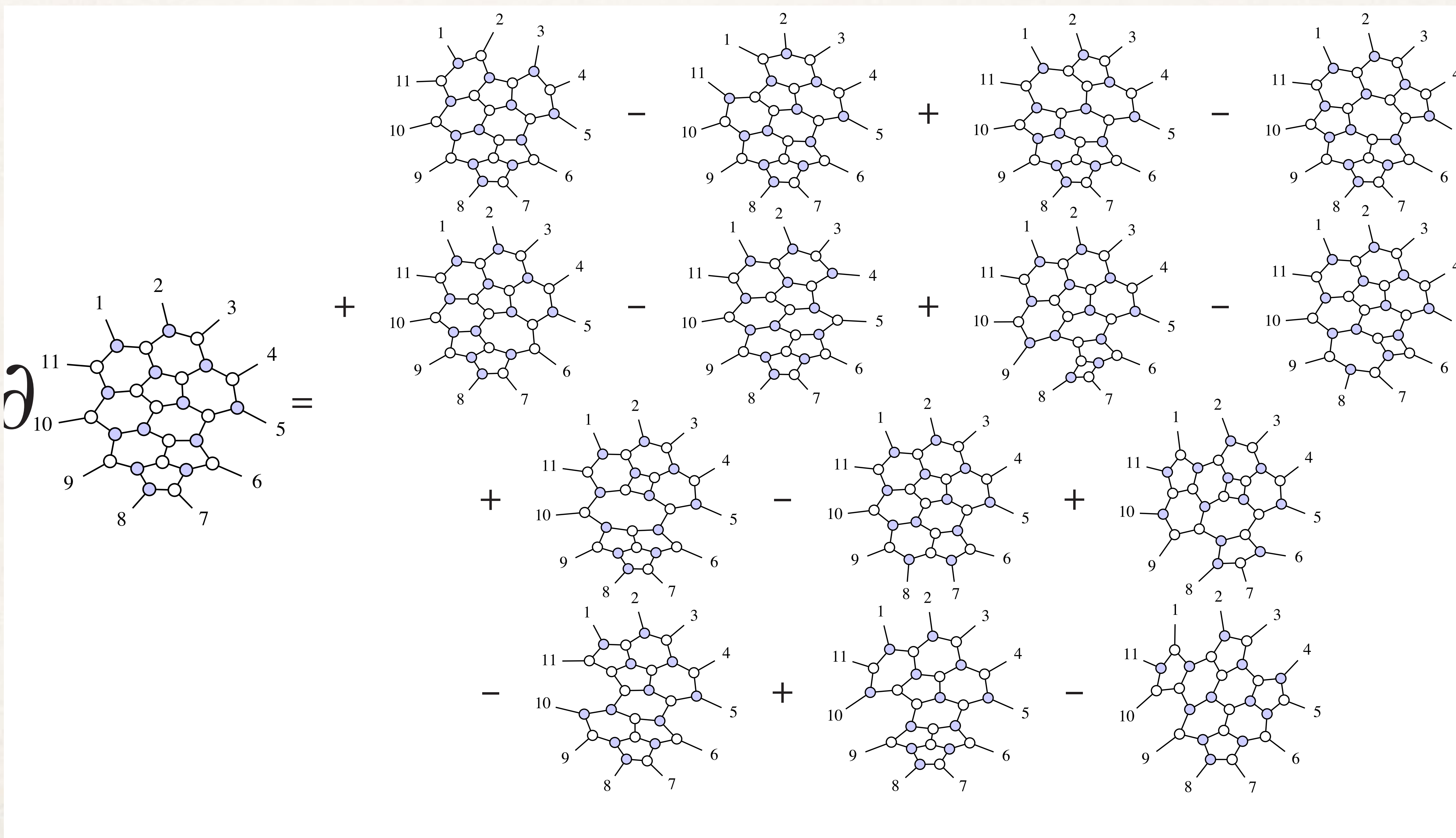
Boundary operator

- Making the configuration more special



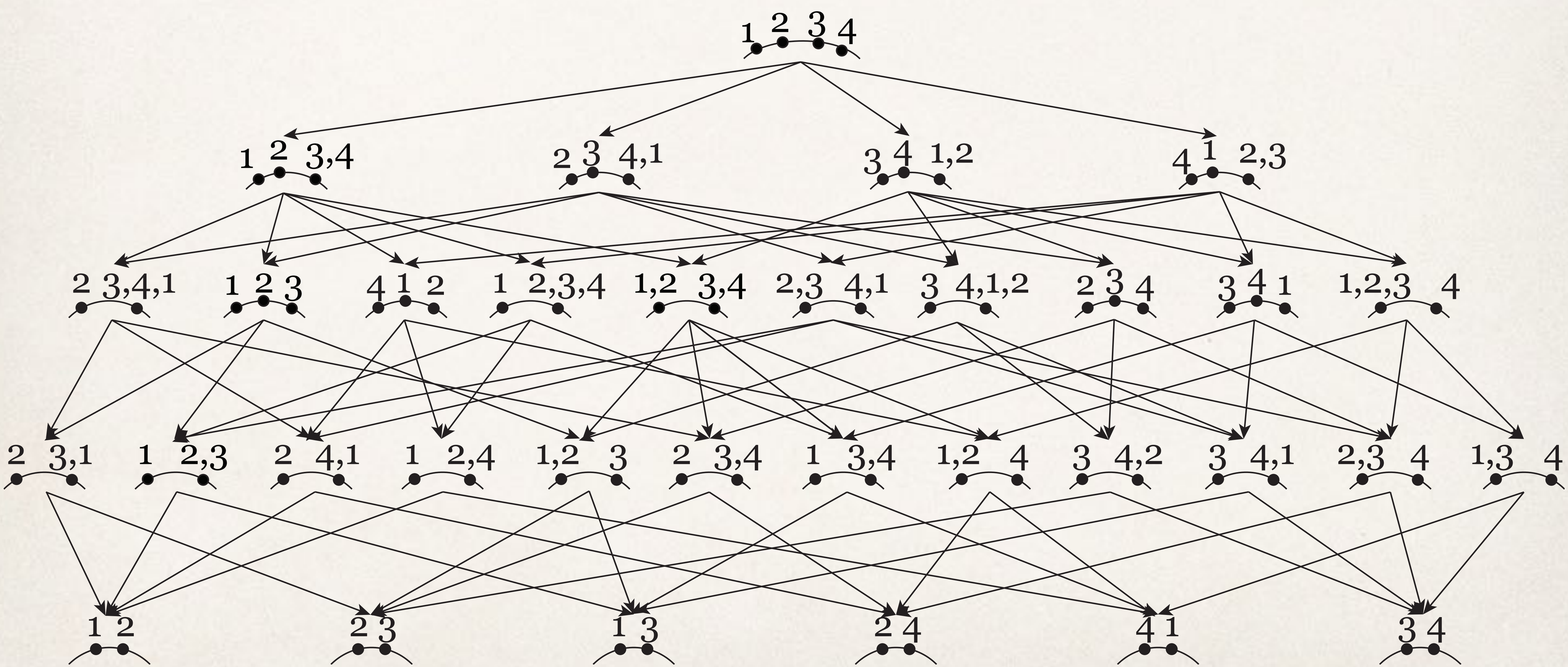
Boundary operator

- Erasing an edge in the plabic graph



Stratification of the positive Grassmannian

- Example of G(2,4):



Summary of positive Grassmannian

Reduced graphs (mod identity moves)

↕

Permutations

↕

Configuration of vectors with linear dependencies

↕

Cells of Positive Grassmannian

Thank you for attention!