Grassmannian Geometry of Scattering Amplitudes

LECTURE 2

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP)

University of California, Davis

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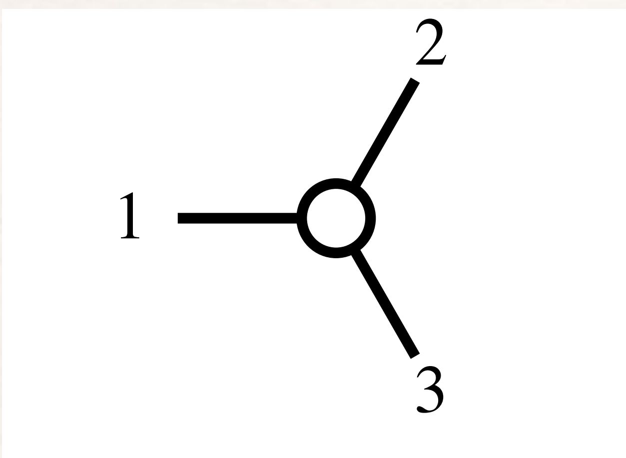
Atoms of amplitudes

New building blocks:
Three point amplitudes

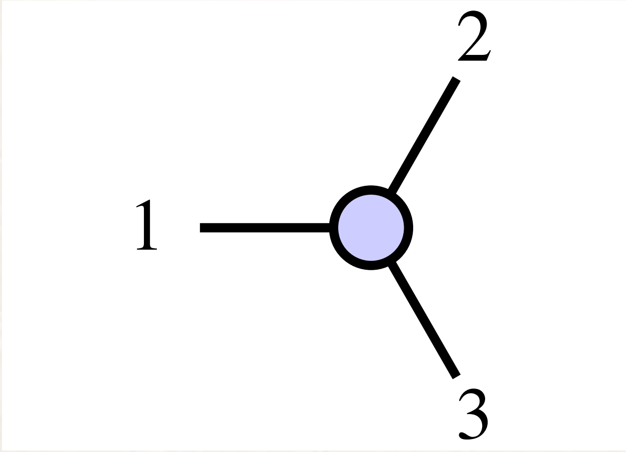
- On-shell, gauge invariant functions
- Fully fixed by Lorentz symmetry
- We want to build scattering amplitudes from them
- We will glue them together and introduce new objects called **on-shell diagrams**

Three point kinematics

- Two options



$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$



$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

<u>Spinor helicity variables</u>

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \widetilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda_{1a} \lambda_{2b}$$

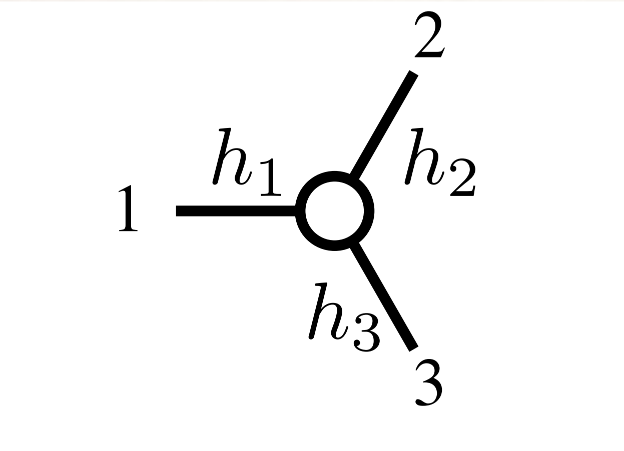
$$[12] = \epsilon_{\dot{a}\dot{b}} \widetilde{\lambda}_{1\dot{a}} \widetilde{\lambda}_{2\dot{b}}$$

Two solutions for 3pt kinematics

$$p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3) = 0$$

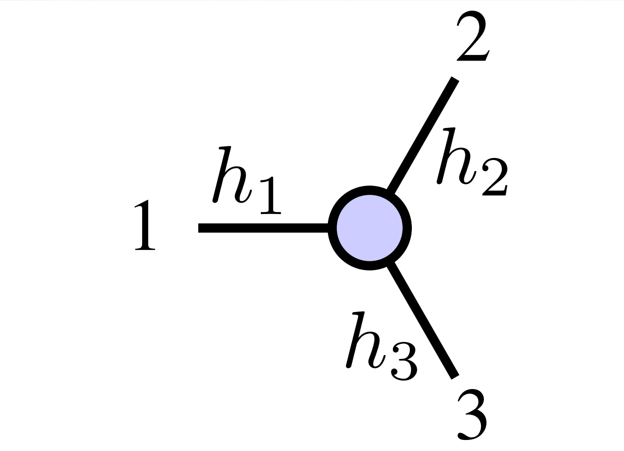
Three point amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$



$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

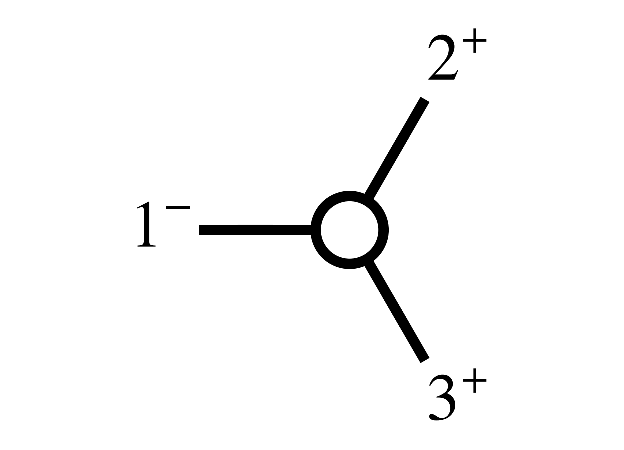
$$h_1 + h_2 + h_3 \leq 0$$

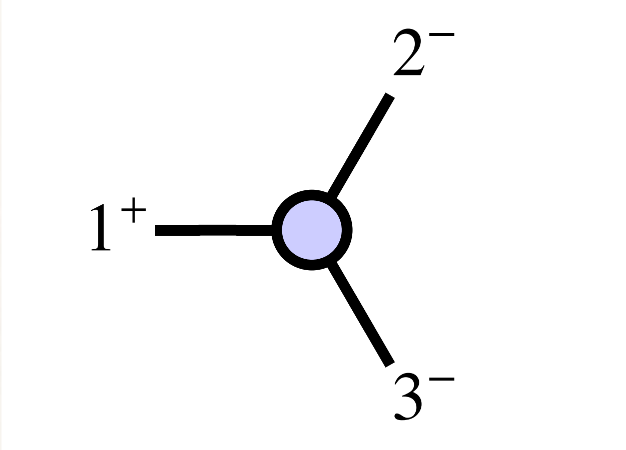
Special case of interest

- To simplify our task we choose a specific spin-1 case

$\lambda_1 \sim \lambda_2 \sim \lambda_3$

$$A_3^{(1)} = \frac{[23]^3}{[31][12]}$$





$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$

$$A_3^{(2)} = \frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle}$$

- Maximal supersymmetry: no need to specify helicities

- We consider SU(N) $\mathcal{N} = 4$ SYM theory

Amplitudes in N=4 SYM

- $\mathcal{N}=4$ superfield

$$\Phi = G_+ + \tilde{\eta}_A \Gamma_A + \frac{1}{2}\tilde{\eta}^A\tilde{\eta}^B S_{AB} + \frac{1}{6}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\overline{\Gamma}^D + \frac{1}{24}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\tilde{\eta}^D G_-$$

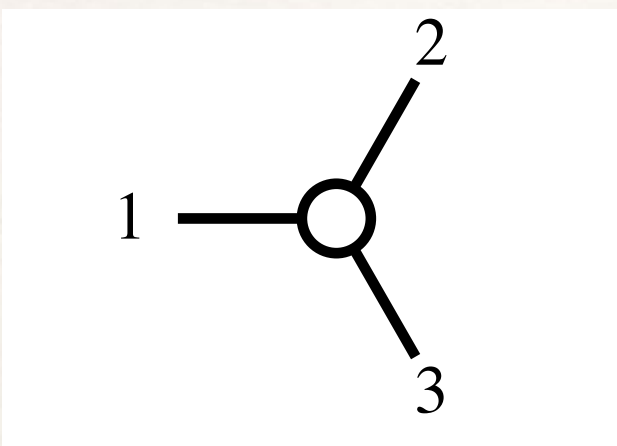
- Superamplitudes: $\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}$

Component amplitudes with power $\tilde{\eta}^{4k}$

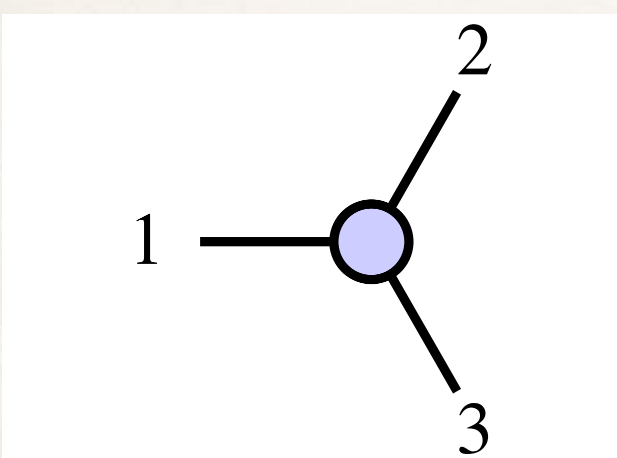
Contain amplitudes with k negative and (n-k) positive helicity gluons but also many others

Three point amplitudes

- In $\mathcal{N}=4$ SYM: super-amplitudes



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$



$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

$\widetilde{\eta}_k$ fermonic variable in the superfield

Easy book-keeping

Planar limit

- At tree-level

$$\mathcal{M}_n = \sum_{\sigma} \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n}) \times \mathcal{A}(123 \ldots n)$$

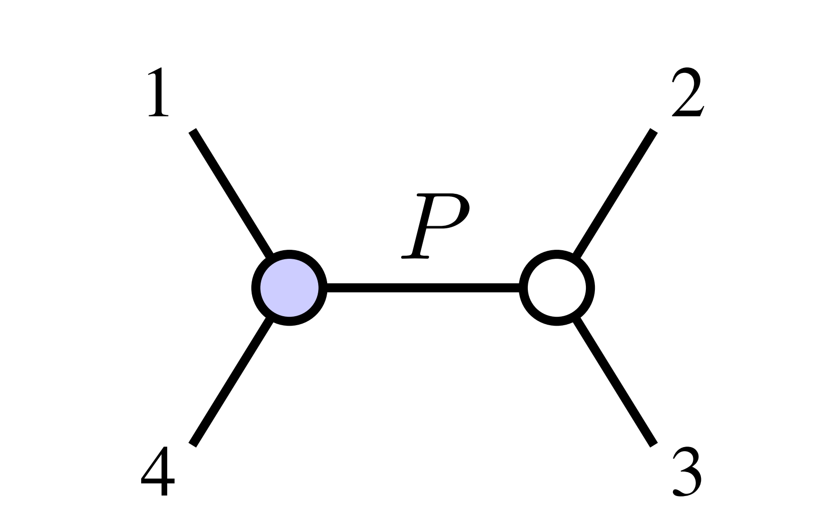
Supersymmetry irrelevant: massless QCD and N=4 SYM identical

- At loop level we get multiple traces too
  - Consider the planar limit $N \to \infty$
  - Single trace dominates, cyclic ordering
  - Planar $\mathcal{N} = 4$ SYM theory is special

On-shell diagrams

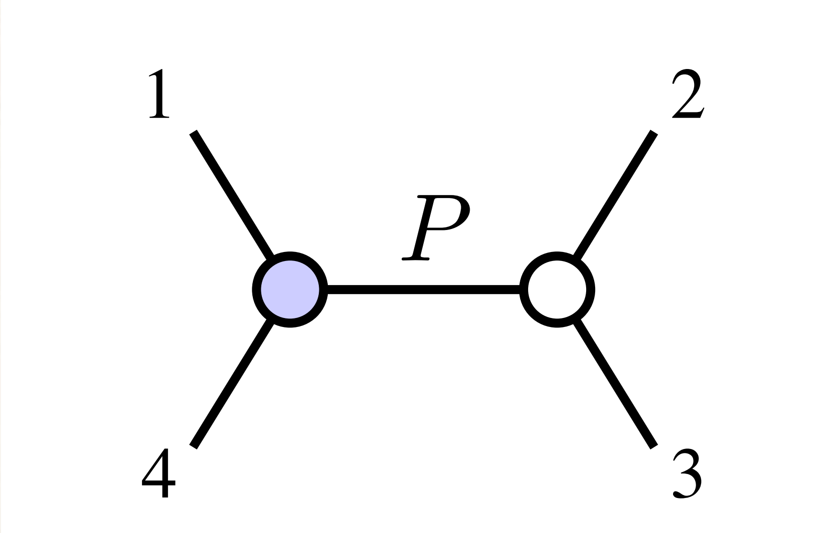
Gluing three point amplitudes

- Let us build a diagram



Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

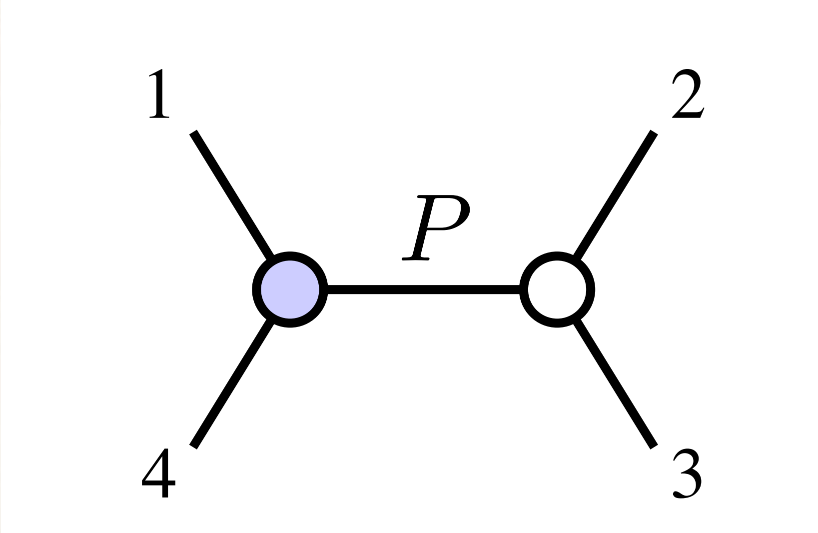
$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

$$= \frac{\delta^4(p_1+p_4+P)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_4\widetilde{\eta}_4+\lambda_P\widetilde{\eta}_P)}{\langle 14\rangle\langle 4P\rangle\langle P1\rangle} \times \frac{\delta^4(p_2+p_3-P)\delta^4(\widetilde{\eta}_P[23]+\widetilde{\eta}_2[3P]+\widetilde{\eta}_3[P2]}{[23][3P][P2]}$$

also $\lambda_P \sim \lambda_2 \sim \lambda_3$ and $\widetilde{\lambda}_1 \sim \widetilde{\lambda}_4 \sim \widetilde{\lambda}_P$

Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

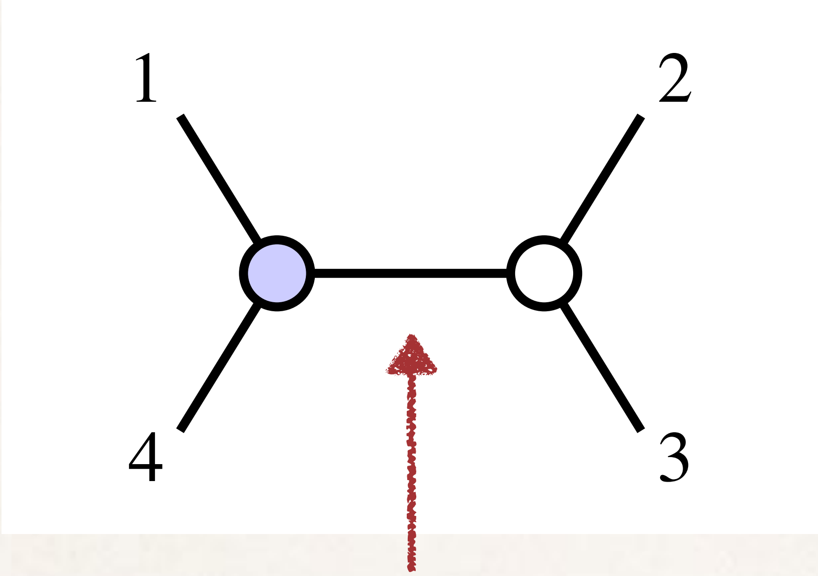
$$= \frac{\delta^4(p_1+p_2+p_3+p_4)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3+\lambda_4\widetilde{\eta}_4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \delta((p_2+p_3)^2)$$

$$= \mathcal{A}_4(1234) \times \delta((p_2+p_3)^2)$$

Four point tree level amplitude on factorization channel

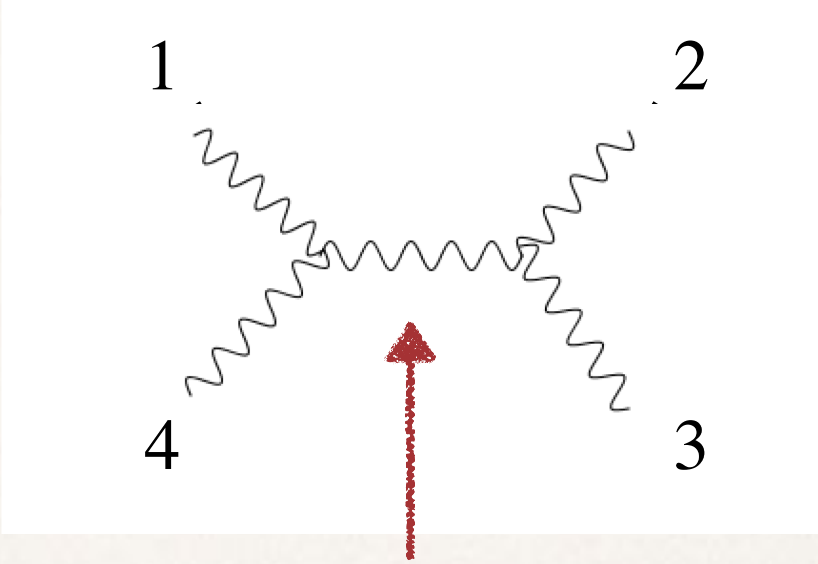
On-shell vs off-shell

- Different from Feynman diagrams



on-shell

- gauge-invariant
- factorization of amplitude

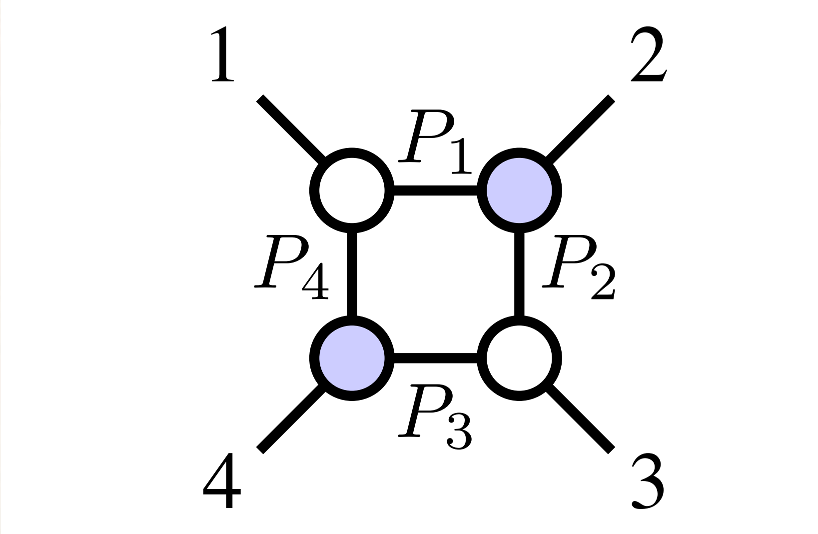


off-shell

- not gauge-invariant
- contributes to the amplitude

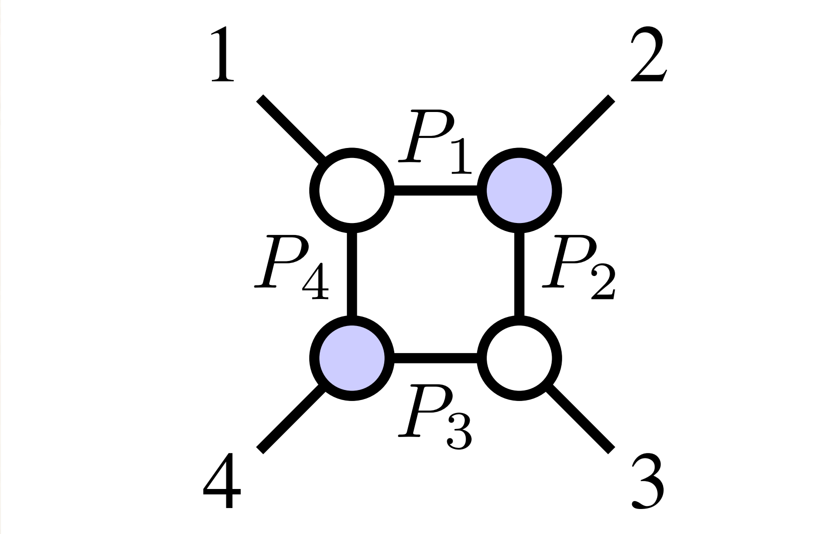
Gluing three point amplitudes

- Let us build a diagram



Gluing three point amplitudes

- Let us build a diagram

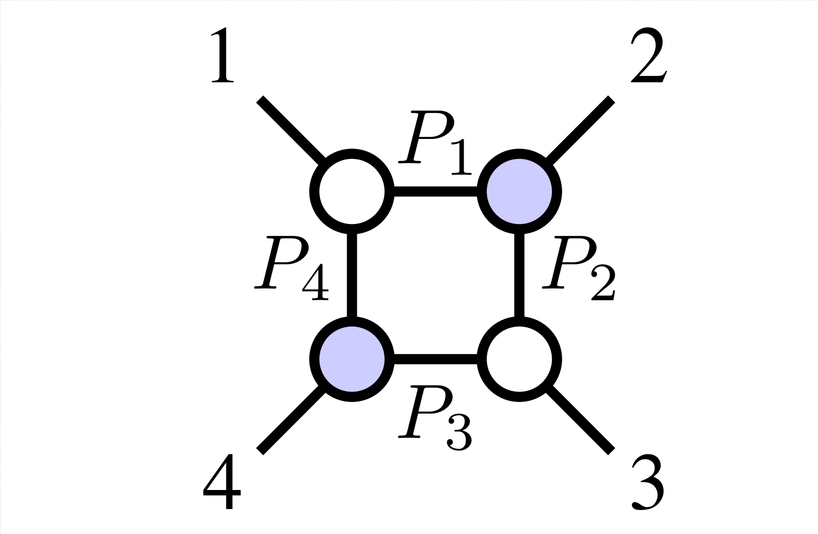


Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

Gluing three point amplitudes

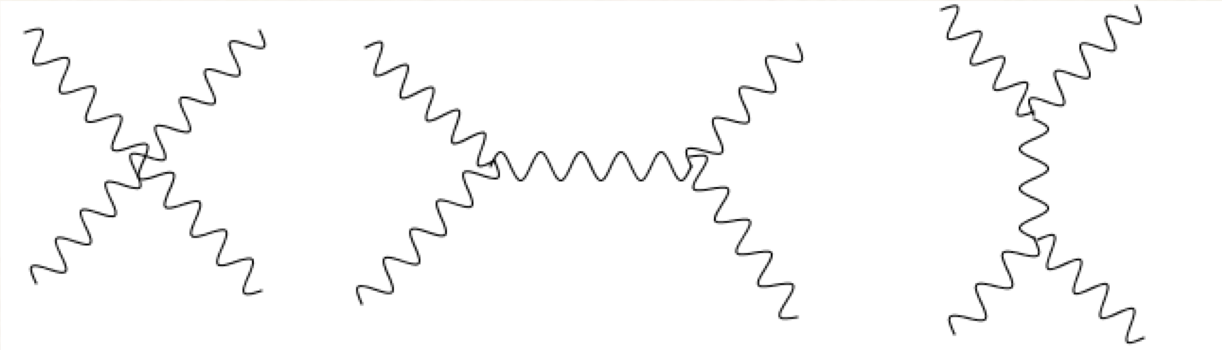
- Let us build a diagram



Multiply four three point amplitudes

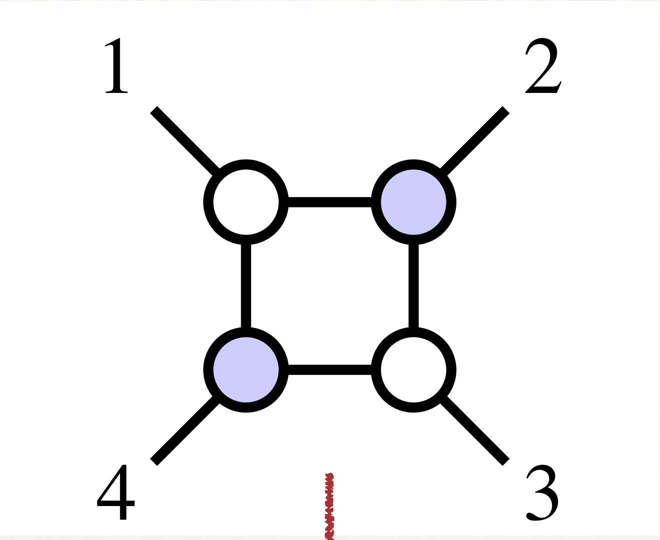
$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

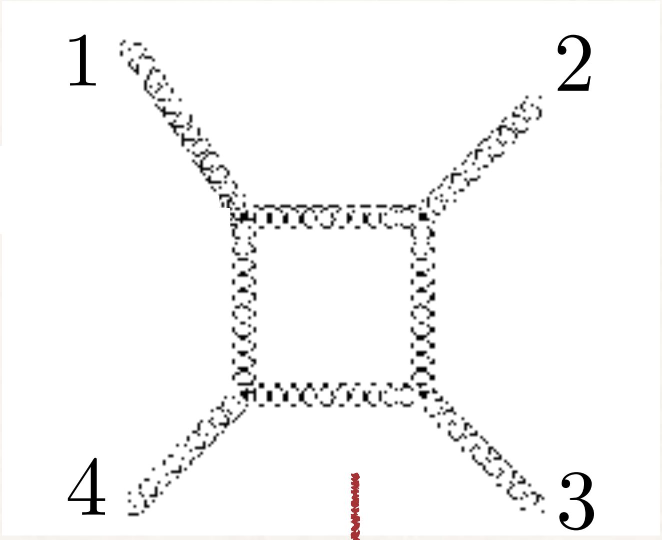
$$\boxed{= \mathcal{A}_4(1234)}$$



On-shell vs off-shell

- Comparison to similarly looking Feynman diagram



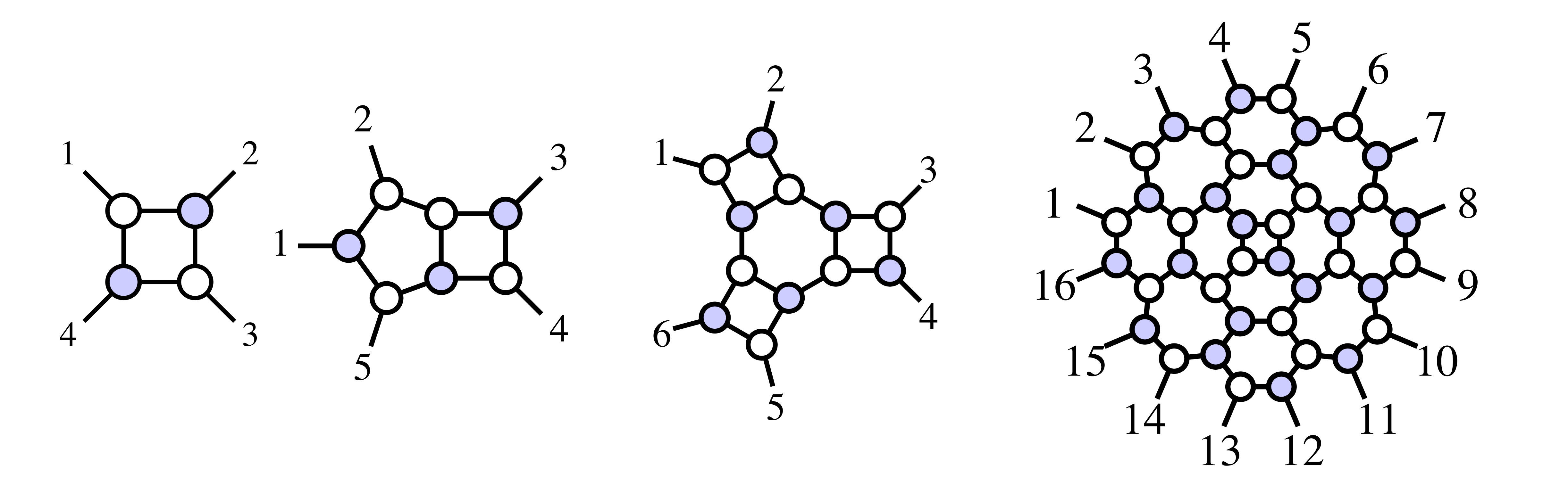


- product of four 3pt amplitudes
- also a cut of 1-loop amplitude: 4 propagators on-shell
- no unspecified parameters left

- one-loop diagram
- loop momentum with 4 unspecified parameters

On-shell diagrams

- Draw arbitrary graph with three point vertices

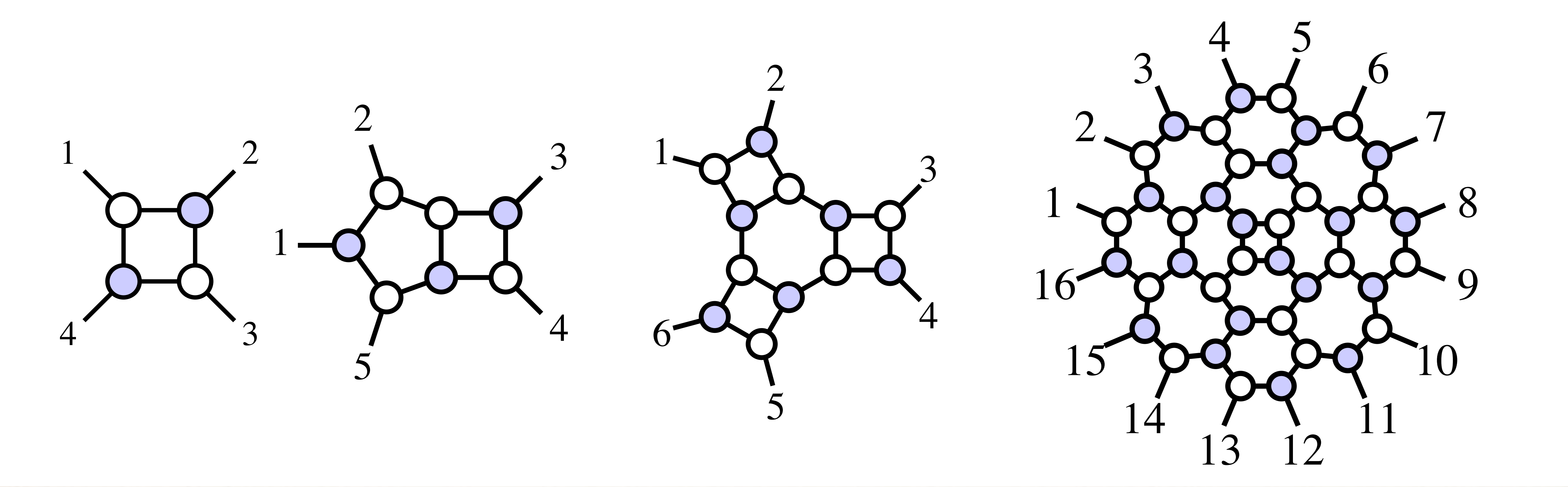


Products of three point amplitudes

$$\begin{cases} P > 4L & \text{Extra delta functions} \\ P = 4L & \text{Function of external data only} \\ P < 4L & \text{Unfixed parameters (forms)} \end{cases}$$

On-shell diagrams

- Draw arbitrary graph with three point vertices



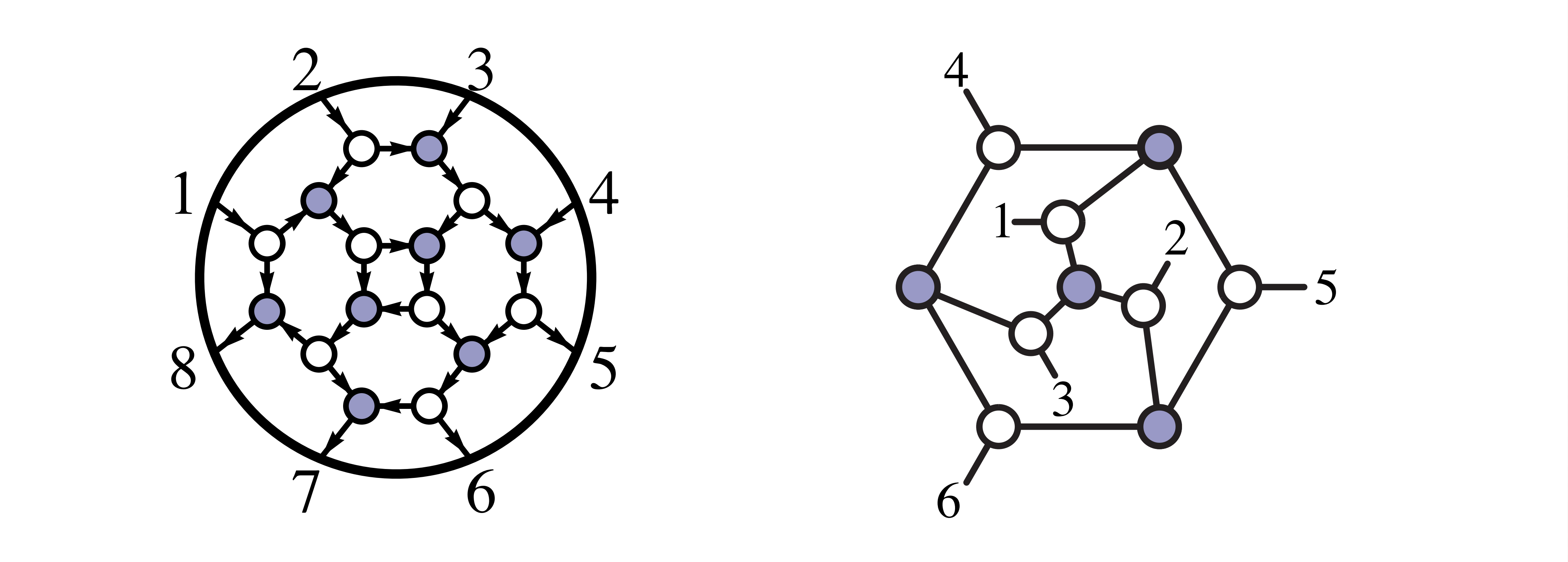
On-shell diagrams with $P \leq 4L$ are cuts of the amplitude

- Parametrized by n, k

$$k = 2B + W - P$$

On-shell diagrams in general

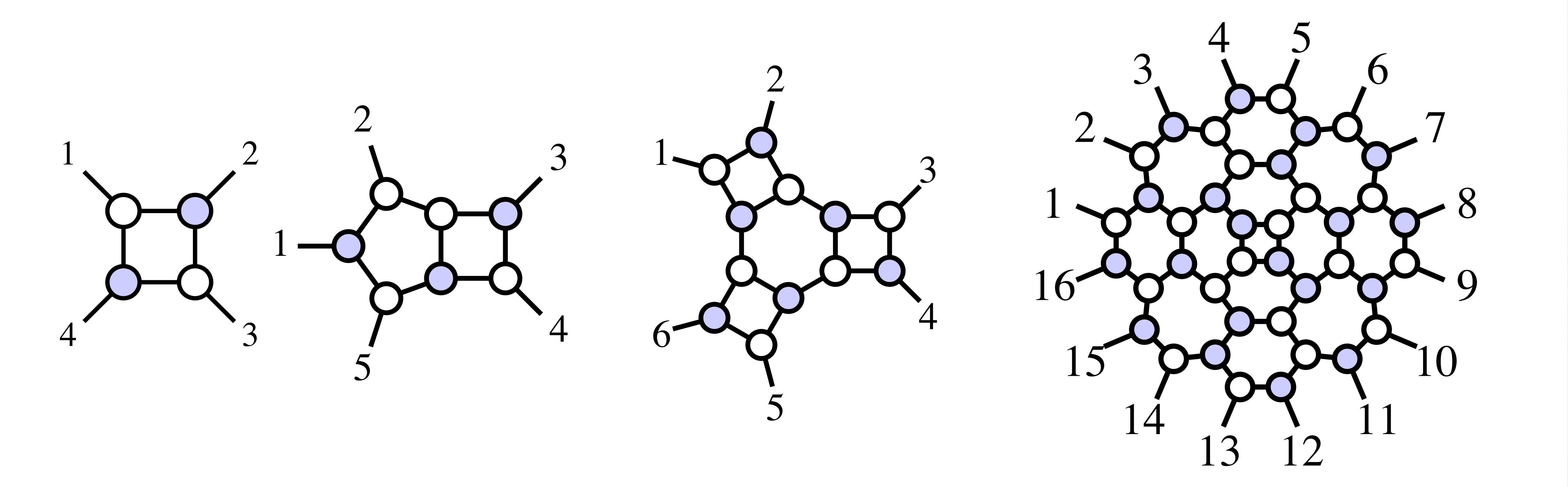
- Not limited just to planar N=4 SYM theory



- Planar, non-planar, susy, non-susy, gluons, gravitons, electrons, quarks…..

On-shell diagrams

- Stay in planar N=4 SYM theory

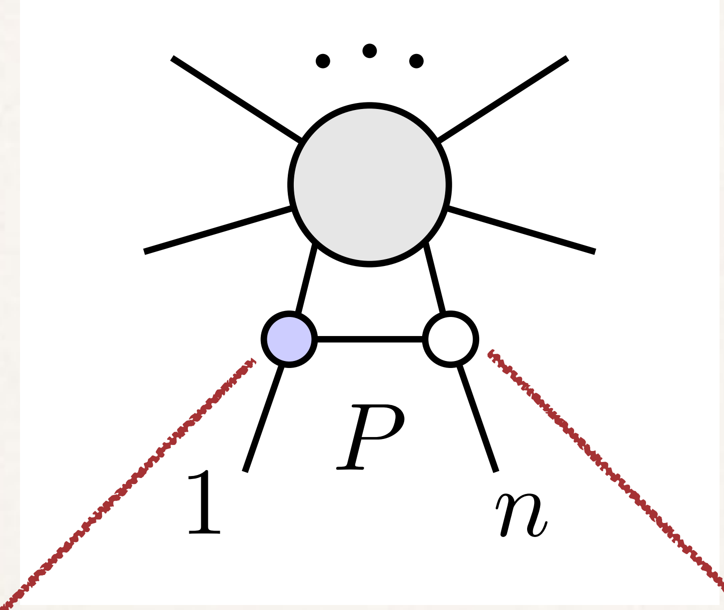


Question: Can we build amplitude from on-shell diagrams?

Recursion relations

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

$\downarrow$

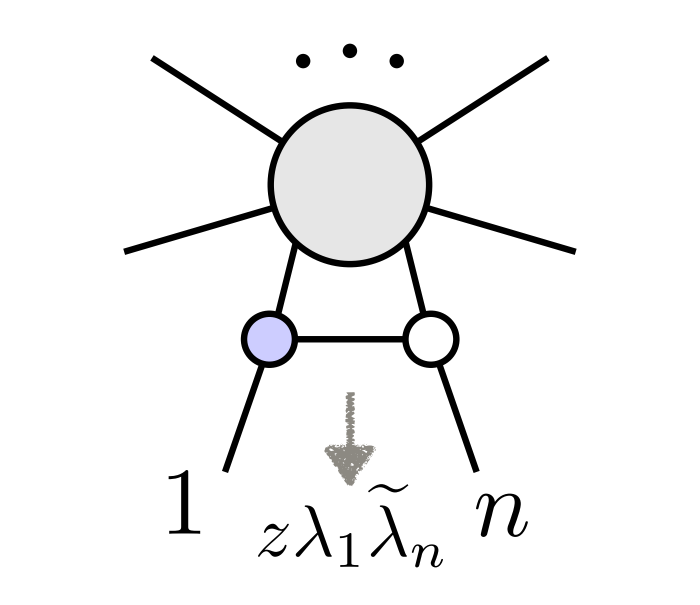
Adding one parameter

$$\tilde{\lambda}_P \sim \tilde{\lambda}_1 \qquad \lambda_P \sim \lambda_n$$

$$P = z\lambda_n\tilde{\lambda}_1$$

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

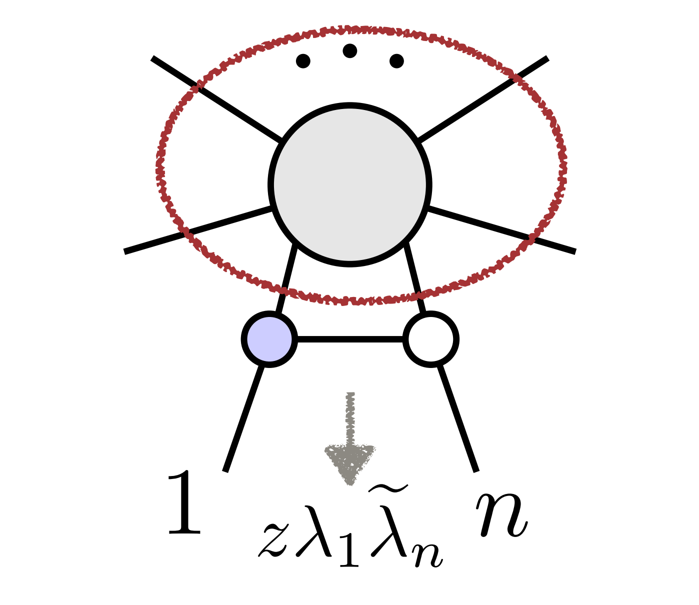
↓

Adding one parameter

New formula: $$K_1(z) = \frac{dz}{z} K_0(z)$$

BCFW shift

- Consider following diagram



One more loop

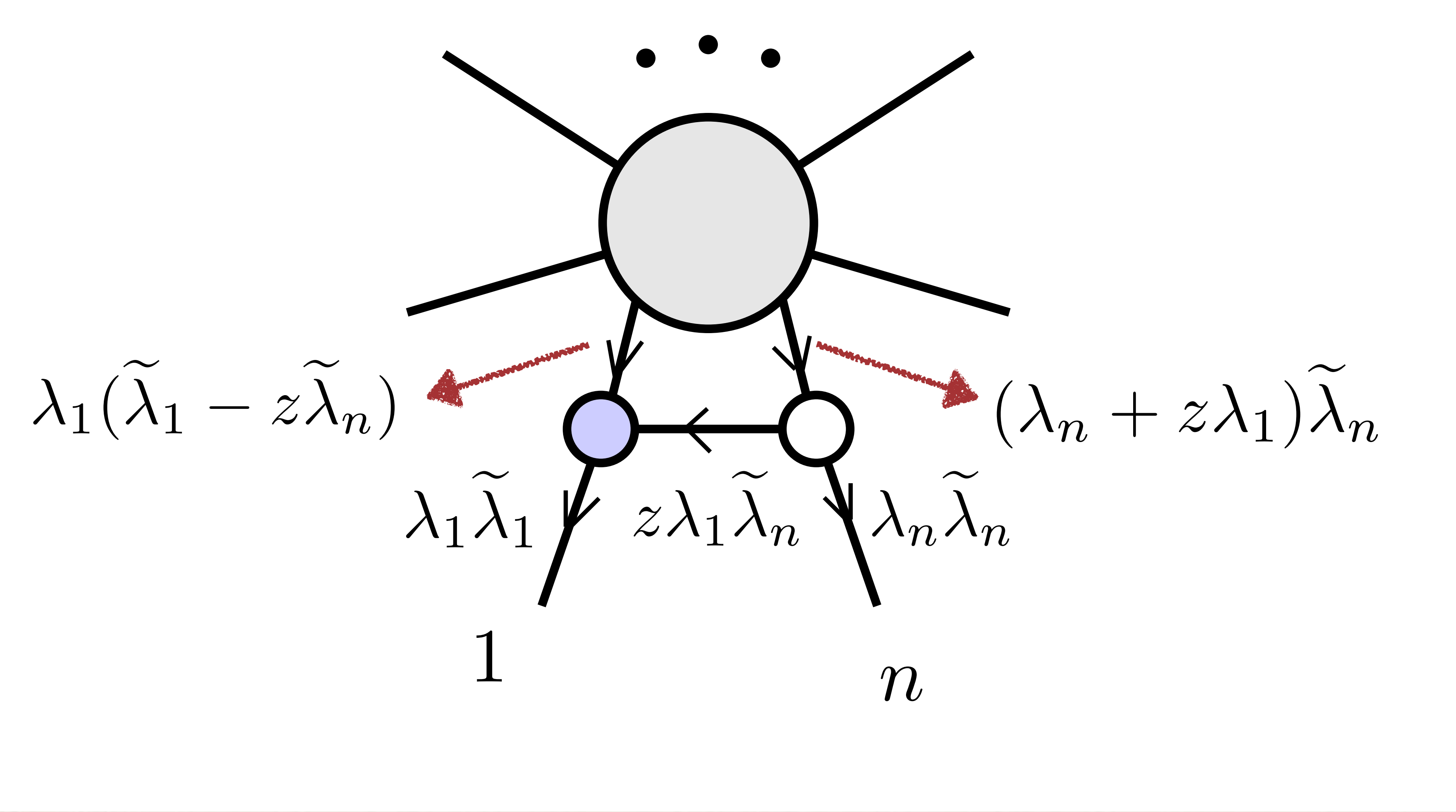
Three more on-shell conditions

Adding one parameter

New formula: $$K_1(z) = \frac{dz}{z} K_0(z)$$ What is this?

BCFW shift

- Consider following diagram



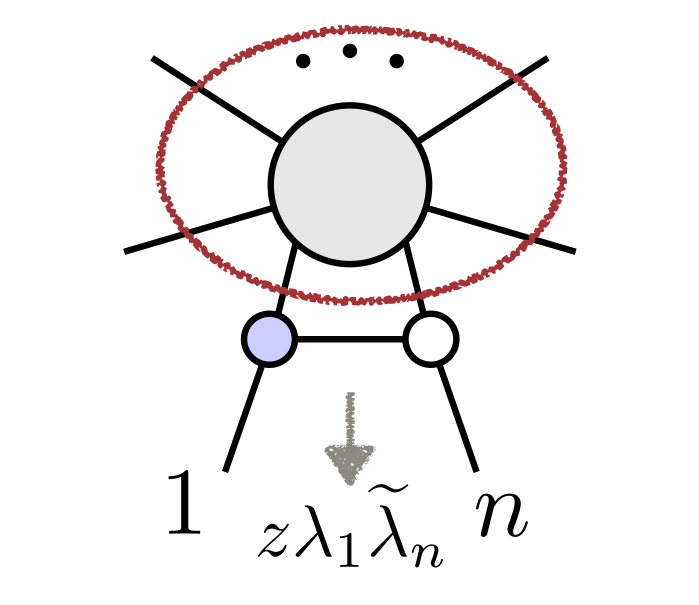
Effectively a shift

$$\lambda_n \to \lambda_n + z\lambda_1$$

$$\widetilde{\lambda}_1 \to \widetilde{\lambda}_1 - z\widetilde{\lambda}_n$$

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

Adding one parameter

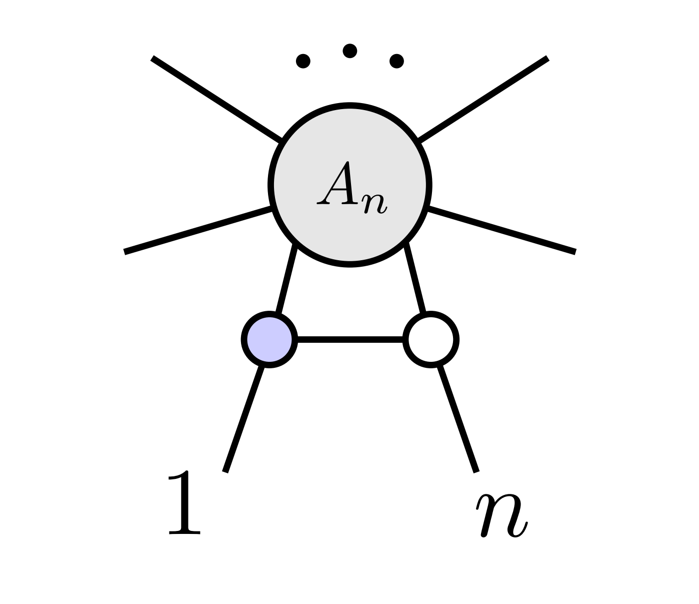
New formula: $K_1(z) = \frac{dz}{z} K_0(z)$

Old on-shell diagram with shift

$$\lambda_n \to \lambda_n + z\lambda_1$$
$$\widetilde{\lambda}_1 \to \widetilde{\lambda}_1 - z\widetilde{\lambda}_n$$

Cauchy formula

- Suppose the blob is the amplitude



Shifted amplitude $= \mathcal{A}_n(z)$

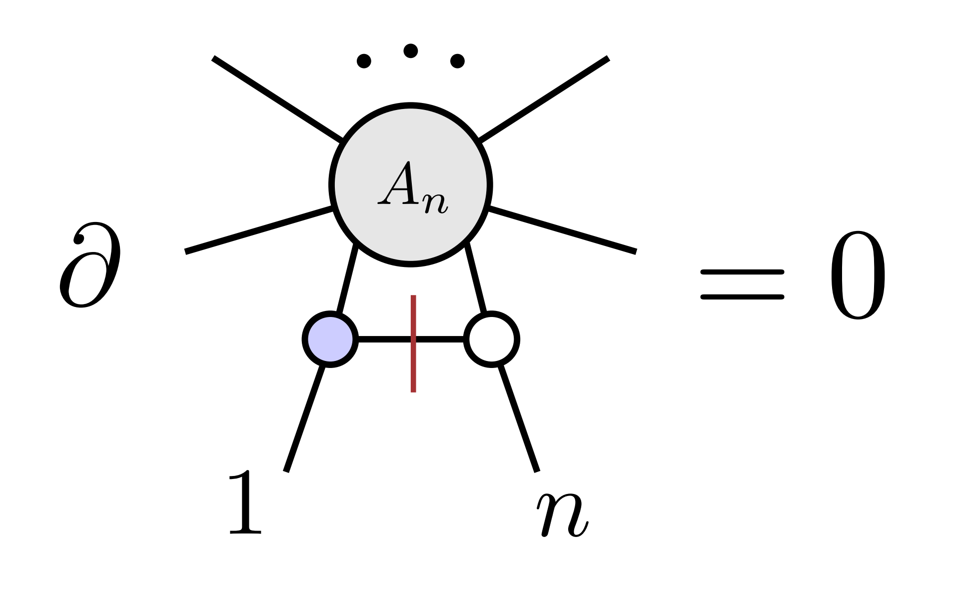
$$\lambda_n \to \lambda_n + z\lambda_1$$
$$\tilde{\lambda}_1 \to \tilde{\lambda}_1 - z\tilde{\lambda}_n$$

- Cauchy formula $\quad \partial\, \mathcal{A}_n(z) = 0$

Take the residue on $z = z_k \quad \leftrightarrow \quad$ Erase an edge in the diagram

Cauchy formula

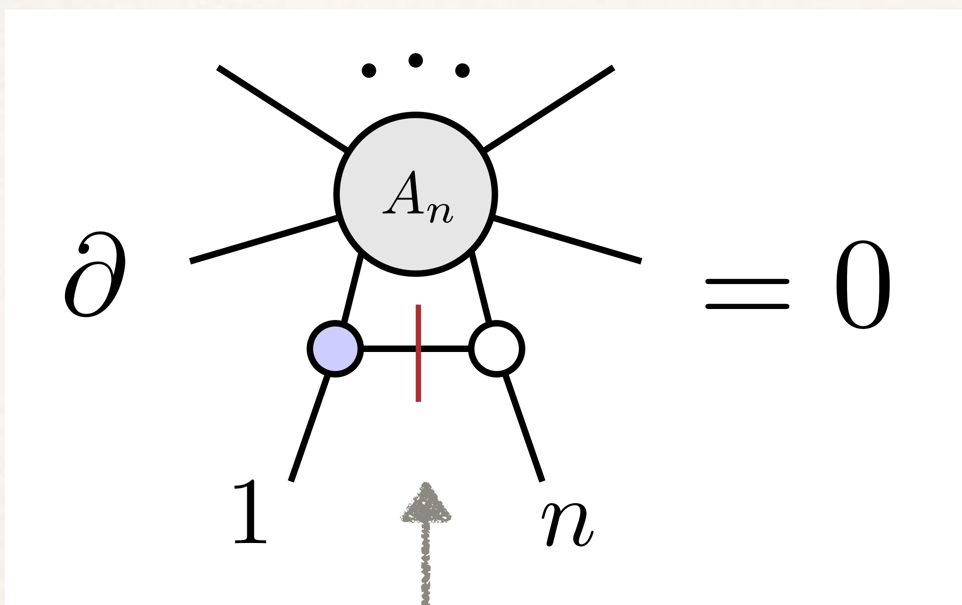
- Suppose the blob is the amplitude



$$\oint \frac{dz}{z} A_n(z) = 0$$

Cauchy formula

- Suppose the blob is the amplitude



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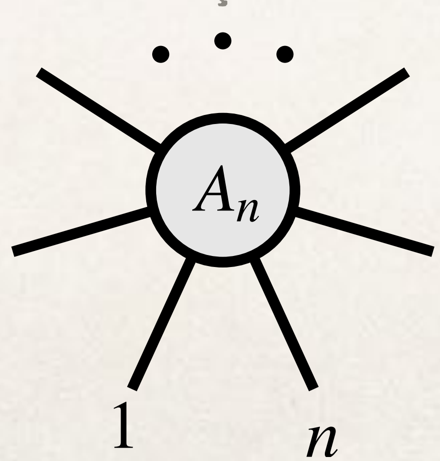
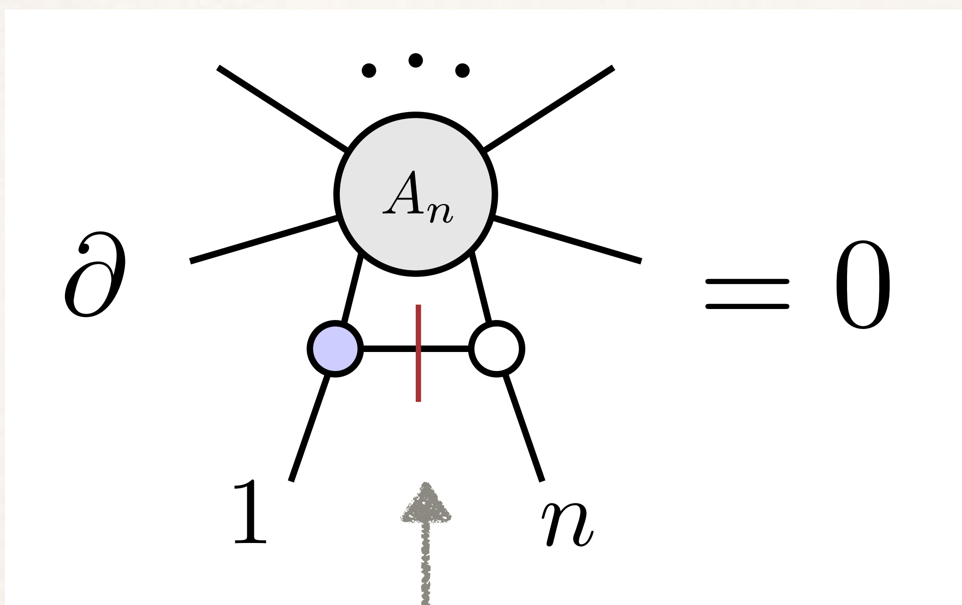


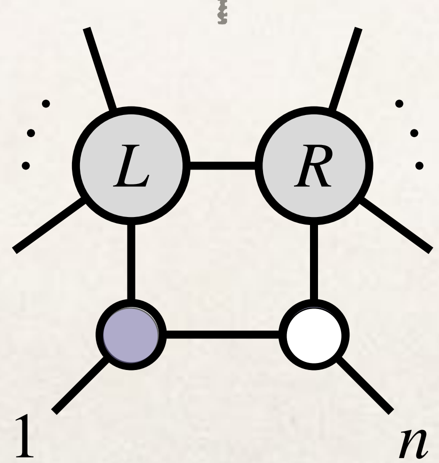
pole at $z = 0$

Cauchy formula

- Suppose the blob is the amplitude



$$\oint \frac{dz}{z} A_n(z) = 0$$

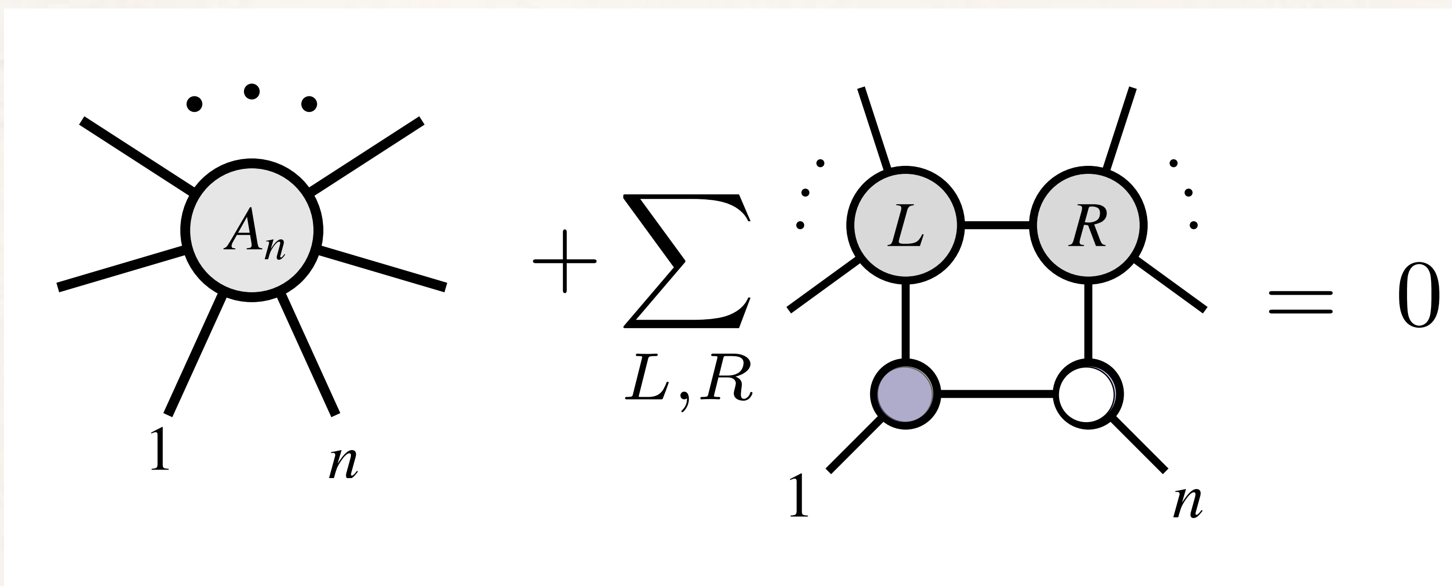


pole at $z = z_k$

come from $P^2(z_k) = 0$

BCFW recursion relations

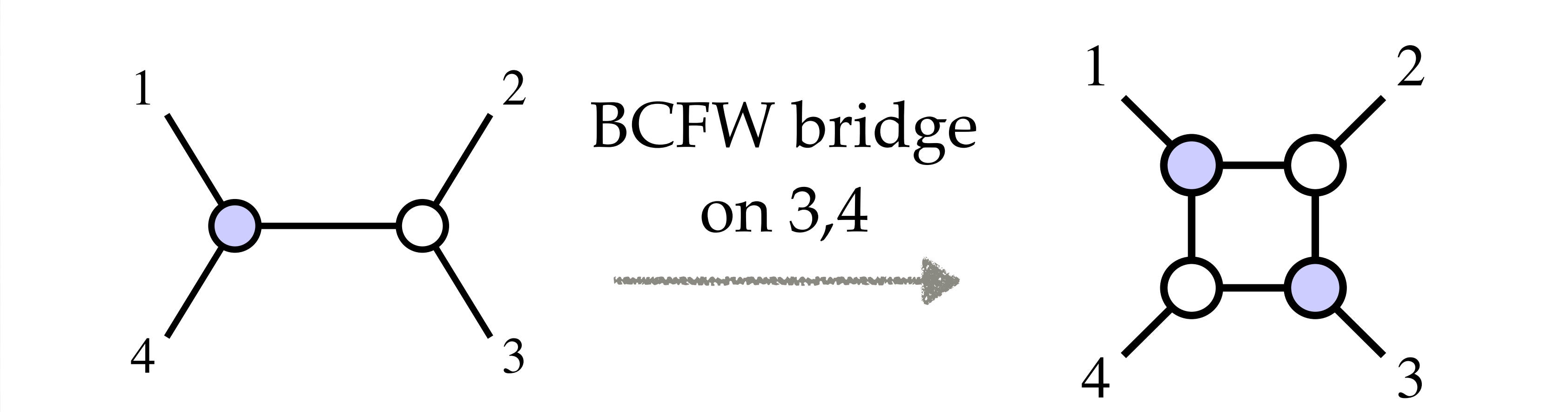
- Recursion relations for amplitude



- Tree-level amplitude = sum of on-shell diagrams

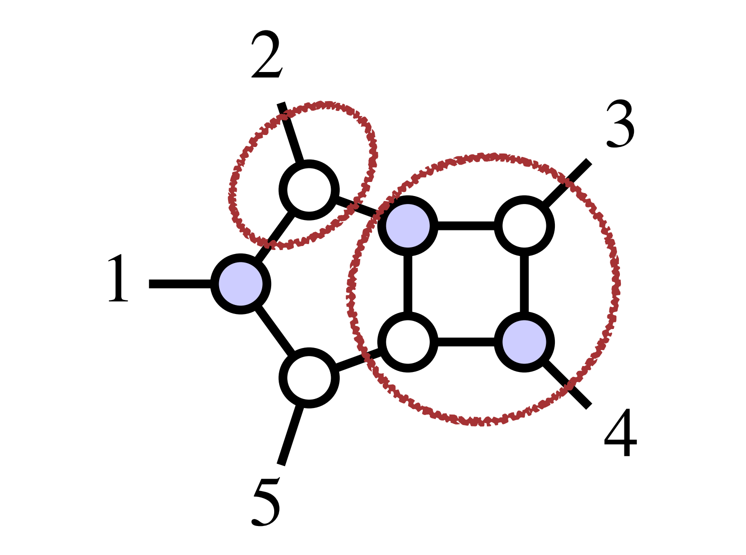
Simple examples

- Four point: only one factorization channel



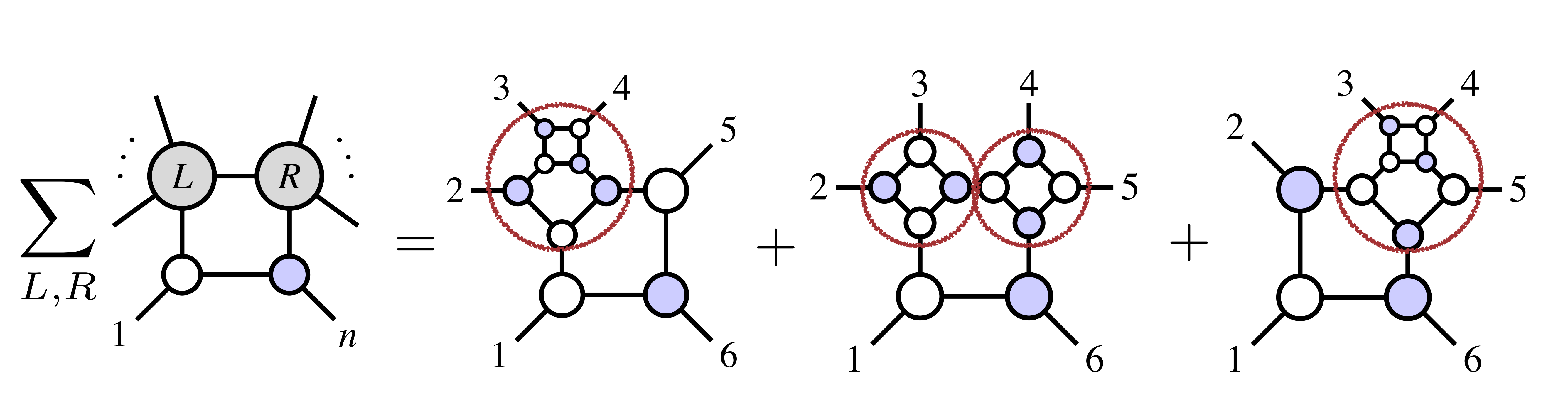
- Five point amplitude

Bridge 5,1 on 3pt
and 4pt amplitudes



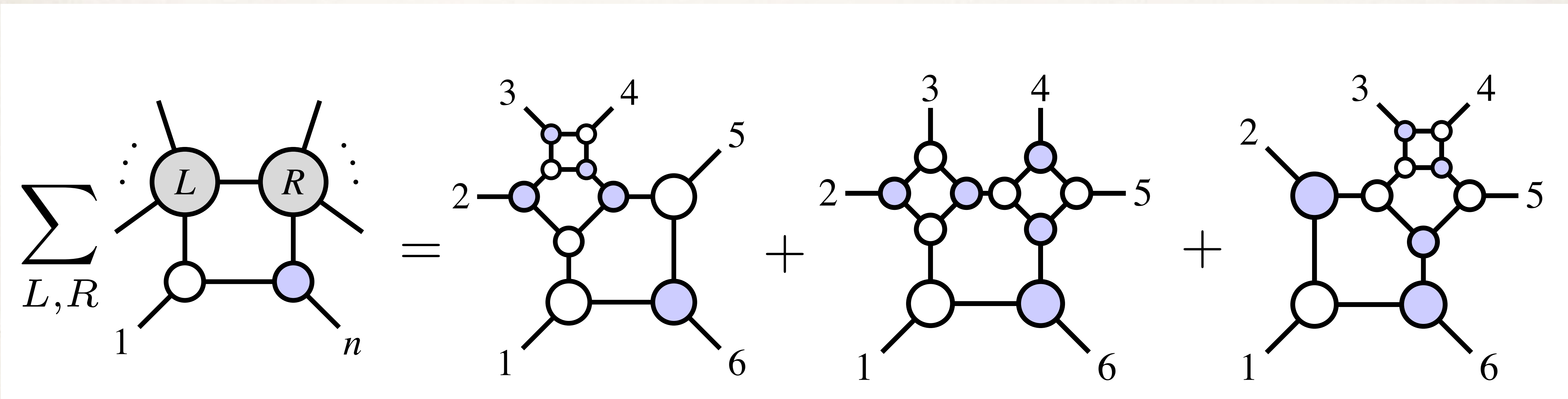
Six point example

- For k=3 we get three diagrams



Six point example

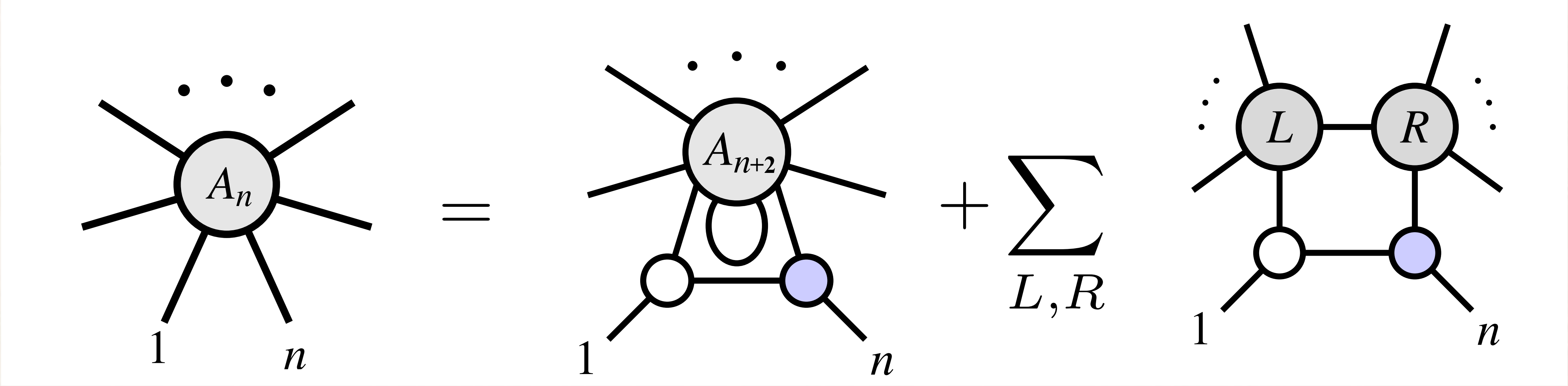
- For k=3 we get three diagrams



- Particular representation depends on the BCFW shift

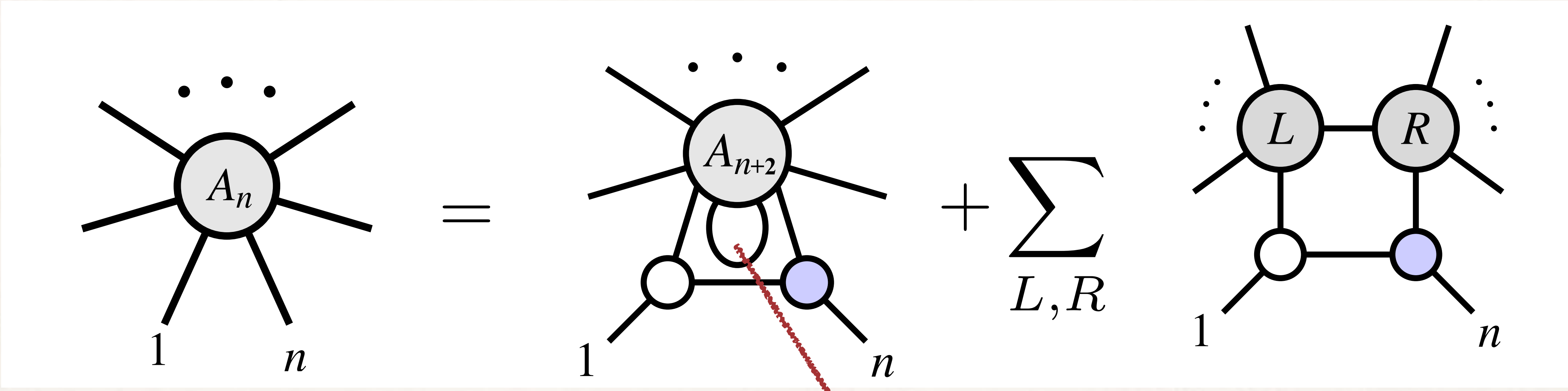
Loop recursion relations

- Recursion relations for ℓ-loop integrand



Loop recursion relations

- Recursion relations for ℓ-loop integrand



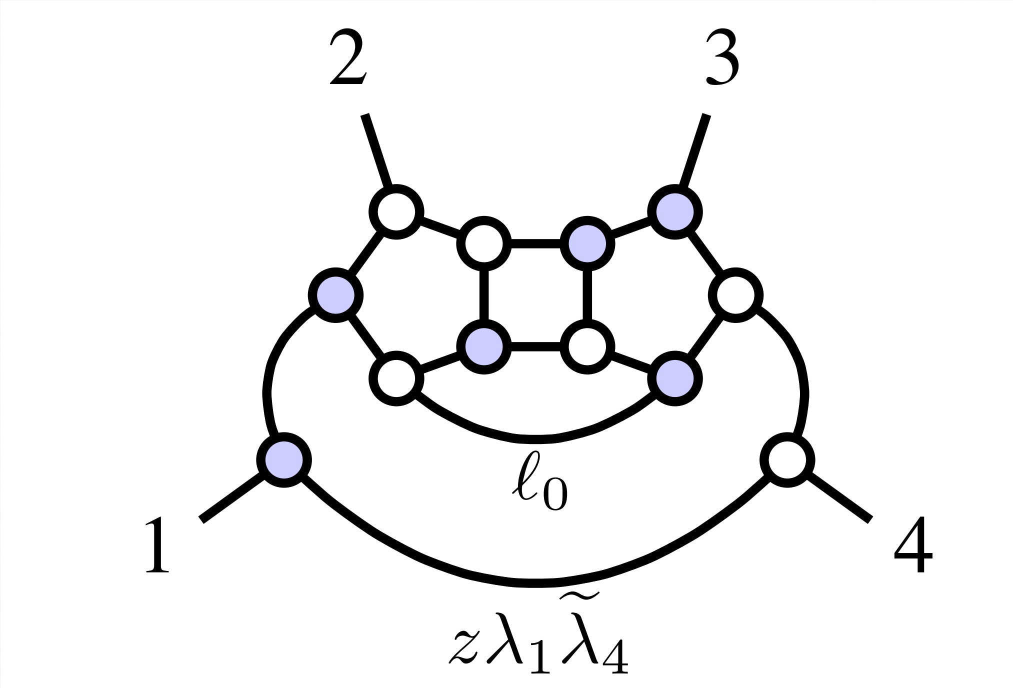
- Loop orders: $(\ell - 1)$ ℓ_1, ℓ_2 $\ell_1 + \ell_2 = \ell$

- New loop momentum $\ell^{(L)} = \ell_0^{(L)} + z\lambda_1\widetilde{\lambda}_n$

$$(\ell_0^{(L)})^2 = 0$$

Four point one loop amplitude

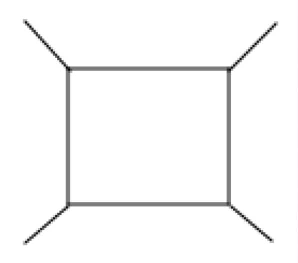
- It is given by one diagram



$$\ell = \ell_0 + z\lambda_1\widetilde{\lambda}_4$$

- 4 complex parameters -> impose reality condition

- 5-loop on-shell diagram = 1-loop off-shell box

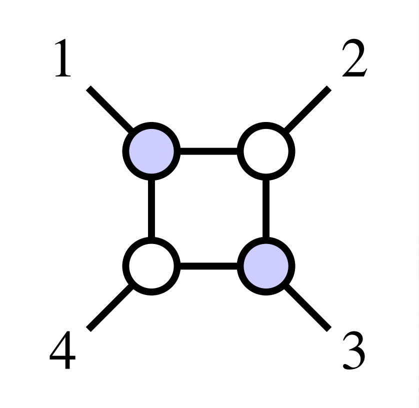


Dimensionality of diagrams

- Tree-level recursion: diagrams with $P = 4L$ contribute

  rational functions of external kinematics

  no delta functions, no free parameters

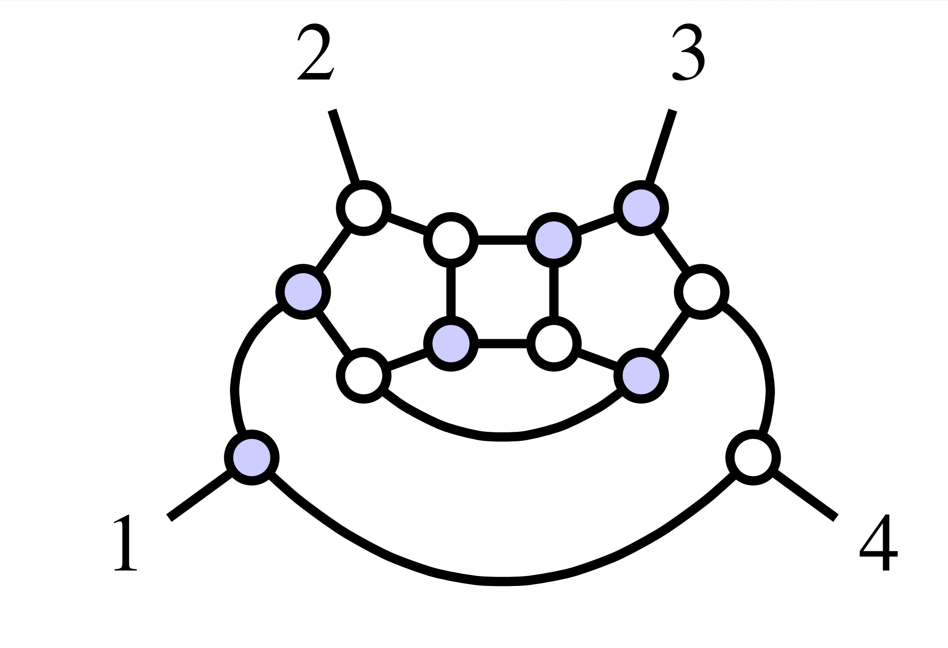


- Loop level: free parameters left

  components of loop momenta

  $\text{free} = 4L - P$

  $P = 16$
  $L = 5$
  $\text{free} = 4$



Identity moves

- On-shell diagrams satisfy identity moves that do not change the expression

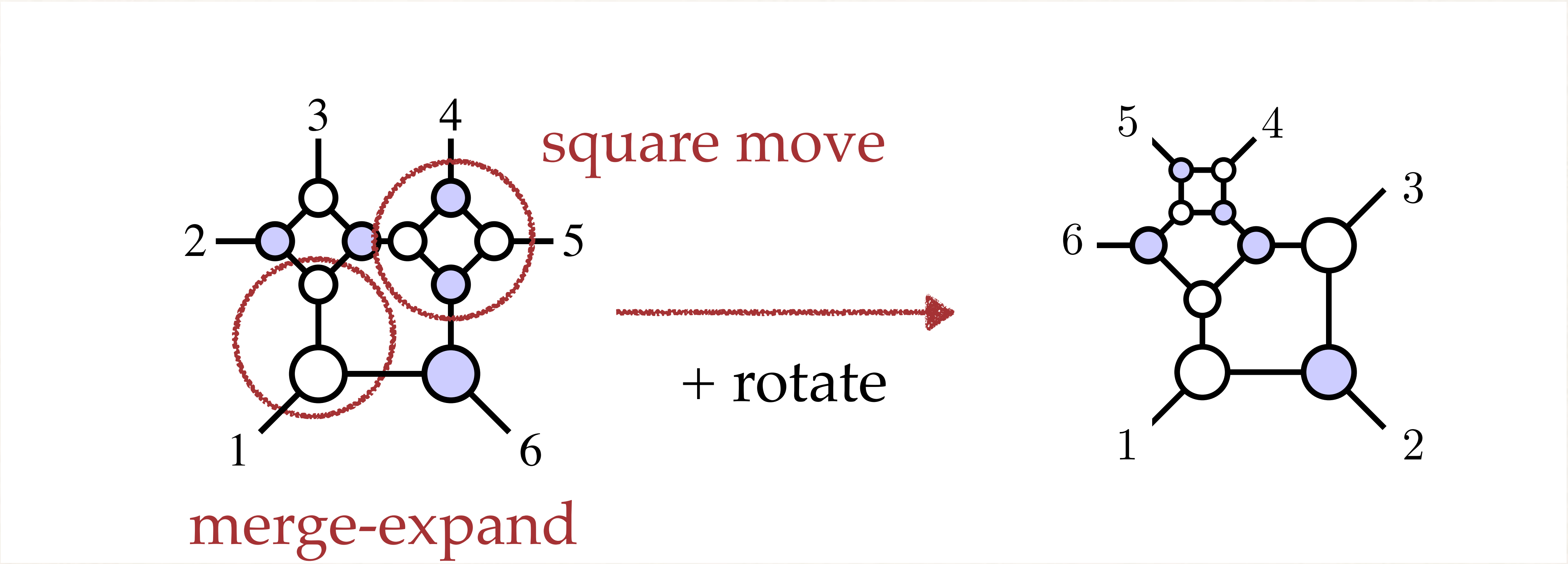
merge-expand

square move

- Anywhere in the graph

Identity moves

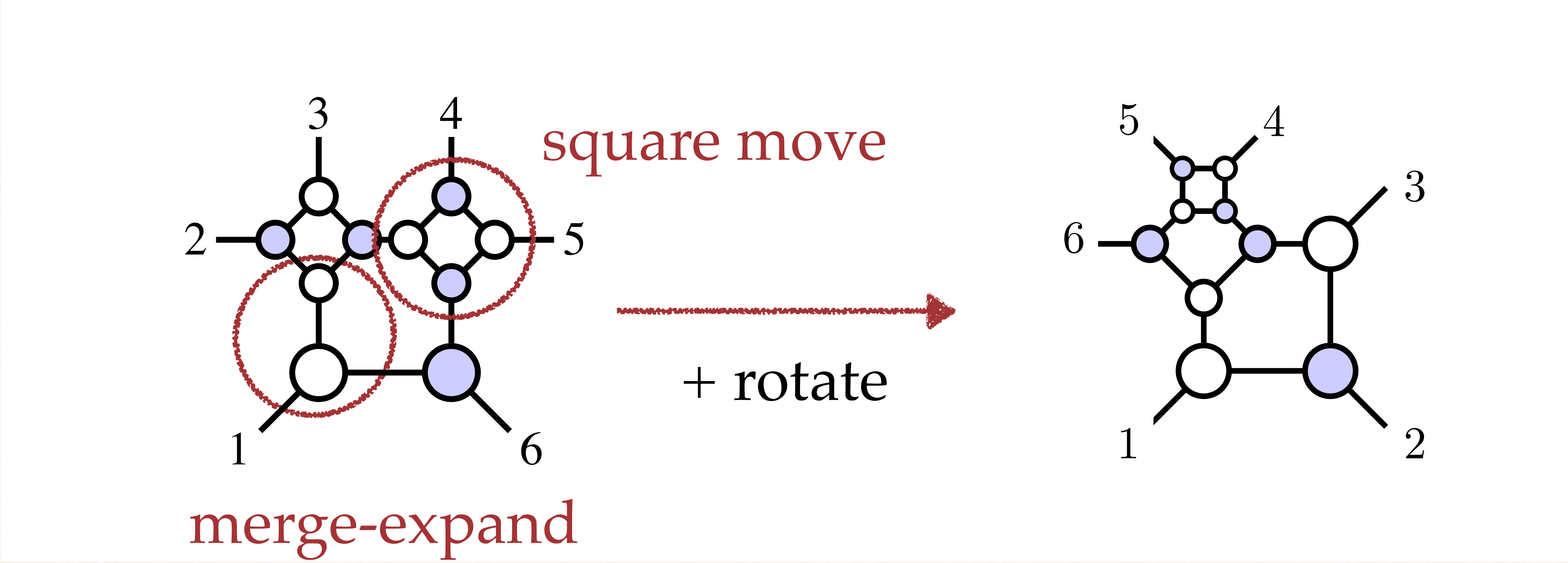
- Example:



- There must be more invariant way how to talk about these diagrams

Identity moves

- Example:



- There must be more invariant way how to talk about these diagrams: indeed there is…. next lecture

Thank you for attention!