Properads and Homotopy Algebras Related to Surfaces

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Properads

DCor := Cor × Cor category of directed corollas

Properad $\mathcal{P}$ consists of

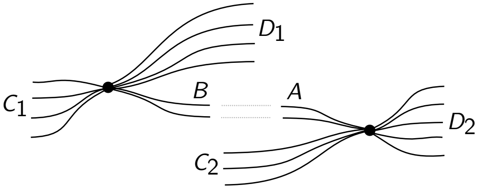
- collection $\{\mathcal{P}(C, D) \mid (C, D) \in \text{DCor}\}$ of dg vector spaces
- two collections of degree 0 morphisms

$$\{\mathcal{P}(\rho, \sigma) : \mathcal{P}(C, D) \to \mathcal{P}(C', D') \mid (\rho, \sigma) : (C, D) \to (C', D')\}$$

$$\{ {}_B\overset{\eta}{\circ}_A : \mathcal{P}(C_1, D_1 \sqcup B) \otimes \mathcal{P}(C_2 \sqcup A, D_2) \to \mathcal{P}(C_1 \sqcup C_2, D_1 \sqcup D_2) \mid \eta : B \xrightarrow{\sim} A\}$$

satisfying the axioms:

1. Σ-bimodule
   $\mathcal{P}((1_C, 1_D)) = 1_{\mathcal{P}(C,D)}, \quad \mathcal{P}((\rho\rho', \sigma'\sigma)) = \mathcal{P}((\rho, \sigma))\, \mathcal{P}((\rho', \sigma'))$
2. equivariance
   $(\mathcal{P}((\rho_1 \sqcup \rho_2|_{C_2}, \sigma_1|_{D_1} \sqcup \sigma_2))\ {}_B\overset{\eta}{\circ}_A = {}_{\sigma_1(B)}\overset{\rho_2\eta\sigma_1^{-1}}{\circ}_{\rho_2(A)} (\mathcal{P}((\rho_1, \sigma_1)) \otimes \mathcal{P}((\rho_2, \sigma_2))$
3. associativity ...



Additional grading by $\mathbb{N}_0$ - **genus** G or by the **Euler characteristic** χ

$$\chi = 2G + |C| + |D| - 2$$

$\Longrightarrow$ components $\mathcal{P}(C, D, \chi)$
We assume only **stable components**, i.e. $\chi > 0$

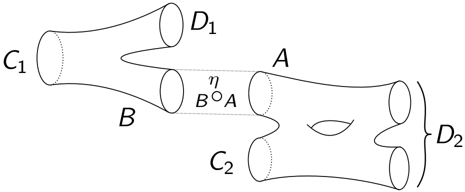
Example: (Closed) **Frobenius properad $\mathcal{F}$**:

$\mathcal{F}(C, D, \chi) = \Bbbk$

$\Longrightarrow$ has trivial differential and Σ-structure

$\Longrightarrow$ ${}_B\overset{\eta}{\circ}_A$ do not depend on sets A, B

Geometrically: 2-dim compact oriented surfaces with punctures in the interior, $G = g_1 + g_2 + |A| - 1$

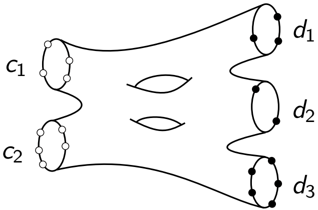


Example: **Open Frobenius properad $\mathcal{OF}$**:

$\mathcal{OF}(C, D, \chi) := \mathrm{Span}_{\Bbbk} \{\{\mathbf{c}_1, \dots \mathbf{c}_p, \mathbf{d}_1, \dots, \mathbf{d}_q\}^g \mid p, q \in \mathbb{N},\ g \in \mathbb{N}_0\}$
where $\mathbf{c}_i, \mathbf{d}_j$ are disjoint cycles in C and D, respectively.

Geometrically: 2-dim compact oriented surfaces with punctures on the boundaries.

In this case, the Euler characteristic is not additive!



Example: **Endomorphism properad $\mathcal{E}_V$:**

For (V, d) dg vector space, $(C, D) \in \text{DCor}$, $\chi > 0$ define

$$\mathcal{E}_V(C, D, \chi) := \text{Hom}_{\mathbb{k}}(\bigodot_D V, \bigodot_C V)$$

For $\bar{f} \in \text{Hom}_{\mathbb{k}}(\bigotimes_D V, \bigotimes_C)$ corresponding to $f \in \text{Hom}_{\mathbb{k}}(\bigodot_D V, \bigodot_C V)$

$$d(\bar{f}) = \sum_{i=0}^{m-1}(1^{\otimes i} \otimes d \otimes 1^{\otimes m-i-1})\bar{f} - (-1)^{|\bar{f}|}\sum_{i=0}^{n-1}\bar{f}(1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1})$$

Algebra over properad is a properad morphism $\alpha : \mathcal{P} \to \mathcal{E}_V$, i.e.

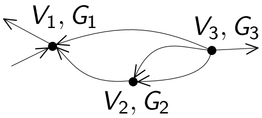
$$\{\alpha(C, D, \chi) : \mathcal{P}(C, D, \chi) \to \mathcal{E}_V(C, D, \chi) \mid (C, D) \in \text{DCor}, \chi > 0\}$$

satisfying:

$$\begin{aligned} \alpha \circ \mathcal{P}(\rho, \sigma) &= \mathcal{E}_V(\rho, \sigma) \circ \alpha \\ \alpha \circ ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{P}} &= ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha) \end{aligned}$$

Cobar complex

Directed graph G



Assign a non-negative integer $G := \dim_{\mathbb{Q}} H_1(\mathrm{G}, \mathbb{Q}) + \sum_i G_i$
The stable graph satisfies for every vertex V_i

$$2(G_i - 1) + |C_i| + |D_i| > 0$$

Cobar complex of properad $\mathcal{P}$

- Elements are iso class of G with "decoration" by element

$$(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n)$$

$$\partial_{C\mathcal{P}} = d_{P^{\#}} \otimes 1 + \sum_{\substack{C_1 \sqcup C_2 = C \\ D_1 \sqcup D_2 = D \\ \chi_1, \chi_2 > 0 \\ \chi}} \frac{1}{|A|!} \left({}^{(C_1, D_1 \sqcup B, \chi_1)} \eta \ {}_{B}\circ_{A} \ {}^{(C_2 \sqcup A, D_2, \chi_2)} \right)^{\#}_{P} \otimes (\uparrow V \wedge \cdot)$$

Theorem: Algebra over the cobar complex $C\mathcal{P}$

Algebra over $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection $\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi)^{\#} \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathrm{DCor}, \chi > 0\}$ of deg 1 linear maps s.t.

$$\mathcal{E}_V(\rho,\sigma) \circ \alpha(C,D,\chi) = \alpha(C',D',\chi) \circ \mathcal{P}(\rho^{-1},\sigma^{-1})^{\#}$$

$$d \circ \alpha = \alpha \circ d_{\mathcal{P}^{\#}} + \sum \frac{1}{|A|!} ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha) \circ ({}_B\overset{\eta}{\circ}{}_A)^{\#}_{\mathcal{P}}$$

By isomorphism

$$\mathrm{Hom}_{\Sigma_C \times \Sigma_D}(\mathcal{P}(C,D,\chi)^{\#}, \mathcal{E}_V(C,D,\chi)) \xrightarrow{\cong} {}^{\Sigma_C}(\mathcal{P}(C,D,\chi) \otimes \mathcal{E}_V(C,D,\chi))^{\Sigma_D}$$

we can rewrite algebra over $C\mathcal{P}$ as element

$$L \in \prod_{\substack{|C|,|D| \\ \chi > 0}} {}^{\Sigma_C}(\mathcal{P}(C,D,\chi) \otimes \mathcal{E}_V(C,D,\chi))^{\Sigma_D}$$

satisfying **Master equation** $d(L) + L \circ L = 0$ with differential

$$d = d_{\mathcal{P}} \otimes 1_{\mathcal{E}_V} - 1_{\mathcal{P}} \otimes d_{\mathcal{E}_V}$$

The invariants are isomorphic to coinvariants so we get an isomorphism

$$\prod_{\substack{|C|,|D|\\ \chi>0}} {}^{\Sigma_C}\left(\mathcal{P}(C,D,\chi)\otimes\mathcal{E}_V(C,[n],\chi)\right)^{\Sigma_n} \cong$$

$$\prod_{\substack{|C|,|D|\\ \chi>0}} \left(\mathcal{P}(C,D,\chi)_{\Sigma_C}\otimes_{\Sigma_D}(V^{\otimes C}\otimes(V^{\#})^{\otimes D}\right)$$

with “transfered” differential and composition map

The algebras over the cobar complex $\mathcal{CF}$ of the (closed) Frobenius properad are IBL_∞-algebras.

Shortly: L is simplified to the form

$$L = \sum_{m,n,\chi} \sum_{I,J} \frac{1}{m!n!} f_I^{\chi,J} (a_J \otimes \phi^I)$$

where $m = |C|, n = |D|$, $f_I^{\chi,J} = (\bar{\alpha}(p^{\#}_{m,n,\chi}))_I^J$, $\{p^{\#}_{m,n,\chi}\}$ is a dual basis to $\mathbb{k}$-basis of $\mathcal{P}(C, D, \chi)$, $\{a_i\}$ is homogeneous basis of V and $\{\phi_i\}$ is its dual basis.

And similarly ...

Theorem: Algebra over the cobar complex $\mathcal{COF}$ of the open Frobenius properad is described by a degree one element L of $T^{\mathrm{cyc}}(V) \otimes T^{\mathrm{cyc}}(V^{\#})$ such that $L \circ L = 0$

$$L = \sum_{p,q,g} \sum_{\substack{I_1|I_2|\cdots|I_q \\ J_1|J_2|\cdots|J_p}} \frac{1}{p!q!\prod_s' l_s k_s} f^{(g,p,q)J_1|J_2|\cdots|J_p}_{I_1|I_2|\cdots|I_q} a_{J_1|J_2|\cdots|J_p} \otimes \phi^{I_1|I_2|\cdots|I_q}$$

where

$$I_1|I_2|\cdots|I_q := i_1 \cdots i_{l_1}|i_{l_1+1} \cdots i_{l_1+l_2}|\cdots|i_{l_1+\ldots l_{q-1}+1} \cdots + i_n$$

denotes multi-index of cyclic words.