BGG sequences on curved manifolds

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Natural / invariant differential operators

Conformally invariant operators – Killing fields

(M, g) (pseudo-)Riemannian manifold gives conformal class $[g]$
equivalence relation $g \simeq \widetilde{g} = \Omega^2 g$ for some positive $\Omega \in \mathcal{C}^\infty(M)$

$$\mathcal{L}_X g = \lambda g$$

equivalent to

$$\text{trace-free part of } \nabla_{(a} X_{b)} = 0$$

Conformally invariant operators – Yamabe operator

$$Y = \Delta - \frac{n-2}{4(n-1)} R$$

acting on conformal densities of weight $w = 1 - \frac{n}{2}$

$$f \rightsquigarrow \widetilde{f} = \Omega^w f$$

Solution spaces

For open $U \subseteq M$:

$$\dim\{f \in \mathcal{C}^\infty(U) : Yf = 0\} = \infty$$

$$\dim\{X \in \mathcal{X}(U) : \text{trace-free part of } \nabla_{(a}X_{b)} = 0\} \leq \frac{(n+2)(n+1)}{2}$$

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maximum attained on $\mathrm{SO}(1, n+1)/P$ where solutions arise from the action of $\mathrm{SO}(1, n+1)$

Conformally invariant operators – nonexistence

$$\Delta^3 + \text{ l.o.t.}$$

There is no sixth order conformally invariant differental operator on M^4 whose principal part is third power of the Laplace operator. [Graham1992]

Invariant differential operators

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$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

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- From invariance we equivalently have homomorphism $J^k\mathbb{V} \to \mathbb{W}$, where $J^k\mathbb{V}$ denotes the algebraic jet prolongation of $\mathbb{V}$.
- Passing to dual maps and taking the limit $k \to \infty$ we get

$$\mathrm{Hom}_{\mathfrak{p}}\left(\mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right)$$

BGG resolutions and Lie algebra (co)homology

G semisimple, P parabolic, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_+$, $\lambda \in \mathfrak{h}^*$ $\mathfrak{g}$-integral, dominant
$\rightsquigarrow L_\lambda$ finite-dimensional $\mathfrak{g}$-representation

BGG resolution

$$\cdots \to \bigoplus_{w \in W^{\mathfrak{l},i}} M(w \cdot \lambda) \to \cdots \bigoplus_{w \in W^{\mathfrak{l},1}} M(w \cdot \lambda) \to M(\lambda) \to L_\lambda$$

$$M(w \cdot \lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{F}_{w \cdot \lambda}$$

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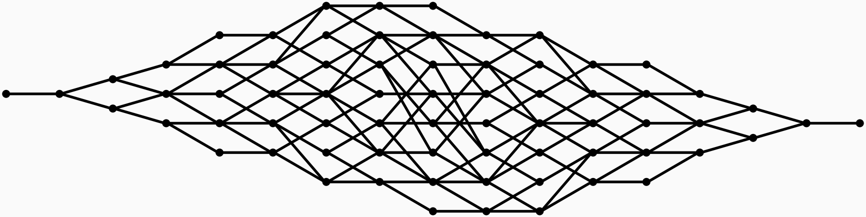
Kostant's theorem on nilpotent cohomology

$$H^i(\mathfrak{p}_+, L_\lambda) = \bigoplus_{w \in W^{\mathfrak{l},i}} \mathbb{F}_{w \cdot \lambda} = H_i(\mathfrak{p}_+, L_\lambda)$$

Nilpotent cohomology / BGG resolution for $\mathrm{SU}(2,2)$

$$(0,\ 0,\ 0) \longrightarrow (1,\ -2,\ 1) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} (2,\ -3,\ 0) \\ \\ (0,\ -3,\ 2) \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} (1,\ -4,\ 1) \longrightarrow (0,\ -4,\ 0)$$

The BGG graph for $\mathrm{SU}(4,4)$



Cartan geometries aka curved setting

Cartan geometry modeled on (G, P) is a principal P-bundle $\mathcal{G} \to M$ with a choice of Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

This generalizes $G \to G/P$ to a principal P-bundle $\mathcal{G} \to M$ and the Maurer-Cartan form $\omega_{MC} = g^{-1}\mathrm{d}g$ to ω.

define the curvature of ω by

$$R^{\omega} = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

(Maurer–Cartan equation: $R^{\omega_{MC}} = 0$)

Cartan geometries – Examples

Riemannian manifolds:
(M, g)

$$\rightsquigarrow (\mathrm{SO}(n+1), \mathrm{SO}(n))$$

Conformal structures:
$(M, [g])$ where $g_1, g_2 \in [g]$ if there exists $\varphi > 0$ such that $g_1 = \varphi g_2$

$$\rightsquigarrow (\mathrm{SO}(p+1, q+1), \mathrm{SO}(p, q) \ltimes \mathbb{R}^{p+q})$$

Projective structures:
$(M, [\nabla])$ where $\nabla_1, \nabla_2 \in [\nabla]$ if they have the same unparametrized geodesics

$$\rightsquigarrow (\mathrm{SL}(n+1), \mathrm{SL}(n) \ltimes \mathbb{R}^n)$$

contact structures, Grassmanian geometries, CR-geometries, Cartan's $(2, 3, 5)$ distributions, ...

Cartan geometries – tractor bundles and connections

$(\mathcal{G} \to M, \omega)$

For any P-module $\mathbb{V}$ we get associated bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ over M and out of ω we get a connection ∇^ω and twisted deRham operator

$$(d^\omega s)(u) = \sum_i \epsilon^i \wedge (\nabla^\omega_{e_i} s)(u) + \partial\, s(u) - \sum_{i<j} \epsilon^i \wedge \epsilon^j \wedge \kappa(e_i, e_j) \lrcorner\, s(u)$$

where ∂ is a Lie algebra cohomology differential.

The twisted deRham sequence:

$$0 \to \Omega^0(M, \mathcal{V}) \to \Omega^1(M, \mathcal{V}) \to \Omega^2(M, \mathcal{V}) \to \dots$$

Kostant's theorem on nilpotent cohomology

Start with finite-dimensional $\mathfrak{g}$-representation L_λ.

1. Find invariant positive definite inner product on cochain spaces.

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Hodge decomposition:

$$C = \mathrm{im}\,\partial \oplus \ker \Box \oplus \mathrm{im}\,\delta$$

$$\ker \delta = \mathrm{im}\,\delta \oplus \ker \Box \qquad \ker \partial = \mathrm{im}\,\partial \oplus \ker \Box\,.$$

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L-invariant projector onto $\ker\Box$

$$\mathrm{Id} - \Box^{-1}\,\Box$$

Kostant + Cartan = Calderbank–Diemer

Replace L-invariant projector onto $\ker \Box$

$$\mathrm{Id} - \Box^{-1}\,\Box = \mathrm{Id} - \Box^{-1}(\partial\,\delta + \delta\,\partial) = \mathrm{Id} - \partial\,\Box^{-1}\,\delta + \Box^{-1}\,\delta\,\partial$$

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$$\mathbb{V} \rightsquigarrow \mathcal{V}$$

$$\partial \rightsquigarrow d^{\omega}$$

$$\Box \rightsquigarrow \Box_{\omega} == \delta\, d^{\omega} + d^{\omega}\,\delta$$

$$\Box^{-1}\,\delta \rightsquigarrow Q = \Box_{\omega}^{-1}\,\delta$$

$$\Pi^{\omega} = \mathrm{Id} - Q d^{\omega} - d^{\omega}\, Q$$

Π^ω calculus

The operator $\Pi_k^\omega : \Omega^k(M, \mathcal{V}) \to \Omega^k(M, \mathcal{V})$ has the following properties.

1. The operator Π_k^ω vanishes on $\mathrm{im}\,\delta$ and maps into $\ker\delta$:
$$\Pi_k^\omega \circ \delta = 0 \quad \& \quad \delta \circ \Pi_k^\omega = 0.$$
2. The operator Π_k^ω induces identity on the homology bundles $\mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})$:
$$\Pi_k^\omega = \mathrm{Id} \mod \mathrm{im}\,\delta\,.$$
3. The commutator of d^ω and Π^ω equals to the commutator of Q and R
$$d^\omega \circ \Pi_k^\omega - \Pi_{k+1}^\omega \circ d^\omega = Q \circ R - R \circ Q,$$
where R is the curvature operator defined by $R(s) = (d^\omega \circ d^\omega)(s)$.
4. For $k = 0$ and in the flat case, the operator is actually a projection:
$$(\Pi_k^\omega)^2 = \Pi_k^\omega + Q \circ R \circ Q.$$
5. $$\Pi_k^\omega \circ \Box_\omega = -Q \circ R \circ \delta \quad \& \quad \Box_\omega \circ \Pi_k^\omega = -\,\delta \circ R \circ Q.$$

Calderbank–Diemer construction of BGG sequences [CD01]

$$\Pi^\omega = \mathrm{Id} - Qd^\omega - d^\omega Q$$

in the flat case:

- differential projection Π^ω onto a subspace of $\ker\delta$ complementary to $\mathrm{im}\,\delta$
- Π^ω is a chain map between twisted deRham complexes $d^\omega : \Omega^\bullet\mathcal{V} \to \Omega^{\bullet+1}\mathcal{V}$ which is homotopic to the identity, the chain-homotopy being the operator $Q : \Omega^\bullet\mathcal{V} \to \Omega^{\bullet-1}\mathcal{V}$

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The BGG operator $D_k^{\mathbb{V}} : \mathcal{C}^\infty(M, \mathcal{H}_k(\mathfrak{p}_+, \mathbb{V})) \to \mathcal{C}^\infty(M, \mathcal{H}_{k+1}(\mathfrak{p}_+, \mathbb{V}))$ is then defined as

$$D_k := \mathrm{proj} \circ \Pi^\omega_{k+1} \circ d^\omega \circ \Pi^\omega_k \circ \mathrm{rep},$$

where proj is the algebraic projection on homology and rep is a choice of representative in the homology class.

Properties of BGG operators

$$D_{k+1} D_k = \mathrm{proj} \circ \Pi^{\omega}_{k+2} \circ R \circ \Pi^{\omega}_{k} \circ \mathrm{rep}$$

In the flat case

1. $\ker D_0 \simeq \ker \nabla^{\omega}$
2. the sequence

$$D_{\bullet} : \mathcal{C}^{\infty}(M, \mathcal{H}_{\bullet}(\mathfrak{p}_+, \mathbb{V})) \to \mathcal{C}^{\infty}(M, \mathcal{H}_{\bullet+1}(\mathfrak{p}_+, \mathbb{V}))$$

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In general, one can modify ∇^{ω} with curvature terms so that $\ker D_0 \simeq \ker \nabla^{\omega}$

Leibniz rule

For a $(\mathfrak{g}, P)$-map $\mu : \mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{V}$ we can use wedge product to define bi-differential operators

$$\diamond : \mathcal{C}^\infty(M, \mathcal{H}_k(\mathbb{W}_1)) \otimes \mathcal{C}^\infty(M, \mathcal{H}_l(\mathbb{W}_2)) \to \mathcal{C}^\infty(M, \mathcal{H}_{k+l}(\mathbb{V}))$$

by

$$\alpha \diamond \beta = \mathrm{proj} \circ \Pi^\omega_{k+l} \circ \wedge \left(\Pi^\omega_k \circ \mathrm{rep}\,\alpha, \Pi^\omega_l \circ \mathrm{rep}\,\beta\right)$$

and then

$$D_{k+l}(\alpha \diamond \beta) = (D_k\alpha) \diamond \beta + (-1)^k \alpha \diamond D_l\beta +$$
$$\Pi^\omega_{k+l+1}\big((QR\Pi^\omega_k\alpha) \wedge \Pi^\omega_l\beta + (-1)^k \Pi^\omega_k\alpha \wedge (QR\Pi^\omega_l\beta) - RQ(\Pi^\omega_k\alpha \wedge \Pi^\omega_l\beta)\big)$$

In the flat case the product $\diamond$ descends to cup product in cohomology.

Multidifferential operations and relations

For any $(\mathfrak{g}, P)$-equivariant linear map $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3 \to \mathbb{W}$ one can define multidifferential map

$$\mathcal{C}^\infty(M, \mathcal{H}_k(\mathbb{V}_1)) \times \mathcal{C}^\infty(M, \mathcal{H}_l(\mathbb{V}_2)) \times \mathcal{C}^\infty(M, \mathcal{H}_m(\mathbb{V}_3)) \to \mathcal{C}^\infty(M, \mathcal{H}_{k+l+m-1}(\mathbb{W}))$$

which is related to Massey products in the flat case and which is compatible with Leibniz rule.

One can continue in this manner and obtain (curved) A_∞ or L_∞ algebra realized by multi-differential operators.

Thank you for attention!

Inverse of $\Box_\omega$

Since ∂ is hidden in d^ω we get $\Box_\omega = \Box(\mathrm{Id} - N)$ where

$$N = \mathrm{Id} - \Box^{-1}\Box_\omega = \sum_i \epsilon^i \nabla^\omega_{e_i}$$

is a degree 1 map with respect to a naturally defined grading and so we can get the inverse by Neumann series

$$\Box_\omega^{-1} = (\mathrm{Id} - N)^{-1}\Box^{-1} = (\sum_{k\geq 0} N^k)\Box^{-1}$$

provided

1. the P-module $\mathbb{V}$ has lowest / highest weight and
2. the inverse of $\Box$ exists (e.g. by Kostant's algebraic Hodge theory)