BV formalism, quantum L_∞ algebras and the homological perturbation lemma

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Overview

- *Homological perturbation theory*: perturbing chain complexes.
- In this talk: interpreting the *Batalin-Vilkovisky effective action* via the homological perturbation lemma.
- Motivation: quantum L_∞ algebras and their minimal models.

Definition

A *dg odd symplectic vector space* V is a dg vector space (V, Q); together with a bilinear form ω that is ghost degree -1, antisymmetric, non-degenerate and satisfies $\omega(Q \otimes 1 + 1 \otimes Q) = 0$.

space of functionals

Definition

The *space of functions* $\mathcal{F}(V)$ on V is

$$\mathcal{F}(V) := \widehat{\mathrm{Sym}^\bullet}(V^*)[[\hbar]] \, .$$

with a Batalin-Vilkovisky operator (ϕ^i : coordinates on V)

$$\Delta F := \frac{1}{2} \sum_{i,j} (-1)^{|\phi^i|} \omega^{ij} \frac{\partial_L^2 F}{\partial \phi^i \partial \phi^j} \, ,$$

We have $\mathcal{F}(V)$, Δ and the associated $\{-,-\}$. Define

$$S_{\mathrm{free}} := \omega(Q-,-) \in \mathcal{F}(V) \quad \Longrightarrow \quad Q^{\text{transpose}} = \{S_{\mathrm{free}}, -\}$$

a quadratic functional S_{free} and a differential $\{S_{\mathrm{free}}, -\}$ on $\mathcal{F}(V)$.

Definition

A quantum L_∞ algebra is given by degree 0 element $S_{\mathrm{int}} \in \mathcal{F}(V)$ s.t.

$$\Delta e^{(S_{\mathrm{free}}+S_{\mathrm{int}})/\hbar} = 0 \quad \Longleftrightarrow \quad \hbar \Delta S_{\mathrm{int}} + \{S_{\mathrm{free}}, S_{\mathrm{int}}\} + \frac{1}{2}\{S_{\mathrm{int}}, S_{\mathrm{int}}\} = 0$$

i.e. the *quantum master equation* holds.

Generalization of L_∞ algebra on V: operations $l_n^g : V^{\otimes n} \to V$ are

$$(l_n^g)^{\text{transpose}} = \{S_{\mathrm{int}}^{[n+1,g]}, -\} : \; V^* \to (V^*)^{\otimes n}.$$

in physics, the cohomology H of V w.r.t Q are the physical fields

- We want to find an effective action $e^{W/\hbar} \in \mathcal{F}(H)$. H inherits ω_H from $V \implies$ we have Δ' on $\mathcal{F}(H)$.
- We choose a Hodge decomposition of V compatible with ω

$$V = H \oplus \underbrace{\overbrace{\operatorname{Im} Q \oplus C}^{\text{homotopy } k}}_{Q} \underset{i}{\overset{p}{\rightleftarrows}} (H, \text{differential} = 0)$$

  where $Qk + kQ = ip - 1$, i, p are chain maps etc.
- extend to $\mathcal{F}(V)$ to get a so-called *special deformation retract*

$$K \circlearrowright (\mathcal{F}(V), \{S_{\text{free}}, -\}) \underset{I}{\overset{P}{\rightleftarrows}} (\mathcal{F}(H), 0)$$

- This is the setting for the *homological perturbation lemma*.

Theorem (HPL, due to Brown; Shih)

For data (a special deformation retract)

$$K \circlearrowright (\mathcal{F}(V), \{S_{\mathrm{free}}, -\}) \underset{I}{\overset{P}{\rightleftarrows}} (\mathcal{F}(H), E = 0)$$

and a small perturbation δ *of differential s.t.* $(\{S_{\mathrm{free}}, -\} + \delta)^2 = 0$, *there is a perturbed special deformation retract*

$$K' \circlearrowright (\mathcal{F}(V), \{S_{\mathrm{free}}, -\} + \delta) \underset{I'}{\overset{P'}{\rightleftarrows}} (\mathcal{F}(H), E')$$

where e.g. $P' = P(1 - \delta K)^{-1}$.

We use $\delta_1 = \hbar\Delta$ and $\delta_2 = \hbar\Delta + \{S_{\mathrm{int}}, -\}$ as perturbations.

perturbing by $\delta_1 = \hbar\Delta$:

Theorem

For $F \in \mathcal{F}(V)$

$$P_1(F) = \int_{C \subset V} F e^{S_{\text{free}}/\hbar} .$$

This implies that

$$W := \hbar \log P_1(e^{S_{\text{int}}/\hbar})$$

satisfies the quantum master equation on the homology H.

W is called the *effective action*. Gives a *minimal model* of the qL_∞ algebra S_1.

perturbing by $\delta_2 = \hbar\Delta + \{S_{\text{int}}, \}$:

Theorem

For $F \in \mathcal{F}(V)$

$$P_2(F) = e^{-W/\hbar} \int_C F e^{(S_{\text{free}} + S_{\text{int}})/\hbar}$$

i.e. it's the normalized path integral. The perturbed differential on $\mathcal{F}(H)$ is

$$E_2 = \hbar\Delta' + \{W, -\}' .$$

P_2 intertwines two twisted BV operators

Homotopies

HPL implies that $e^{S_{\mathrm{int}}/\hbar}$ and $I_1 P_1(e^{S_{\mathrm{int}}/\hbar}) = e^{W/\hbar}$ are homotopic.

Theorem

Define $e^{A(t)/\hbar} = (1-t)e^{S_{\mathrm{int}}/\hbar} + te^{W/\hbar}$. *Then the hamiltonian flow* Φ_t *of the function*

$$-\hbar e^{-A(t)/\hbar} K_1 e^{S_{\mathrm{int}}/\hbar}$$

interpolates between S_{int} *and* W.
More precisely,

$$(\Phi_t^{-1})^*(e^{(S_{\mathrm{free}}+S_{\mathrm{int}})/\hbar}\mathrm{d}^{\frac{1}{2}}V) = e^{(S_{\mathrm{free}}+A(t))/\hbar}\mathrm{d}^{\frac{1}{2}}V\,.$$

For $t=1$, the right hand side is $e^{(S_{\mathrm{free}}+W)/\hbar}\mathrm{d}^{\frac{1}{2}}V$

Odd symplectic category – a sketch

We want to define morphisms of quantum L_∞ algebras.

Definition (Ševera)

The quantum odd symplectic category QOSC is the "category" where objects are odd symplectic manifolds.

The space of morphisms is given by $\mathrm{QOSC}(V_1, V_2) := \mathrm{Dens}^{\frac{1}{2}}(\bar{V}_1 \times V_2)$, which carries a BV structure. Composition is given by integration.

- The space of morphisms carries a natural BV operator, due to Khudaverdian.
- Solution to a quantum master equation on $V \iff$ closed element $e^{S/\hbar}\mathrm{d}^{\frac{1}{2}}V$ in $\mathrm{QOSC}(\mathrm{pt}, V) = \mathrm{Dens}^{\frac{1}{2}}(V)$.

Definition

Morphism of quantum L_∞ algebras A morphism between two quantum L_∞ algebras $(V_1, S_1) \to (V_2, S_2)$ is a morphism $X \in \mathrm{QOSC}(V_1, V_2)$ s.t.

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{e^{S_1/\hbar}} & V_1 \\ & {\scriptstyle e^{S_2/\hbar}} \searrow & \downarrow X \\ & & V_2 \end{array}$$

In other words, X is a semidensity on $\bar{V}_1 \times V_2$ such that

$$\int_{V_1} e^{S_1/\hbar} X = e^{S_2/\hbar}.$$

In our case, $X = \delta_{C \times H_{\mathrm{diag}}}$. Since X is closed, so is the effective action.

Conclusion

- BV effective action on odd symplectic vector spaces can be computed via the homological perturbation lemma.
- HPL also gives homotopy between the original and effective action.
- Viewing S_{int} as a quantum L_∞ algebra, this gives a *decomposition theorem*.
- (WIP) This constructs a morphism in the quantum odd symplectic category.

Thank you for your attention!

Homological perturbation lemma

Consider a special deformation retract

$$h \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (W, e) \qquad (1)$$

Let δ be a perturbation of d (i.e. $(d+\delta)^2 = 0$) which is small in the sense that

$$(1 - \delta h)^{-1} := \sum_{i=0}^{\infty} (\delta h)^i$$

is a well defined linear map $V \to V$.
Then

$$h' \circlearrowright (V, d') \underset{i'}{\overset{p'}{\rightleftarrows}} (W, e')$$

is a special deformation retract.

Homological perturbation lemma

Denote $A := (1-\delta h)^{-1}\delta$ and

$$\begin{aligned}
d' &:= d + \delta, \\
e' &:= e + p(1-\delta h)^{-1}\delta i = e + p\delta(1-h\delta)^{-1} i, \\
p' &:= p + p(1-\delta h)^{-1}\delta h = p(1-\delta h)^{-1}, \\
i' &:= i + h(1-\delta h)^{-1}\delta i = (1-h\delta)^{-1} i, \\
h' &:= h + h(1-\delta h)^{-1}\delta h = h(1-\delta h)^{-1},
\end{aligned}$$