On homotopy moment maps for Lie 2-algebras

Leyli Mammadova

KU Leuven

September 26, 2018

Joint work with Marco Zambon

Outline

Set-up

- M a symplectic or n-plectic **connected** manifold.
- $\mathfrak{g}$ a Lie algebra acting on M **effectively** and via Hamiltonian vector fields:

$$\mathfrak{g} \to \mathfrak{X}_{\mathrm{Ham}}(M)$$
$$x \mapsto v_x$$

Symplectic geometry

Let M be a manifold, and $\omega \in \Omega^2(M)$ a **symplectic**, i.e., closed and non-degenerate, form.

Definition

$X \in \mathfrak{X}(M)$ is a **Hamiltonian vector field** corresponding to $f \in C^\infty(M)$, denoted by X_f, if $df = -i_X\omega$

Remark

$C^\infty(M)$, equipped with the Poisson bracket, is a Lie algebra, called the algebra of **observables**

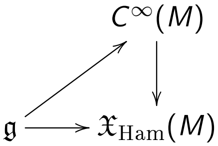
Symplectic geometry: moment map

Definition

A **(co)moment** map for $\mathfrak{g}$ is a Lie algebra morphism

$$f : \mathfrak{g} \to C^\infty(M)$$

such that $v_x = X_{f(x)}$.



2-plectic geometry

Definition

Let M a manifold, and $\omega \in \Omega^3(M)$ such that

$$d\omega = 0$$

and

$$i_v\omega = 0 \iff v = 0.$$

Then ω is an **2-plectic form** form, and M is a **2-plectic manifold**.

Definition

A 1-form $\alpha \in \Omega^1(M)$ is **Hamiltonian** if there exists $X_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -i_{X_\alpha}\omega.$$

2-plectic geometry: examples

- An oriented 3-dimensional manifold M with volume form ω_{vol}
- $\wedge^2 T^*M$ with

$$\omega = -d\theta,$$

where $\theta|_{(m,\alpha)}(v_1, v_2) = \alpha(\pi_* v_1, \pi_* v_2)$

- G compact semi-simple Lie group with

$$\omega = \langle \theta, [\theta, \theta] \rangle,$$

where $\langle \ , \ \rangle$ is an Ad-invariant inner product, and θ is the Maurer Cartan form.

Lie 2-algebra

Definition

A **Lie 2-algebra** is a graded vector space $V_1[1] \oplus V_0$ together with maps

$$\begin{aligned} \delta &: V_1[1] \to V_0 \\ [\ ,\ ] &: \wedge^2 V_0 \to V_0 \\ &: V_1[1] \wedge V_0 \to V_1[1] \\ [\ ,\ ,\ ] &: \wedge^3 V_0 \to V_1[1] \end{aligned}$$

satisfying the "higher Jacobi identities".

The Lie 2-algebra of observables

Theorem (Baez, Hoffnung, Rogers 2008)

Let (M, ω) be a 2-plectic manifold. There is a Lie 2-algebra structure on the graded vector space $C^\infty(M)[1] \oplus \Omega^1_{Ham}(M)$, where for $f \in C^\infty(M), \alpha_i \in \Omega^1_{Ham}(M)$

$$\delta f = df$$

$$[\alpha_1, \alpha_2] = \omega(v_{\alpha_1}, v_{\alpha_2}, \cdot)$$

$$[\alpha_1, \alpha_2, \alpha_3] = -\omega(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}).$$

We will denote this Lie 2-algebra by $L_\infty(M, \omega)$.

Lie algebra moment map

Let (M, ω) be a 2-plectic manifold.

Definition (Callies, Fregier, Rogers, Zambon)

A (homotopy) moment map for $\mathfrak{g}$ is an L_∞-algebra morphism

$$f : \mathfrak{g} \to L_\infty(M, \omega)$$

such that $df_1(x) = -i_{v_x}\omega \; \forall x \in \mathfrak{g}$.

Lie algebra moment map: existence and obstruction

Let $p \in M$, and $\widetilde{\omega}_p \in \wedge^3 \mathfrak{g}^*$ a cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$.

Proposition (Callies, Fregier, Rogers, Zambon 2016)

If there exists a moment map for $\mathfrak{g}$, then $[\widetilde{\omega}_p]_{\mathfrak{g}} = 0 \in H^3(\mathfrak{g})$. Conversely, if $[\widetilde{\omega}_p]_{\mathfrak{g}} = 0$ and $H^1(M) = 0$, then there exists a moment map for the action of $\mathfrak{g}$ on M.

Lie 2-algebra moment map

Let $(\mathfrak{h}[1] \oplus \mathfrak{g}, \delta, [\ ,\ ], [\ ,\ ,\ ])$ be a Lie 2-algebra, (M, ω) a 2-plectic manifold.

Definition

A moment map for $\mathfrak{h}[1] \oplus \mathfrak{g}$ is an L_∞-morphism

$$f : \mathfrak{h}[1] \oplus \mathfrak{g} \to L_\infty(M, \omega)$$

such that $df_1(x) = -i_{v_x}\omega \ \forall x \in \mathfrak{g}$.

A closer look at the Lie 2-algebra

We will consider the case where $\delta \equiv 0$, i.e, the Lie 2-algebra $(\mathfrak{h}[1] \oplus \mathfrak{g}, [\ ,\ ], [\ ,\ ,\ ])$. This data is equivalent to to:

- Lie algebra $\mathfrak{g}$
- A representation $\rho : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$ given by

  $$\rho(x)h = [x, h]$$

  for $x \in \mathfrak{g}, h \in \mathfrak{h}$
- A 3-cocycle $c : \wedge^3 \mathfrak{g} \to \mathfrak{h}$ for the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$ given by

  $$c(x, y, z) = [x, y, z].$$

Why Lie 2-algebra moment maps?

- No need to restrict ourselves to a Lie algebra, since we already have an L_∞-morphism.
- There always exists a Lie 2-algebra moment map for a special Lie 2-algebra $\mathbb{R}[1] \oplus \mathfrak{g}$, provided $H^1(M) = 0$.

Existence and obstruction

Question: For which Lie 2-algebras does there exist a moment map?

Proposition

If there exists a moment map for $\mathfrak{h}[1] \oplus \mathfrak{g}$*, then* $[\widetilde{\omega}_p]_E = 0 \in H^3(E, d_E)$.
Conversely, if $[\widetilde{\omega}_p]_E = 0$ *and* $H^1(M) = 0$*, then there exists a moment map for* $\mathfrak{h}[1] \oplus \mathfrak{g}$.

Remark

This proposition encodes a constructive way to obtain moment maps for $\mathfrak{h}[1] \oplus \mathfrak{g}$.
Moreover, when $H^1(M) = 0$, any moment map for $\mathfrak{h}[1] \oplus \mathfrak{g}$ is co-homologous to one obtained as in the proposition.

Existence and obstruction

Question: For which Lie 2-algebras does there exist a moment map?

Proposition

Assume $[\widetilde{\omega}_p]_{\mathfrak{g}} \neq 0$. *If* $\mathfrak{h}[1] \oplus \mathfrak{g}$ *admits a moment map, then* $\mathfrak{h}[1] \oplus \mathfrak{g}$ *has a quotient which is* L_∞*-quasi-isomorphic to* $\mathbb{R}[1] \oplus_{-\widetilde{\omega}_p} \mathfrak{g}$ *by a morphism that is identity on* $\mathfrak{g}$.
The converse holds if $H^1(M) = 0$.

Remark

We assume $[\widetilde{\omega}_p]_{\mathfrak{g}} \neq 0$, because otherwise there exists a $\mathfrak{g}$-moment map. Note that, in that case, there exists a moment map for any $\mathfrak{h}[1] \oplus \mathfrak{g}$, since *proj* : $\mathfrak{h}[1] \oplus \mathfrak{g} \to \mathfrak{g}$ is an L_∞-morphism.

Existence and obstruction

Let $\mathfrak{h}_{red} := \mathfrak{h}/[\mathfrak{g}, \mathfrak{h}]$, and $c_{red} = pr \circ c$, where $pr : \mathfrak{h} \to \mathfrak{h}_{red}$. Define

$$\begin{aligned} \Psi : \mathfrak{h}_{red}^* &\to H^3(\mathfrak{g}) \\ \xi &\mapsto [\xi \circ c_{red}]_{\mathfrak{g}} \end{aligned}$$

Proposition

$[\widetilde{\omega}_p]_E = 0 \iff [\widetilde{\omega}_p]_{\mathfrak{g}}$ lies in the image of Ψ.

Corollary

Assume $[\widetilde{\omega}_p]_{\mathfrak{g}} \neq 0$.

- *If $[c_{red}]_{\mathfrak{g}} = 0$, then there is no moment map for $\mathfrak{h}[1] \oplus \mathfrak{g}$*
- *If $[c_{red}]_{\mathfrak{g}} \neq 0$, $H^3(\mathfrak{g}) \cong \mathbb{R}$, $H^1(M) = 0$, then there exists a moment map for $\mathfrak{h}[1] \oplus \mathfrak{g}$.*

References

- John C Baez and Alissa S Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory Appl. Categ **12** (2004), no. 15, 492–528.
- John C Baez, Alexander E Hoffnung, and Christopher L Rogers, *Categorified symplectic geometry and the classical string*, Communications in Mathematical Physics **293** (2010), no. 3, 701.
- Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001.
- Martin Callies, Yael Fregier, Christopher L. Rogers, and Marco Zambon, *Homotopy moment maps*, Advances in Mathematics **303** (2016), 954–1043.
- Yaël Frégier, Camille Laurent-Gengoux, and Marco Zambon, *A cohomological framework for homotopy moment maps*, J. Geom. Phys. **97** (2015), 119–132.
- Leonid Ryvkin and Tilmann Wurzbacher, *Existence and unicity of co-moments in multisymplectic geometry*, Differential Geom. Appl. **41** (2015), 1–11.

Thank you!