The low-dimensional algebraic cohomology of the Virasoro algebra

Based on arXiv:1805.08433 and on-going work with Martin Schlichenmaier

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Motivation

- Virasoro algebra (=central extension of Witt algebra): very important ∞-dimensional Lie algebra, omnipresent in 2-dimensional conformal field theory and String Theory; see Kac, Raina and Rozhkovskaya [7].
- Low-dimensional cohomology: interpretation in terms of invariants, outer derivations, extensions, deformations and obstructions, as well as crossed modules $\leftrightarrow$ their knowledge allows a better understanding of the Lie algebra itself.
- Algebraic cohomology (arbitrary maps) vs continuous cohomology (continuous maps): valid for any base field $\mathbb{K}$ with $char(\mathbb{K}) = 0$, independent of any topology chosen, independent of any concrete realization of the Lie algebra.

Main Objectives

- Aim: Compute the third algebraic cohomology with values in the adjoint module of the Virasoro algebra.
- Byproduct: third algebraic cohomology with values in the trivial module of the Witt and the Virasoro algebra.
- Known in the case of the adjoint module:
  - ⋆ First algebraic cohomology of the Witt and the Virasoro algebra; see Ecker and Schlichenmaier [2].
  - ⋆ Second algebraic cohomology of the Witt algebra; see Schlichenmaier [9, 8] and also Fialowski [4, 3].
  - ⋆ Second algebraic cohomology of the Virasoro algebra; see Schlichenmaier [9].
  - ⋆ Third algebraic cohomology of the Witt algebra; see Ecker and Schlichenmaier [2].

The Witt algebra

- Witt algebra $\mathcal{W}$ generated as vector space over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ by the elements $\{e_n \mid n \in \mathbb{Z}\}$ satisfying the following Lie structure:

$$[e_n, e_m] = (m - n)e_{n+m}, \qquad n, m \in \mathbb{Z}$$

- $\mathbb{Z}$-graded Lie algebra: $deg(e_n) := n$
- Decomposition of $\mathcal{W}$: $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$, with each $\mathcal{W}_n$ a 1-dimensional homogeneous subspace generated by e_n
- Internally graded: $[e_0, e_n] = ne_n = deg(e_n)e_n$, i.e. e_n is eigenvector of $ad_{e_0} := [e_0, \cdot]$ with eigenvalue n
- Algebraic realization: Lie algebra of derivations of Laurent polynomials $\mathbb{K}[Z^{-1}, Z]$
- Geometrical realization:
  - $\mathbb{K} = \mathbb{C}$, algebra of meromorphic vector fields on $\mathbb{CP}^1$ holomorphic outside of 0 and ∞, with $e_n = z^{n+1}\frac{d}{dz}$
  - Lie algebra of polynomial vector fields on S^1, with $e_n = e^{in\phi}\frac{d}{d\phi}$

The Virasoro algebra

- The Virasoro algebra $\mathcal{V}$ is the universal one-dimensional central extension of the Witt algebra
- Central extension described by short exact sequence:

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} \mathcal{V} \xrightarrow{\pi} \mathcal{W} \longrightarrow 0\,.$$

- Exact sequence $\cdots \stackrel{f_{i-1}}{\rightarrow} M_i \stackrel{f_i}{\rightarrow} M_{i+1} \stackrel{f_{i+1}}{\rightarrow} \ldots \;:\; ker\ f_i = im\ f_{i-1}$.
- As a vector space, $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$ generated by $\hat{e}_n := (0, e_n)$ and $t := (1, 0)$
- Lie structure equation:

$$\begin{aligned} &[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} - \tfrac{1}{12}(n^3 - n)\delta_n^{-m} t, \\ &[\hat{e}_n, t] = [t, t] = 0 \end{aligned}$$

- $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) = 0 \Rightarrow \mathcal{V}$ is $\mathbb{Z}$-graded

The Lie algebra cohomology

- Let $\mathcal{L}$: Lie algebra; M: $\mathcal{L}$-module and $C^q(\mathcal{L}, M)$: vector space of q-multilinear alternating maps with values in M, called q-cochains ($q \in \mathbb{N}$)
  Convention: $C^0(\mathcal{L}, M) := M$
- Coboundary operators δ_q defined by:

$$\forall q \in \mathbb{N}, \qquad \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q \psi \,,$$

$$(\delta_q \psi)(x_1, \dots x_{q+1}) : \; = \sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \; \psi([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1})$$

$$+ \sum_{i=1}^{q+1} (-1)^i \; x_i \cdot \psi(x_1, \dots, \hat{x}_i, \dots, x_{q+1}) \,,$$

  with $x_1, \dots, x_{q+1} \in \mathcal{L}$
- Adjoint module $M = \mathcal{L}$, $x \cdot m = [x, m]$; trivial module $M = \mathbb{K}$, $x \cdot m = 0$
- $\delta_{q+1} \circ \delta_q = 0 \; \forall \; q \in \mathbb{N} \to$ complex of vector spaces:

$$\{0\} \xrightarrow{\delta_{-1}} M \xrightarrow{\delta_0} C^1(\mathcal{L}, M) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{q-2}} C^{q-1}(\mathcal{L}, M) \xrightarrow{\delta_{q-1}} C^q(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} C^{q+1}(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} \dots$$

  where $\delta_{-1} := 0$

The Chevalley-Eilenberg cohomology

- *q*-cocycles: $Z^q(\mathcal{L}, M) := \ker \delta_q$
- *q*-coboundaries: $B^q(\mathcal{L}, M) := \operatorname{im} \delta_{q-1}$
- q^{th} cohomology group of $\mathcal{L}$ with values in M:

$$H^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)$$

- Chevalley-Eilenberg cohomology:

$$H^*(\mathcal{L}, M) := \bigoplus_{q=0}^{\infty} H^q(\mathcal{L}, M)$$

The degree of a homogeneous cochain

- $\mathcal{L}$ graded Lie algebra, M a graded $\mathcal{L}$-module, M internally graded with respect to the same grading element as the Lie algebra $\mathcal{L}$
- **Examples**: adjoint module $M=\mathcal{L}$; trivial module $M=\mathbb{K}$ with $\mathbb{K} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}_n$, $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_n = \{0\}$ for $n \neq 0$
- A q-cochain ψ is homogeneous of degree d if $\exists$ a $d \in \mathbb{Z}$ s.t. for all q-tuple $x_1, \ldots, x_q$ of homogeneous $x_i \in \mathcal{L}_{deg(x_i)}$, we have:

$$\psi(x_1, \ldots, x_q) \in M_n \text{ with } n = \sum_{i=1}^{q} deg(x_i) + d$$

$\rightsquigarrow$ decomposition of cohomology:

$$H^q(\mathcal{L}, M) = \bigoplus_{d\in\mathbb{Z}} H^q_{(d)}(\mathcal{L}, M)$$

- Result by Fuks [5]:

$$H^q_{(d)}(\mathcal{L}, M) = \{0\} \quad \text{for} \quad d \neq 0 \ ,$$
$$H^q(\mathcal{L}, M) = \mathrm{H}^q_{(0)}(\mathcal{L}, M)$$

Main result

Main Theorem

The third algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the adjoint module is one-dimensional, i.e.

$$dim(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1$$

- Use intermediate results

$$\mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W})$$

- and

$$dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K}))(= dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K}))) = 1$$

- Proof of $\mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W})$: uses Hochschild-Serre spectral sequence (c.f. [1]).
- Proof of $dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$: later.

Proof of main theorem (I)

- Short exact sequence $0 \longrightarrow \mathbb{K} \xrightarrow{i} \mathcal{V} \xrightarrow{\pi} \mathcal{W} \longrightarrow 0$ of Lie algebras is also a short exact sequence of $\mathcal{V}$-modules.
- In cohomology, we obtain long exact sequence:

$$\cdots \to \mathrm{H}^2(\mathcal{V},\mathcal{W}) \to \mathrm{H}^3(\mathcal{V},\mathbb{K}) \to \mathrm{H}^3(\mathcal{V},\mathcal{V}) \to \mathrm{H}^3(\mathcal{V},\mathcal{W}) \to \dots .$$

- Second cohomology: $\mathrm{H}^2(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^2(\mathcal{W},\mathcal{W})$ and also $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$ hence $\mathrm{H}^2(\mathcal{V},\mathcal{W}) = 0$ (c.f. [9])
- Third cohomology: $\mathrm{H}^3(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^3(\mathcal{W},\mathcal{W})$ and also $\mathrm{H}^3(\mathcal{W},\mathcal{W}) = \{0\}$ hence $\mathrm{H}^3(\mathcal{V},\mathcal{W}) = \{0\}$ (c.f. [2])

Proof of main theorem (II)

- The long exact sequence becomes a short exact sequence:

$$0 \to \mathrm{H}^3(\mathcal{V}, \mathbb{K}) \to \mathrm{H}^3(\mathcal{V}, \mathcal{V}) \to 0 \,.$$

- Recall 2nd intermediate result:

$$\mathit{dim}(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$$ (proof on next slide)

- By exactness, we obtain the result of the main theorem:

$$\mathit{dim}(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1 \,.$$

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (I)

Theorem

The third cohomology group of the Witt and the Virasoro algebra with values in the trivial module $\mathbb{K}$ is one-dimensional, i.e.:

$$dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$$

- First step : Find a cocycle of $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ that is not a coboundary.
- Second step : There are no other cocycles up to equivalence.

Cocycle condition and coboundary condition

- The condition for a 3-cochain ψ to be a cocycle with values in the trivial module is:

$$
\begin{aligned}
(\delta_3\psi)(x_1,x_2,x_3,x_4) =&\psi([x_1,x_2],x_3,x_4) - \psi([x_1,x_3],x_2,x_4) \\
&+ \psi([x_1,x_4],x_2,x_3) + \psi([x_2,x_3],x_1,x_4) \\
&- \psi([x_2,x_4],x_1,x_3) + \psi([x_3,x_4],x_1,x_2) = 0\,,
\end{aligned}
$$

  where x_1,x_2,x_3,x_4 are elements of $\mathcal{W}$ or $\mathcal{V}$.

- The condition for a 3-cocycle ψ to be a coboundary with values in the trivial module is:

$$
\begin{aligned}
&\psi(x_1,x_2,x_3) = (\delta_2\phi)(x_1,x_2,x_3) \\
\Leftrightarrow\ &\psi(x_1,x_2,x_3) = \phi([x_1,x_2],x_3) + \phi([x_2,x_3],x_1) + \phi([x_3,x_1],x_2)\ ,
\end{aligned}
$$

  where ϕ is a 2-cochain with values in $\mathbb{K}$ and x_1,x_2,x_3 are elements of $\mathcal{W}$ or $\mathcal{V}$.

Inspiration from continuous cohomology (I)

- Let t be the coordinate along S^1. Elements of $Vect(S^1)$: $f(t)\frac{d}{dt}$, with f real-valued smooth function on S^1.
- Continuous cohomology: $dim(\mathrm{H}^3_c(Vect(S^1), \mathbb{R})) = 1$ (see Fuks and Gelfand [6]) .
- Generator given by Godbillon-Vey cocycle (c.f. [6]) :

$$\mathscr{G}\mathscr{V} : \left(f\frac{d}{dt}, g\frac{d}{dt}, h\frac{d}{dt} \right) \mapsto \int_{S^1} \det \begin{pmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{pmatrix} dt$$

with $f, g, h \in C^\infty(S^1)$. Prime = derivative w.r.t. t.

Inspiration from continuous cohomology (II)

- Geometrical realization of the Witt algebra: $\boxed{\tilde{e}_n = ie^{int}\frac{d}{dt}}$.
- We obtain:

$$\mathscr{G}\mathscr{V}(\tilde{e}_n, \tilde{e}_m, \tilde{e}_k) = -\int_{S^1} \det \begin{pmatrix} 1 & 1 & 1 \\ n & m & k \\ n^2 & m^2 & k^2 \end{pmatrix} e^{i(n+m+k)t} dt$$
$$= (n-m)(n-k)(m-k) \int_{S^1} e^{i(n+m+k)t} dt$$

Integral evaluates to zero if $n + m + k \neq 0$, otherwise it yields the value 1.

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (II)

- Trilinear map $\Psi \in \mathrm{H}^3(\mathcal{W},\mathbb{K})$:

$$\Psi : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to \mathbb{K}$$

defined on basis elements e_i as follows:

$$\boxed{\Psi(e_i, e_j, e_k) = (i-j)(j-k)(i-k)\delta_{i+j+k,0}}$$

- Trivial extension to a map of $\mathrm{H}^3(\mathcal{V},\mathbb{K})$:

$$\hat{\Psi} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{K}\,,$$

by setting $\hat{\Psi}(x_1, x_2, x_3) = 0$ whenever one of the elements x_1, x_2 or x_3 is a multiple of the central element t.

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (III)

Proposition 1

The trilinear maps Ψ and $\hat{\Psi}$ define non-trivial cocycle classes of $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{V},\mathbb{K})$, respectively.

Proof.

- Ψ and $\hat{\Psi}$ are cocycles of $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ respectively: shown by direct computation.
- Ψ and $\hat{\Psi}$ are not coboundaries: evaluate at e_{-1}, e_1, e_0 $\rightarrow \Psi(e_{-1}, e_1, e_0) = 2$ but $(\delta_2\Phi)(e_{-1}, e_1, e_0) = 0$ for all 2-cochains $\Phi : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{K}$, shown by direct computation. Similarly for $\hat{\Psi}$.

□

Proof of $dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$ (IV)

- Second step: There are no other non-trivial cocycles than Ψ resp. $\hat{\Psi}$, up to multiples and coboundaries.
  (dimension of $\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})$ is at most one)
- Fuks [5]: We only need to consider degree zero cohomology. Ψ and $\hat{\Psi}$ are of degree zero.
- Let ψ be an arbitrary degree zero 3-cocycle of $\mathcal{V}$ or $\mathcal{W}$.
- Set:

$$\psi' = \psi - \frac{\psi(e_{-1}, e_1, e_0)}{2}\,\hat{\Psi} \quad \text{resp.} \quad \psi' = \psi - \frac{\psi(e_{-1}, e_1, e_0)}{2}\,\Psi$$

- Then $\psi'(e_{-1}, e_1, e_0) = 0$ because
  $\Psi(e_{-1}, e_1, e_0) = \hat{\Psi}(e_{-1}, e_1, e_0) = 2.$

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (V)

Proposition 2

Let ψ be a 3-cocycle for $\mathcal{V}$ or $\mathcal{W}$ with $\boxed{\psi(e_{-1}, e_1, e_0) = 0}$.
Then ψ is a coboundary.

- ψ' fulfills $\psi'(e_{-1}, e_1, e_0) = 0 \Rightarrow \psi' = 0$ up to coboundaries:

$$\boxed{\psi = \frac{\psi(e_{-1}, e_1, e_0)}{2}\,\hat{\Psi} \quad \text{resp.} \quad \psi = \frac{\psi(e_{-1}, e_1, e_0)}{2}\,\Psi}$$

- $\Rightarrow$ Then any 3-cocycle ψ of $\mathcal{V}$ or $\mathcal{W}$ is a multiple of $\hat{\Psi}$ resp. Ψ, up to coboundaries, if Proposition 2 is true

Proof of $dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$ (VI)

- Proof of Proposition 2: elementary but tedious computations.
- Proof in three steps:
- Step 1 Fuks [5]: Reduce to degree zero cochains and cocycles.
- Step 2 Perform cohomological change $\psi \to \psi - \delta_2 \phi$ (Lemma 1)
- Step 3 Use fact that we are dealing with cocycles; i.e. use cocycle conditions (Lemma 2).

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (VII)

- Step 1

- Cochains and cocycles can be defined entirely by a system of coefficients in $\mathbb{K}$.
- Write

$$\psi(e_i, e_j, e_k) := \psi_{i,j,k} \qquad \text{and} \qquad \psi(e_i, e_j, t) := c_{i,j}$$

  with $\psi_{i,j,k}, c_{i,j} \in \mathbb{K}$.
- We have: $\psi_{i,j,k} = 0$ if $i + j + k \neq 0$ and $c_{i,j} = 0$ if $i + j \neq 0$ because we are considering degree zero cochains (c.f. [5])

Proof of $dim(\mathrm{H}^3(\mathcal{W},\mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V},\mathbb{K})) = 1$ (VIII)

- Step 2

Lemma 1

Every 3-cocycle $\psi' \in \mathrm{H}^3(\mathcal{V},\mathbb{K})$ satisfying $\psi'(e_1, e_{-1}, e_0) = 0$ is cohomologous to a 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{V},\mathbb{K})$ with coefficients $c_{i,j}, \psi_{i,j,k} \in \mathbb{K}$ fulfilling:

$$c_{i,j} = \delta_{i,-j}\left(\frac{1}{6}\; i\;(i-1)(i+1)c_{2,-2}\right) \text{ and } \psi_{i,j,1} = 0 \qquad \forall\; i,j \in \mathbb{Z}$$

- Similarly for $\mathcal{W}$
- Proof: elementary algebra but technical; use coboundary conditions

Proof of $dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$ (IX)

- Step 3

Lemma 2

Let $\psi \in \mathrm{H}^3(\mathcal{V}, \mathbb{K})$ be a 3-cocycle such that:

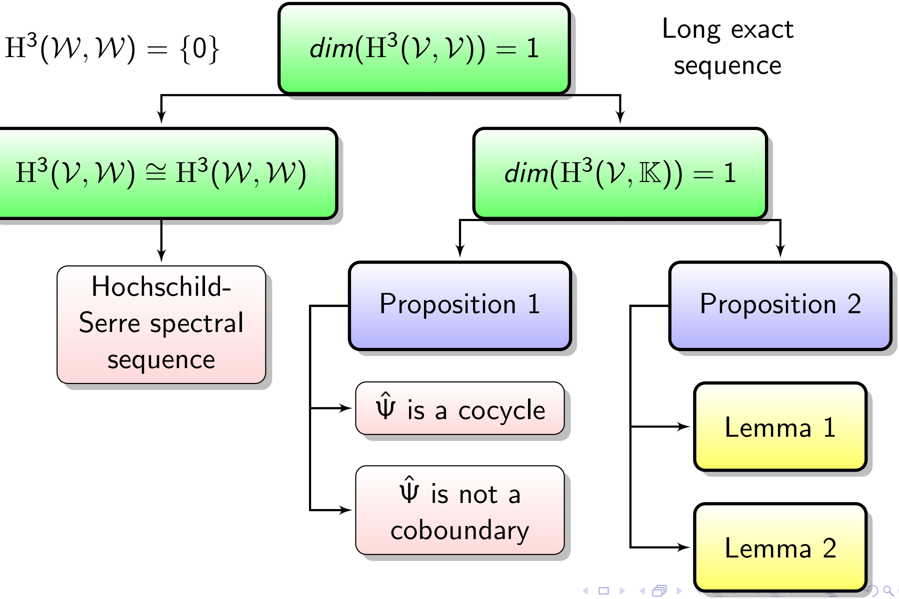
$$c_{i,j} = \delta_{i,-j}\left(\frac{1}{6}(i-1)(i)(i+1)c_{2,-2}\right) \text{ and } \psi_{i,j,1} = 0 \ \forall\ i,j \in \mathbb{Z}\,.$$

Then:

$$\boxed{c_{i,j} = 0 \quad \forall\ i,j \in \mathbb{Z}} \qquad \text{and} \qquad \boxed{\psi_{i,j,k} = 0 \quad \forall\ i,j,k \in \mathbb{Z}}$$

- Similarly for $\mathcal{W}$
- Proof: elementary algebra, but complicated; use cocycle conditions
- $\Rightarrow$ Every 3-cocycle ψ satisfying $\psi(e_1, e_{-1}, e_0) = 0$ is a coboundary

Summary



$H^3(\mathcal{W},\mathcal{W}) = \{0\}$
$dim(H^3(\mathcal{V},\mathcal{V})) = 1$
Long exact sequence
$H^3(\mathcal{V},\mathcal{W}) \cong H^3(\mathcal{W},\mathcal{W})$
$dim(H^3(\mathcal{V},\mathbb{K})) = 1$
Hochschild-Serre spectral sequence
Proposition 1
Proposition 2
$\hat{\Psi}$ is a cocycle
$\hat{\Psi}$ is not a coboundary
Lemma 1
Lemma 2

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Thank you for your attention!