Recall from first lec we

Defn A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by
$\mathcal{A}$: $*$-algebra of bdd operators on
$\mathcal{H}$: Hilbert space
D: ess. self-adj. operator in $\mathcal{H}$
s.t.
- $[D, a]$ extends to bdd commutator $\forall a \in \mathcal{A}$
- $(1+D^2)^{-1}$ is cpt. operator.

grading: γ. $\gamma D = -D\gamma$

real structure: J $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, $J\gamma = \varepsilon'' \gamma J$

$\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$.

N.B. Distance formula makes sense on $S(A)$:

$$d(\varphi, \psi) = \sup_{a \in A} \{ |\varphi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}.$$

for all $\varphi, \psi \in S(A)$. (positive, normed lin. functionals)

Basic example: (M, g) $\mathcal{S}_M$, D_M, γ_M, $J_M \dots$

Finite spaces $X = \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \cdots \quad \underset{N}{\bullet}$

$$X \rightsquigarrow \Big(\mathbb{C}^N, \mathbb{C}^N, \underset{\text{hermitian matrix.}}{D = D^+}\Big) : \text{diagonal matrices} \begin{pmatrix} \lambda_1 & & \phi \\ & \ddots & \\ \phi & & \lambda_N \end{pmatrix}$$

Reconstruction: any metric on $\{1, 2, \ldots, N\}$ can be obtained from such a sp. tr.

Ex. $N = 2$ $\quad \left(\mathbb{C}^2, \mathbb{C}^2, D = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}\right)$

$$d(1,2) = \frac{1}{|c|}$$

Finite noncommutative spaces

$$\begin{pmatrix} a_1 & & & \phi \\ & a_2 & & \\ & & \ddots & \\ \phi & & & a_N \end{pmatrix} \quad \underline{\underline{\text{block}}} \text{ diagonal matrices}$$

$$\begin{cases} \mathcal{A}_F = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C}). \\ \mathcal{H}_F = V \quad \mathbb{C} \text{ vector space.} \\ D_F = D_F^+ \quad \text{herm. matrix} \end{cases}$$

Ex (YM) $\begin{cases} \mathcal{A} = M_n(\mathbb{C}) \\ \mathcal{H} = \mathbb{C}^n \end{cases}, \quad D = D^+$

Ex (SM)

$$A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

$$\mathcal{H} = \mathbb{C}^{96}$$

gen

$$= \Big(\underset{f,\bar{f}}{\mathbb{C}^2} \otimes \underset{L,R}{\mathbb{C}^2} \otimes \big(\underset{\ell}{\mathbb{C}} \oplus \underset{q}{\mathbb{C}^2 \otimes \mathbb{C}^3} \big) \Big)^{\oplus 3}$$

$$D = \begin{pmatrix} S & T \\ T^* & \bar{S} \end{pmatrix} \qquad S = \begin{pmatrix} 0 & Y_{\ell,q} \\ Y_{\ell,q}^* & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & Y_R \\ Y_R & 0 \end{pmatrix}$$

$J: f \leftrightarrow \bar{f}$

$\gamma: L \leftrightarrow R$

Ex (PS)

$$A = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}).$$

$$\mathcal{H} = \underset{f,\bar{f}}{\mathbb{C}^2} \otimes \underset{L,R}{\mathbb{C}^2} \otimes \big(\mathbb{C}^2 \otimes \underset{\ell q}{\mathbb{C}^4} \big)^{\oplus 3}$$

Ex. (AC manifolds): product $M \times F$

$$\left(C^\infty(M) \otimes A_F,\ L^2(S_M) \otimes \mathcal{H}_F,\ D_M \otimes 1 + \gamma \otimes D_F \right)$$

M even-dimensional; $\begin{cases} \gamma_M \otimes 1 + 1 \otimes \gamma_F \\ J_M \otimes 1 + 1 \otimes J_F \end{cases}$

Symmetries of spectral triples.

$(\mathcal{A}, \mathcal{H}, D)$

Most natural : unitary equivalence.

Defn. $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are called unitarily equivalent if there exists isomorphism $\alpha : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2$ and unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.

$$\alpha(a) = U a U^*$$

$$D_2 = U D_1 U^*.$$

Special case: $u \in \mathcal{U}(A)$: $\begin{pmatrix} A_1 = A_2 = A \\ \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \end{pmatrix}$

$$\alpha(u) = u a u^* = \alpha_u(a)$$

inner automorphisms $\nearrow$ $\quad \alpha : \mathcal{U}(A) \to \mathrm{Inn}(A)$.

$$u D u^* = D + u[D, u^*]$$

pure gauge fields

Ex(manifold): $\quad u[D, u^*] = i\gamma^\mu (u \partial_\mu u^*)$

$u = e^{iX}$ $\quad = \gamma^\mu \partial_\mu X$

Ex (AC) $\left(C^\infty(M) \otimes M_N(\mathbb{C}), L^2(S_M) \otimes \mathbb{C}^N, D_M \otimes 1 + \gamma_M \otimes D \right)$

Then $\quad \mathcal{U}(A) = C^\infty(M, \mathcal{U}(n))$

and $\quad u[D_M, u^*]$ is a $\mathcal{U}(n)$ - pure gauge field

Can we extend this mathematical description of pure gauge fields to capture arbitrary gauge fields?