New fuzzy spheres through confining potentials and energy cutoffs

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Introduction

Noncommutative space(time) algebras are introduced and studied:

- To avoid UV divergences in QFT [Snyder 1947].
- As an arena to formulate QG, inducing $\Delta x \gtrsim L_p$ predicted by QG arguments [Mead 1966, Doplicher et al 1994-95].
- As an arena for unification of interactions [Connes-Lott,....]
- ...

Fuzzy spaces are particularly appealing: a FS is a sequence $\mathcal{A}_{n\in\mathbb{N}}$ of *finite-dimensional* algebras such that $\mathcal{A}_n \stackrel{n\to\infty}{\longrightarrow} \mathcal{A}$ ≡algebra of regular functions on an ordinary manifold.
First, seminal example: the Fuzzy Sphere (FS) of Madore [1991]: $\mathcal{A}_n \simeq M_n(\mathbb{C})$, generated by coordinates x^i $(i=1,2,3)$ fulfilling

$$[x^i,x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \quad r^2 := x^i x^i = 1, \qquad n \in \mathbb{N}\setminus\{1\}; \quad (1)$$

(1) are covariant under $SO(3)$, but not under the whole $O(3)$; in particular not under parity $x^i \mapsto -x^i$.

In fact $L^i = x^i\sqrt{n^2-1}/2$ make up the standard basis of $so(3)$ in the irrep (π_l, V_l) characterized by $L^iL^i = l(l+1)$, $l = 2n+1$.
Does the FS approximate the configuration space algebra of a particle on S^2? Problems: a) parity; b) V_l is irreducible, whereas

$$\mathcal{L}^2(S^2) = \bigoplus_{l=0}^{\infty} V_l$$

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$$\mathcal{L}^2(S^2) = \bigoplus_{l=0}^{\infty} V_l = C(S^2) \qquad (2)$$

Here fuzzy approximations of QM on S^d $(d = 2)$ solving a),b):

• Start with ordinary quantum particle in $\mathbb{R}^D$ $(D = d+1)$, under a potential $V(r)$ with a very sharp minimum on the sphere $r = 1$.

• By low enough energy-cutoff $E \leq \overline{E}$ we *'freeze' radial excitations*, make only a finite-dimensional Hilbert subspace $\mathcal{H}_{\overline{E}}$ accessible, and on it the x^i noncommutative *à la Snyder*; the x^i generate the whole algebra of observables. $O(D)$-covariant by construction.

• Making $\overline{E}$, $k := V''(1)/4 \gg 0$ diverge with $\Lambda \in \mathbb{N}$ (while $E_0 = 0$), we get a sequence $\mathcal{A}_\Lambda$ of fuzzy approximations of ordinary QM on S^d.

• On $\mathcal{H}_{\overline{E}}$ the square distance $\mathcal{R}^2$ from the origin is not identically 1, but a function of L^2 which collapses to 1 in the $\Lambda \to \infty$ limit.

Remarks:

- Our construction is inspired by the Landau model: there noncommuting x, y obtained projecting QM with a strong uniform magnetic field B on the lowest energy subspace.
- *Physically sound method*, applicable to more general contexts. Imposing a cutoff $\overline{E}$ on an existing theory:
  • can yield an effective description of a system when our measurements, or the interactions with the environment, cannot bring its state to energies $E > \overline{E}$; or even
  • may be a necessity if we believe $\overline{E}$ represents the threshold for the onset of new physics not accountable by that theory.
- If H is invariant under some symmetry group, then the projection $P_{\overline{E}}$ on $\mathcal{H}_{\overline{E}}$ is invariant as well, and the projected theory will inherit that symmetry.

General framework

Consider a quantum particle in $\mathbb{R}^D$ configuration space with Hamiltonian

$$H = -\frac{1}{2}\Delta + V(r); \qquad (3)$$

we fix the minimum $V_0 = V(1)$ of the the confining potential $V(r)$ so that the ground state has energy $E_0 = 0$. Choose an energy cutoff $\overline{E}$ fulfilling

$$V(r) \simeq V_0 + 2k\,(r-1)^2 \qquad (4)$$

if $V(r) \leq \overline{E}$; so that $V(r)$ has a harmonic behavior for $|r-1| \leq \sqrt{\frac{\overline{E}-V_0}{2k}}$.

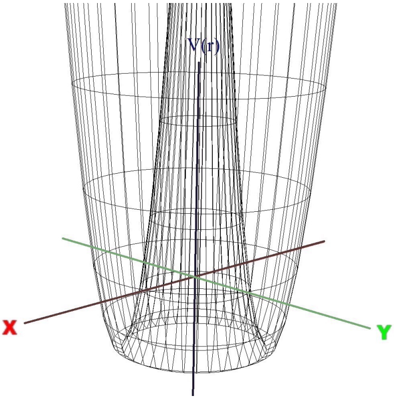
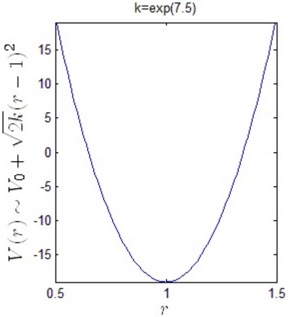


Figure 1: Three-dimensional plot of $V(r)$

Then we restrict to $\mathcal{H}_{\overline{E}} \subset \mathcal{H} \equiv \mathcal{L}^2(\mathbb{R}^D)$ spanned by ψ with $E \leq \overline{E}$. This entails replacing every observable A by $\overline{A}$:

$$A \mapsto \overline{A} := P_{\overline{E}} A P_{\overline{E}},$$



where $P_{\overline{E}}$ is the projection on $\mathcal{H}_{\overline{E}}$. Because of the behavior of $V(r)$ as $k \to +\infty$, we expect that when both k, $\overline{E}$ diverge $\dim(\mathcal{H}_{\overline{E}})$ diverges and we recover standard QM on the sphere S^{D-1}. The Laplacian in D dimensions decomposes as follows

$$\Delta = \partial_r^2 + (D-1)\frac{1}{r}\partial_r - \frac{1}{r^2}L^2. \qquad (5)$$

where $L_{ij} := ix^j\partial_i - ix^i\partial_j$ are the angular momentum components (in normalized units), and $L^2 = L_{ij}L_{ij}$ is the square angular momentum, i.e. the Laplacian on the sphere S^{D-1}.

$H, L_{ij}, P_{\overline{E}}$ commute. As known, the eigenvalues of L^2 are $j\,(j+D-2)$; the Ansatz $\psi = g(r)Y(\varphi,...)$ (Y are eigenfunctions of L^2 and of the elements of a Cartan subalgebra of $so(D)$; $r, \varphi, ...$ are polar coordinates) transforms the eigenvalue equation $H\psi = E\psi$ into this auxiliary ODE in the unknown $g(r)$:

$$\left[-\partial_r^2 - \frac{D-1}{r}\partial_r + \frac{j\,(j+D-2)}{r^2} + V(r)\right] g(r) = Eg(r); \qquad (6)$$

we must stick to solutions g leading to square-integrable ψ. To obtain the lowest eigenvalues we don't need to solve it exactly: condition (4) allows us to approximate (6) with the eigenvalue equation of a $1-$dimensional harmonic oscillator, by Taylor expanding $V(r)$, $1/r$, $1/r^2$ around $r=1$.

D=3: $O(3)$-covariant fuzzy sphere

Ansatz $\psi=\frac{f(r)}{r}Y_l^m(\theta,\varphi)$. Y_l^m are the spherical harmonics:

$$L^2\,Y_l^m(\theta,\varphi)=l(l+1)Y_l^m(\theta,\varphi)\,,\qquad L_3\,Y_l^m(\theta,\varphi)=mY_l^m(\theta,\varphi)\,,$$

$l\in\mathbb{N}_0$, $m\in\mathbb{Z}, |m|\le l$. Under assumption (4) the harmonic oscillator approximation of (6) admits the (Hérmite) eigenfunctions

$$f_{n,l}(r)=N_{n,l}e^{-\frac{(r-\widetilde{r}_l)^2\sqrt{k_l}}{2}}H_n\left((r-\widetilde{r}_l)\sqrt[4]{k_l}\right),\qquad n=0,1,....$$

with $k_l:=2k+3l(l+1)$, $\widetilde{r}_l=\frac{2k+4l(l+1)}{2k+3l(l+1)}$. $E_{0,0}=0\Rightarrow V_0=-\sqrt{2k}+O(1)$; then the energies associated to $\psi_{n,l,m}=\frac{f_{n,l}(r)}{r}Y_l^m(\theta,\varphi)$ are

$$E_{n,l}=2n\sqrt{2k}+l(l+1)+O\left(1/\sqrt{2k}\right)$$

$E_{0,l}=l(l+1)=:E_l$ are the eigenvalues of the Laplacian L^2 on S^2, while $E_{n,l}\to\infty$ as $k\to\infty$ if $n>0$.

We can eliminate the latter (constrain $n = 0$) imposing a cutoff $\mathbf{E \leq \overline{E} < 2\sqrt{2k}}$. And setting $\overline{E} \equiv \Lambda(\Lambda+1)$ we obtain

$$\mathbf{E \leq \Lambda(\Lambda+1) \equiv \overline{E} < 2\sqrt{2k}}. \quad (7)$$

i.e. we project the theory on the subspace $\mathcal{H}_\Lambda \subset \mathcal{L}^2(\mathbb{R}^3)$ spanned by

$$\psi_l^m := \psi_{0,l,m} \simeq \frac{N_l}{r} e^{-\frac{(r-\tilde{r}_l)^2\sqrt{k_l}}{2}} Y_l^m(\theta,\varphi), \quad (8)$$
$$|m| \leq l, \quad l \leq \Lambda.$$

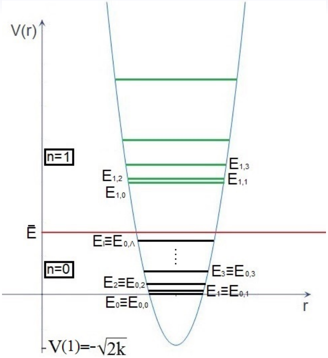


Figure 2: Two-dimensional plot of $V(r)$ including the energy-cutoff

Clearly $\dim(\mathcal{H}_\Lambda) = (\Lambda+1)^2$. Let $x^0 := z$, $x^\pm := \frac{x \pm iy}{\sqrt{2}}$. The action of $x^a = r\frac{x^a}{r}$ $(a = -, 0, +)$ on ψ_l^m factorizes into the one of r on $\frac{f_{0,l}(r)}{r}$ and the one of $\frac{x^a}{r}$ on Y_l^m.

After projection we find

$$\overline{x}^a \psi_l^m = c_l A_l^{a,m} \psi_{l-1}^{m+a} + c_{l+1} A_{l+1}^{-a,m+a} \psi_{l+1}^{m+a},$$
$$c_0 = c_{\Lambda+1} = 0, \qquad c_l = \sqrt{1 + \tfrac{l^2}{k}} \quad 1 \le l \le \Lambda \tag{9}$$

up to $O\left(1/k^{\frac{3}{2}}\right)$, and $A_l^{a,m}, B_l^{a,m}$ are the coefficients determined by

$$\frac{x^a}{r} Y_l^m = A_l^{a,m} Y_{l-1}^{m+a} + A_{l+1}^{-a,m+a} Y_{l+1}^{m+a}.$$

At leading order the $\overline{L}_i, \overline{x}^i$, $i \in \{1,2,3\}$, fulfill

$$\prod_{l=0}^{\Lambda}\left[\overline{L}^2 - l(l+1)I\right] = 0, \qquad \prod_{m=-l}^{l}\left(\overline{L}_3 - mI\right)\widetilde{P}_l = 0, \tag{10}$$

$$\overline{L}_i^{\dagger} = \overline{L}_i, \quad \left[\overline{L}_i, \overline{L}_j\right] = i\varepsilon^{ijh}\overline{L}_h, \qquad \overline{x}^{i\dagger} = \overline{x}^i, \quad \overline{x}^i\overline{L}_i = 0, \tag{11}$$

$$\underbrace{[\overline{L}_i, \overline{x}^j] = i\varepsilon^{ijh}\overline{x}^h}_{Snyder-like}, \qquad [\overline{x}^i, \overline{x}^j] = i\varepsilon^{ijh}\underbrace{\left(-\frac{1}{k} + K\widetilde{P}_\Lambda\right)\overline{L}_h}_{Snyder-like}, \tag{12}$$

where $K = \frac{1}{k} + \frac{1+\frac{\Lambda^2}{k}}{2\Lambda+1}$, $\overline{L}^2 := \overline{L}_i\overline{L}_i = \overline{L}_a\overline{L}_{-a}$ is L^2 projected on $\mathcal{H}_\Lambda$, and $\widetilde{P}_l$ is the projection on its eigenspace with eigenvalue $l(l+1)$. Moreover, the square distance from the origin is

$$\mathcal{R}^2 := \overline{x}^i\overline{x}^i = 1 + \frac{\overline{L}^2 + 1}{k} - \left[1 + \frac{(\Lambda+1)^2}{k}\right]\frac{\Lambda+1}{2\Lambda+1}\widetilde{P}_\Lambda. \tag{13}$$

These relations are *exact* if we adopt (9) as *definitions* of $\overline{x}^a$.

To obtain a fuzzy space we can choose k as a function of Λ fulfilling (7); one possible choice is $k = \Lambda^2(\Lambda+1)^2$, and the commutative limit will be $\Lambda \to +\infty$.

Some remarks...

- $[\overline{x}, \overline{x}] = ...$ and $[\overline{L}, \overline{x}] = ...$ are Snyder-like: $[\overline{x}, \overline{x}] = -L/k$ (plus term containing $\widetilde{P}_\Lambda$) and vanish as $\Lambda \to \infty$; $\psi_l^m \to \delta(r-1) Y_l^m$.
- Hence (10-12) are covariant under the whole $O(3)$, including parity $\overline{x}_i \mapsto -\overline{x}_i$, $\overline{L}_i \mapsto \overline{L}_i$, contrary to Madore FS.
- $\mathcal{R}^2 \neq 1$; its eigenvalues slightly grow with l (for each fixed Λ), but collapse to 1 as $\Lambda \to \infty$.
- The ordered monomials in $\overline{x_i}, \overline{L_i}$ make up a basis of the $(\Lambda+1)^4$-dim vector space $\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda) \simeq M_{(\Lambda+1)^2}(\mathbb{C})$ ($\widetilde{P}_l$ can be expressed as polynomials in $\overline{L}^2$).
- Actually, $\overline{x}_i$ *generate* the $*$-algebra $\mathcal{A}_\Lambda$ (also the $\overline{L}_i$ can be expressed as a non-ordered polynomial in the $\overline{x}_i$).

Realization of the algebra of observables through $Uso(4)$

$so(4) \simeq su(2) \oplus su(2)$ is spanned by $\left\{E_i^1, E_i^2\right\}_{i=1}^3$ fulfilling

$$[E_i^1, E_j^2] = 0, \qquad [E_i^1, E_j^1] = i\varepsilon^{ijk} E_k^1, \qquad [E_i^2, E_j^2] = i\varepsilon^{ijk} E_k^2. \quad (14)$$

$L_i := E_i^1 + E_i^2$, $X_i := E_i^1 - E_i^2$ make up alternative basis of $so(4)$:

$$[L_i, L_j] = i\varepsilon^{ijk} L_k, \qquad [L_i, X_j] = i\varepsilon^{ijk} X_k, \qquad [X_i, X_j] = i\varepsilon^{ijk} L_k. \quad (15)$$

The L_i close another $su(2)$. Passing to generators labelled by $a \in \{-, 0, +\}$,

$$[L_+, L_-] = L_0, \quad [L_0, L_\pm] = \pm L_\pm = [X_0, X_\pm], \quad [X_+, X_-] = L_0, \quad (16)$$

$$[L_\pm, X_\mp] = \pm X_0, \quad [L_0, X_\pm] = \pm X_\pm = [X_0, L_\pm], \quad [L_a, X_a] = 0 \quad (17)$$

(no sum over a), where $L^2 = L_i L_i = L_a L_{-a}$, $X^2 = X_i X_i = X_a X_{-a}$.

In the tensor product representation $\boldsymbol{\pi}_\Lambda := \pi_{\frac{\Lambda}{2}} \otimes \pi_{\frac{\Lambda}{2}}$ of $Uso(4) \simeq Usu(2) \otimes Usu(2)$ on the Hilbert space $\mathbf{V}_\Lambda := V_{\frac{\Lambda}{2}} \otimes V_{\frac{\Lambda}{2}}$ it is $C^1 := E_i^1 E_i^1 = \frac{\Lambda}{2}(\frac{\Lambda}{2}+1) = E_i^2 E_i^2 =: C^2$, or equivalently

$$X \cdot L = L \cdot X = 0, \qquad X^2 + L^2 = \Lambda(\Lambda+2) \tag{18}$$

(we have dropped the symbols $\boldsymbol{\pi}_\Lambda$). $\mathbf{V}_\Lambda$ admits an orthonormal basis consisting of common eigenvectors of L^2 and L_0:

$$L_0 \,|l,m\rangle = m\,|l,m\rangle\,, \qquad L^2\,|l,m\rangle = l(l+1)\,|l,m\rangle \tag{19}$$

with $0 \le l \le \Lambda$ and $|m| \le l$. $\mathbf{V}_\Lambda, \mathcal{H}_\Lambda$ have the same dimension $(\Lambda+1)^2$ and decomposition in irreps of the L_i subalgebra; we identify them setting $\psi_l^m \equiv |l,m\rangle$. The action of X^a on $\mathbf{V}_\Lambda$ reads

$$X^a\,|l,m\rangle = d_l A_l^{a,m}\,|l-1,m+a\rangle + d_{l+1} B_l^{a,m}\,|l+1,m+a\rangle \tag{20}$$
$$d_l := \sqrt{(\Lambda+1)^2 - l^2}$$

We can naturally realize $\overline{L}_a, \overline{x}^a$ in $\boldsymbol{\pi}_\Lambda[Usu(2)\otimes Usu(2)]$.
Define $\lambda := \frac{\sqrt{4L^2+1}-1}{2}$; then $\lambda\,|l,m\rangle = l\,|l,m\rangle$. The Ansatz

$$\overline{L}_a = L_a, \qquad \overline{x}^a = g(\lambda)\, X^a\, g(\lambda), \tag{21}$$

fulfills (9) and therefore (10-12) provided

$$\begin{aligned} g(l) &= \sqrt{\frac{\prod_{h=0}^{l-1}(\Lambda+l-2h)}{\prod_{h=0}^{l}(\Lambda+l+1-2h)} \prod_{j=0}^{\left[\frac{l-1}{2}\right]} \frac{1+\frac{(l-2j)^2}{k}}{1+\frac{(l-1-2j)^2}{k}}} \\ &= \sqrt{\frac{\Gamma\left(\frac{\Lambda+l}{2}+1\right)\Gamma\left(\frac{\Lambda-l+1}{2}\right)}{\Gamma\left(\frac{\Lambda+1+l}{2}+1\right)\Gamma\left(\frac{\Lambda-l}{2}+1\right)} \frac{\Gamma\left(\frac{l}{2}+1+\frac{i\sqrt{k}}{2}\right)\Gamma\left(\frac{l}{2}+1-\frac{i\sqrt{k}}{2}\right)}{\sqrt{k}\,\Gamma\left(\frac{l+1}{2}+\frac{i\sqrt{k}}{2}\right)\Gamma\left(\frac{l+1}{2}-\frac{i\sqrt{k}}{2}\right)}} \end{aligned} \tag{22}$$

The inverse of (21) is clearly $X^a = [g(\lambda)]^{-1}\,\overline{x}^a\,[g(\lambda)]^{-1}$.
We have thus explicitly constructed a *-algebra map

$$\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C}) \simeq \boldsymbol{\pi}_\Lambda[Uso(4)], \quad N := (\Lambda+1)^2. \tag{23}$$

As known, the group of $*$-automorphisms of $M_N(\mathbb{C}) \simeq \mathcal{A}_\Lambda$ is $SU(N)$ and

$$b \to gbg^{-1}, \qquad b \in \mathcal{A}_\Lambda, \quad g \in SU(N).$$

A special role is played by the subgroup $SO(4)$ acting through the representation $\boldsymbol{\pi}_\Lambda$, namely $g = \boldsymbol{\pi}_\Lambda\left[e^{i\alpha}\right]$, $\alpha \in so(4)$.

$O(3) \subset SO(4)$ plays the role of isometry subgroup.

In particular, choosing $\alpha = \alpha_i L_i$ ($\alpha_i \in \mathbb{R}$) the automorphism amounts to a $SO(3)$ transf. (a rotation in 3-dimensional space).

An $O(3)$ transformation with determinant -1 in the $X^1X^2X^3$ space is parity $(L_i, X^i) \mapsto (L_i, -X^i)$, or equivalently $E_i^1 \leftrightarrow E_i^2$ (this is the only automorphism of $so(4)$, corresponding to the exchange of the two nodes in the Dynkin diagram).

Convergence to $O(3)$-equivariant quantum mechanics on S^2 as $\Lambda \to \infty$

Define $O(3)$-equivariant embedding $\mathcal{I}: \mathcal{H}_\Lambda \hookrightarrow \mathcal{L}^2(S^2) \equiv \mathcal{H}_s$ by $\mathcal{I}(\psi_l^m) := Y_l^m$; below drop $\mathcal{I}$ and identify $\psi_l^m = Y_l^m$.
$P_\Lambda \phi \to \phi$ in the $\mathcal{H}_s$-norm $\| \, \|$: $\mathcal{H}_\Lambda$ 'invades' $\mathcal{H}_s$ as $\Lambda \to \infty$.

$\mathcal{I}$ induces an embedding $\mathcal{J}: \mathcal{A}_\Lambda \hookrightarrow B[\mathcal{H}_s]$.
$\overline{L_i} = L_i$ on $\mathcal{H}_\Lambda$, and $\overline{L}_i \phi \to L_i \phi$ as $\Lambda \to \infty$, $\forall \phi \in D(L_i)$

Bounded (continuous) functions f on S^2, acting as multiplication operators $f\cdot : \phi \in \mathcal{H}_s \mapsto f\phi \in \mathcal{H}_s$, make up a subalgebra $B(S^2)$ [resp. $C(S^2)$] of $B[\mathcal{H}_s]$. Fuzzy analog of vector space $B(S^2)$:

$$\mathcal{C}_\Lambda := \left\{ \sum_{l=0}^{2\Lambda} \sum_{m=-l}^{l} f_l^m \widehat{Y}_l^m \, , \; f_l^m \in \mathbb{C} \right\} = \bigoplus_{l=0}^{2\Lambda} V_l \subset \mathcal{A}_\Lambda, \tag{24}$$

$$\text{where} \qquad \widehat{Y}_l^m := M_l \sqrt{\frac{(l+m)!\,2^{l-m}}{(2l)!\,(l-m)!}} \, L_-^{l-m} (\overline{x}^+)^l \tag{25}$$

are the fuzzy analogs of $Y_l^m \cdot \in B(S^2)$.

We first show $\overline{x}^i\phi \to (x^i/r)\phi$.

Moreover, $\forall f\cdot \in B(S^2)$ let $\hat{f}_\Lambda := \sum_{l=0}^{2\Lambda}\sum_{|m|\le l} f_l^m \widehat{Y}_l^m \in \mathcal{C}_\Lambda$.

Proposition. Choose $k(\Lambda) \geq 2^{3\Lambda+3}\Lambda^{\Lambda+5}(\Lambda+1)$. Then $\hat{f}_\Lambda \to f\cdot$, $\widehat{(fg)}_\Lambda \to fg\cdot$, $\hat{f}_\Lambda\hat{g}_\Lambda \to fg\cdot$ strongly as $\Lambda \to \infty$, $\forall f\cdot, g\cdot \in B(S^2)$.

Final remarks and conclusions

For $d=2$ we have built a sequence $(\mathcal{A}_\Lambda, \mathcal{H}_\Lambda)$ of finite-dim, $O(D)$-covariant ($D=d+1$) approximations of QM of a spinless particle on the sphere S^d; $\mathcal{R}^2 \gtrsim 1$ collapses to 1 as $\Lambda \to \infty$.

Achieved imposing $E \le \Lambda(\Lambda+d-1)$ on QM of a particle in $\mathbb{R}^D$ subject to a sharp confining potential $V(r)$ on the sphere $r=1$.

$\mathcal{A}_\Lambda$ are fuzzy approximations of the *whole algebra of observables* of the particle on S^d (phase space algebra).

$\mathcal{A}_\Lambda \simeq \pi_\Lambda[Uso(D+1)]$, with a suitable irrep π_Λ of $Uso(D+1)$ on $\mathcal{H}_\Lambda$.

$\mathcal{H}_\Lambda$ carries a *reducible* representation of the $Uso(D)$ subalgebra generated by the $\overline{L}_{ij}$: $\mathcal{H}_\Lambda = \bigoplus$irreps fulfilling $L^2 \le \Lambda(\Lambda+d-1)$.

The same decomposition holds for the subspace $\mathcal{C}_\Lambda \subset \mathcal{A}_\Lambda$ of completely symmetrized polynomials in the $\overline{x}^i$.

As $\Lambda \to \infty$ these resp. become the decompositions (2) of $\mathcal{L}^2(S^d)$ and of $C(S^d)$ acting on $\mathcal{L}^2(S^d)$.

Comparison with literature

The fuzzy spheres of dimension $d=4$ [Grosse, Klimcik, Presnajder 1996], $d \geq 3$ [Ramgoolam 2001, Dolan, O'Connor 2003, ...], are based on $End(V)$ where V carries a particular *irrep* of $SO(d+1)$; $\mathcal{R}^2$ is central, can be set=1. Snyder-like commutation relations, hence $O(d+1)$-covariant.

In [Steinacker 2016-17] fuzzy 4-spheres S^4_N through reducible repr. of $Uso(5)$ obtained decomposing irreps π of $Uso(6)$ with suitable highest weights (N, n_1, n_2); so $End(V) \simeq \pi[Uso(6)]$, in analogy with our result. The elements X^i of a basis of $so(6) \setminus so(5)$ (as a vector space) play the role of noncommuting cartesian coordinates. Hence, the $SO(5)$-scalar $\mathcal{R}^2 = X^i X^i$ is no longer central, but its spectrum is still very close to 1 *only if* $N \gg n_1, n_2$;
if $n_1 = n_2 = 0$ then $\mathcal{R}^2 \equiv 1$ ($\Rightarrow$ irrep), and one recovers the fuzzy 4-sphere [Grosse, Klimcik, Presnajder 1996].

Here $\mathcal{R}^2 \simeq 1$ is guaranteed by adopting $\overline{x}^i = g(L^2) X^i g(L^2)$ rather than X^i as noncommutative cartesian coordinates, and $\mathcal{R}^2 = \overline{x}^i \overline{x}^i$.

Bonus slide: Coherent states

It's interesting to look for the states $\chi \in \mathcal{H}_\Lambda$ minimizing the uncertainty on the localization. We suggest that we have to minimize

$$\begin{aligned} \Delta\mathcal{R}^2_\chi := & \left\langle \chi \left| \sum_{i=1}^{3} \left(\Delta\overline{x}^i\right)^2 \right| \chi \right\rangle \\ = & \left\langle \chi \left| \sum_{i=1}^{3} \overline{x}^i\overline{x}^i \right| \chi \right\rangle - \sum_{i=1}^{3} \left(\langle \chi | \overline{x}^i | \chi \rangle\right)^2 = \\ = & \langle \chi \left| \mathcal{R}^2 \right| \chi \rangle - \sum_{i=1}^{3} \left(\langle \chi | \overline{x}^i | \chi \rangle\right)^2 , \end{aligned} \tag{26}$$

where $\|\chi\| = 1$. It's easy to see that (26) is $O(3)$-covariant, and we've proved that our coherent states are more "localized" than the Perelomov coherent states of the Madore's fuzzy sphere.