# Looking back to the Moyal revolution

José M. Gracia-Bondía

QSpace Training School, CCBPP, Spain

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# Topics

- Success of the "Moyal" paradigm
- A classical statistical mechanics look to it
- The "functional" rather more than the "deformation" approach
- Traciality
- Extension of the Moyal product by duality and nets of Hilbert algebras
- The covariant context (Fourier–Kirillov–Moyal)
- Relativistic particles
- The NCG connection
- "Physical Wigner functions" in quantum chemistry

# Moyal technology

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The ambit of its applications, including to mathematics, besides quantum mechanics proper – and its foundational issues – today encompasses:

- Quantum optics
- The theory of sound
- Quantum chemistry
- Non-commutative geometry
- Non-commutative field theory
- Deformation theory
- Special function theory
- Harmonic analysis (...)

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An advantage of the Moyal scheme for quantum mechanics that quantum and classical averages obey the same rule. I choose to look back at one mathematical aspect at the root of the success – that is **traciality**.

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To finish I revisit *physical assumptions* behind the success, pursuing the vision of the old masters, with examples in realms of relativistic and statistical physics.

# Traciality in a nutshell I

A tracial (Stratonovich–Weyl) quantizer is an **self-adjoint operator-valued distribution** $\Omega(x)$ relating a classical system on a phase space $X$ with operators on an associated Hilbert space $\mathcal{H}$, verifying

$$\operatorname{Tr}\Omega(x) = 1; \tag{1}$$

$$\operatorname{Tr}\Big(\Omega(x)\Omega(x')\Big) = \delta(x - x'). \tag{2}$$

These are not trace-class operators in general; the trace is understood in a distributional sense! If one uses the family $\Omega$ (the "quantizer") to convert a "symbol" on $X$ into an operator on $\mathcal{H}$ by the rule

$$a \mapsto \int_X a(x)\Omega(x) =: Q(a),$$

then, from (1) to begin with, $TrQ(a) = \int_X a(x)$ – I suppress the measure on $X$ in the notation.

# Traciality in a nutshell II

The inverse map is given by

$$a(x) = \mathrm{Tr}\Big(\Omega(x)Q(a)\Big);$$

so $\Omega(.)$ is also the dequantizer!

Moreover, we have a **Hilbert algebra** relation:

$$\mathrm{Tr}\Big(Q(a)Q(b)) = \int_X a(x)b(x).$$

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Traciality moreover yields mathematical dividends, often useful in physics. The main one: they lead most naturally to algebras of unbounded operators – the humble position and momentum operators are unbounded...

# The main trick

... is to push an **extension of quantization** to distributions using the duality allowed by (2). Let

$$\text{(Moyal product) } a \times b(x) := \mathrm{Tr}[Q(x)Q(a)Q(b)];$$
$$\text{then it holds } \int_X a \times b(x) = \int_X a(x)b(x).$$

One proves $\mathcal{S} \times \mathcal{S} = \mathcal{S}$ (Schwartz functions). In view of the above $\mathcal{S}' \times \mathcal{S}$ and $\mathcal{S} \times \mathcal{S}'$ are defined.

By the way: for this (Heisenberg) case $\Omega(x) = 2\Pi(x)$, with $\Pi$ the parity operator on phase space (Grossmann and Royer).

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By the way, too: it is perfectly true that

$$f \times g = fg + \frac{i}{2}\{f, g\} + \cdots$$

convergent under favourable conditions envisageable here.

# Vast $*$-spaces

It is natural to introduce the multiplier spaces:

$M_R := \{S \in \mathcal{S}' : f \times S \in \mathcal{S},\ \forall f \in \mathcal{S}\}; \quad M_L := \{S \in \mathcal{S}' : S \times f \in \mathcal{S}\};$

as well as the $*$-algebra $M := M_R \cap M_L$. These spaces of distributions are $*$-algebras under $\times$ that coincide with their strong biduals:

$$(M_L')' = M_L; \quad (M_R')' = M_R;$$

moreover, among other nice properties:

$$M_{L,R}' \times M_{L,R}' = M_{L,R}'.$$

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In the eighties we saw how to use the properties of the so-extended Moyal product to prove Schwartz's crowning kernel theorem quite in a simple way.

# Fourier–Kirillov–Moyal theory

Usually, $X$ is an homogeneous $G$-space, and then $\mathcal{H}$ is a representation space, say for a unirrep $U$. Then we ask covariance from the quantizer:

$$\Omega(g \cdot x) = U(g)\Omega(x)U^\dagger(g).$$

The harmonic analysis (Plancherel theory) of general Lie groups is a desperately abstract branch of mathematics. It can be made more concrete by identifying the coadjoint orbits related to unirreps and defining the Fourier–Plancherel transform by means of the scalar kernel:

$$E(x, g) = \mathrm{Tr}\Big(\Omega(x)U(g)\Big)$$

In this way, most of the nicer properties of standard Fourier theory are recovered.
Mind you! The matter is more complicated for **non-unimodular** groups – as we learned from affine groups some time ago.

# Group orbits with Fourier–Kirillov–Moyal kernels

- The "Fourier–Kirillov–Moyal paradigm" (FKM) holds for every orbit of any compact group.
- For $SU(2)$ (spin) the Moyal representation is a roaring success, since it was introduced almost at the same time that quantum optics & MRI practitioners abandoned the quantum-mechanical description for more visual ones.
- With suitable modifications, FKM works for some non-unimodular groups with simple systems of coadjoint orbits. There one must consider **right** and **left** kernels.

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- With suitable modifications, FKM works for some non-unimodular groups with simple systems of coadjoint orbits. There one must consider **right** and **left** kernels.
- Nilpotent groups: done in all generality by Pedersen.
- Discrete series of $SL(2, \mathbb{R})$: existence proof by the Unterbergers.
- Groups with not-simply connected coadjoint orbits are notoriously difficult.

# Other FKM tales

- Physically interesting are the coadjoint orbits of semi-direct product groups, like the Poincaré groups. For the orbits corresponding to massive particles of the ordinary Poincaré group we have proved the existence of Moyal (i.e., tracial) quantizers.

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- The Moyal representation of the relativistic particle case is based on a hyperbolic reflection:

$$M_p\xi := 2\frac{(p\xi)p}{p^2} - \xi,$$

where both $p$ and $\xi$ stand for 4-momentum, respectively in phase space and as wave function coordinate. This melds well with the spinor formulation, allowing for extensions to higher spins.

# Coda

- Nobody has succeeded yet to do the same for massless particle orbits.

  Upping the stakes, Lizzi, Várilly, Vitale and myself did look last year at the coadjoint orbit picture for the so-called unbounded helicity particles of Wigner – this works out fine, yielding a curious duality with the magnetic monopole. But that has not helped much in today's respect for now.

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- As mentioned, the span of related applications of this and related circles of ideas nowadays is vast: non-formal deformation theory , time-frequency analysis , non-commutative geometry. . .

# Talking about non-commutative geometry. . .

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- Perhaps the mathematical creature with more (disputing) fathers I know of is the "fuzzy sphere". In fact I hold that the fuzzy sphere was introduced *avant la lettre* by Stratonovich (1956). This paper was the germ for our ideas on the general Moyal representation for spin.

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- Alain Connes initially took with some scepticism the idea that Moyal planes could be spectral triples – of a non-compact sort. Teaming our efforts, we (the Marseilles' group of Kastler's disciples, Várilly and myself) were able to show precisely that, now fifteen years ago.

# Physical Wigner functions I

In any physical theory, one should pay due attention to the states. Long ago, Várilly and myself built up a machine that fabricates (generally mixed) states ("**Wigner functions**") being positive both in the standard and in the Moyal sense (that is, the corresponding trace-class operator is positive).

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It is not hard to prove that symmetry **or** anti-symmetry conditions for a 2-body problem demand of the Wigner function:

$$W(\boldsymbol{R},\boldsymbol{r};\boldsymbol{P},\boldsymbol{p}) = W(\boldsymbol{R},-\boldsymbol{r};\boldsymbol{P},-\boldsymbol{p})$$

for $\boldsymbol{R}$ the extracule and $\boldsymbol{r}$ the intracule coordinates, in chemists′ jargon.

# Physical Wigner functions II

To distinguish between the two cases, we may show that $\tilde{W}_{\boldsymbol{R},\boldsymbol{P}}(\boldsymbol{v},\boldsymbol{p}) = \pm\tilde{W}_{\boldsymbol{R},\boldsymbol{P}}(\boldsymbol{p},\boldsymbol{v})$, on using two momentum-like (or equivalently position-like) variables.

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Now, for a more realistic situation, consider spin Wigner functions: a 1-body atomic Wigner function in matrix form would be of the form

$$\begin{pmatrix} W^{\uparrow_1\uparrow_{1'}}(\boldsymbol{x};\boldsymbol{p}) & W^{\uparrow_1\downarrow_{1'}}(\boldsymbol{x},\boldsymbol{p}) \\ W^{\downarrow_1\uparrow_{1'}}(\boldsymbol{x},\boldsymbol{p}) & W^{\downarrow_1\downarrow_{1'}}(\boldsymbol{x},\boldsymbol{p}) \end{pmatrix};$$

and a 2-body atomic Wigner distribution:

$$\begin{pmatrix} W^{\uparrow_1\uparrow_2\uparrow'_1\uparrow'_2}(1,2) & W^{\uparrow_1\uparrow_2\uparrow'_1\downarrow'_2}(1,2) & W^{\uparrow_1\uparrow_2\downarrow'_1\uparrow'_2}(1,2) & W^{\uparrow_1\uparrow_2\downarrow'_1\downarrow'_2}(1,2) \\ W^{\uparrow_1\downarrow_2\uparrow'_1\uparrow'_2}(1,2) & W^{\uparrow_1\downarrow_2\uparrow'_1\downarrow'_2}(1,2) & W^{\uparrow_1\downarrow_2\downarrow'_1\uparrow'_2}(1,2) & W^{\uparrow_1\downarrow_2\downarrow'_1\downarrow'_2}(1,2) \\ W^{\downarrow_1\uparrow_2\uparrow'_1\uparrow'_2}(1,2) & W^{\downarrow_1\uparrow_2\uparrow'_1\downarrow'_2}(1,2) & W^{\downarrow_1\uparrow_2\downarrow'_1\uparrow'_2}(1,2) & W^{\downarrow_1\uparrow_2\downarrow'_1\downarrow'_2}(1,2) \\ W^{\downarrow_1\downarrow_2\uparrow'_1\uparrow'_2}(1,2) & W^{\downarrow_1\downarrow_2\uparrow'_1\downarrow'_2}(1,2) & W^{\downarrow_1\downarrow_2\downarrow'_1\uparrow'_2}(1,2) & W^{\downarrow_1\uparrow_2\downarrow'_1\downarrow'_2}(1,2) \end{pmatrix}.$$

# Spin Wigner functions

Wigner himself criticized the redundancy in the last formula, in one of his last papers. Now, since the 1-body function has a scalar and a vector part, simply on the basis of:

$$\big([\mathbf{1}]\oplus[\mathbf{3}]\big)^{\otimes 2} = 2[\mathbf{1}]\oplus 3[\mathbf{3}]\oplus[\mathbf{5}];$$

we see that the Wigner function multiplet has only six components, or strata under rotations, and under exchange of $\boldsymbol{v}$ and $\boldsymbol{p}$ the correct signs are, going from scalar to quadrupole, respectively: $(+,-,-,-,+,+)$ – I omit the details.

So there is still life in the old subject of Wigner functions, nowadays being required by quantum chemistry applications. . .

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