Double parton distributions

Evolution, initial conditions and transverse momentum dependence

Anna Stasto

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Outline

- Evolution equations for double Parton Distribution Functions (dPDFs)
- Sum rules
- Initial conditions
- Examples for the single channel: gluons
- Transverse momentum dependence in dPDFs

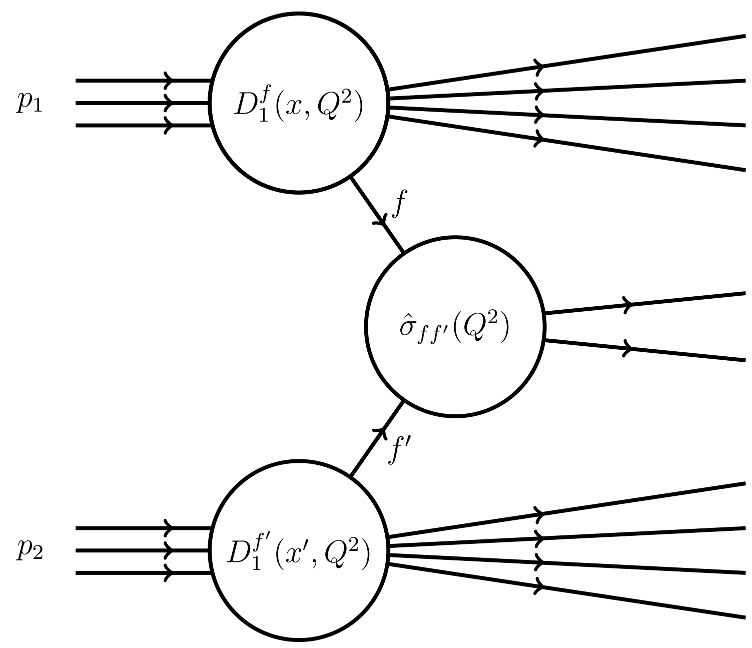
References:

Initial conditions: Golec-Biernat, Lewandowska, Serino, Snyder, AS; 1507.08583

Unintegrated dPDFs: Golec-Biernat, AS; 1611.02033, 1801.00018

Single scattering process

Single parton scattering: one hard process



Single collinear PDF:

$$D_1^f(x,Q^2)$$

Partonic cross section:

$$\hat{\sigma}^{ff'}(x_1,x_2,Q^2)$$

Collins, Soper, Sterman

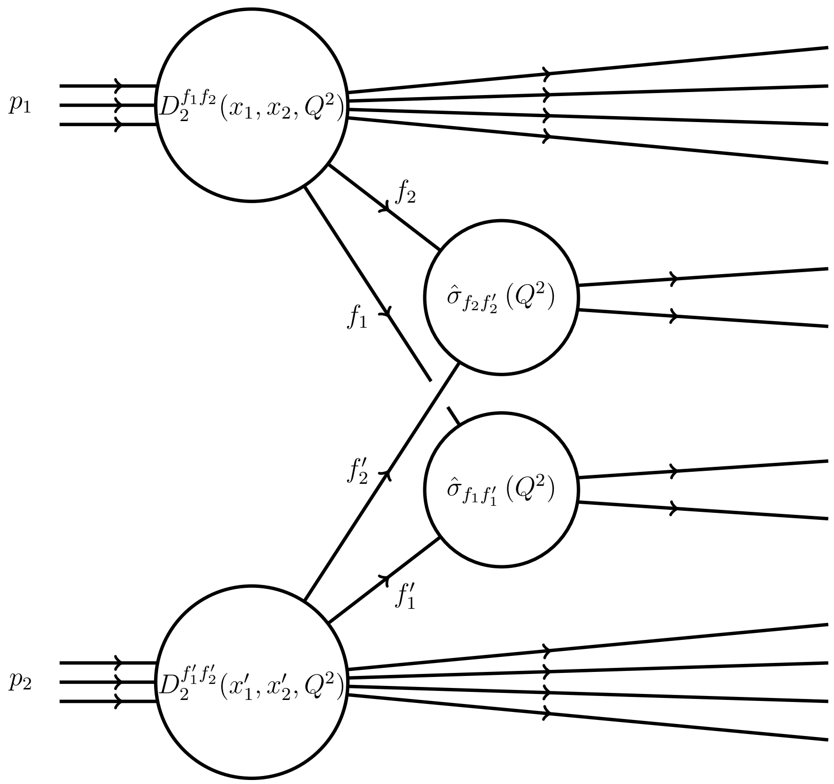
Collinear factorization:

Given the presence of the hard scale, the cross section (up to power corrections) can be factorized into perturbatively calculable partonic cross section and non-perturbative parton distribution functions.

$$\sigma = \sum_{ff'} \int dx_1 dx_2 D_1^f(x_1,Q^2) \hat{\sigma}^{ff'}(x_1,x_2,Q^2) D_1^{f'}(x_2,Q^2)$$

Double scattering process

Double parton scattering: two hard processes



Two types of partons: f_1, f_2

Two momentum fractions: x_1, x_2 $\quad x_1 + x_2 \leq 1$

Two hard scales: $Q_1, Q_2 \gg \Lambda_{QCD}$

Relative transverse momentum: Δ

Double PDF (DPDF): $D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; \Delta)$

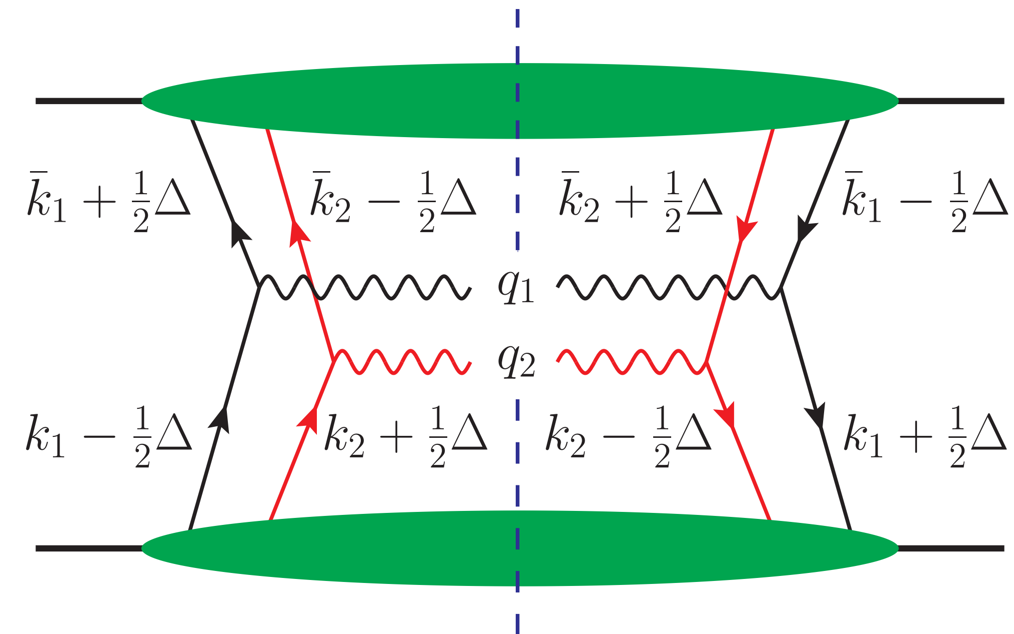
Factorization formula(?):

$$\sigma = \int dx_1 dx_2 dx_1' dx_2' d^2\Delta D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; \Delta) \hat{\sigma}^{f_1 f_1'}(\hat{s}_1, Q_1^2) \hat{\sigma}^{f_2 f_2'}(\hat{s}_2, Q_2^2) D_2^{f_1' f_2'}(x_1', x_2', Q_1^2, Q_2^2; -\Delta)$$

Diehl,Gaunt,Ostermeier,Ploessl,Schaefer Important steps towards the proof in double Drell-Yan process.

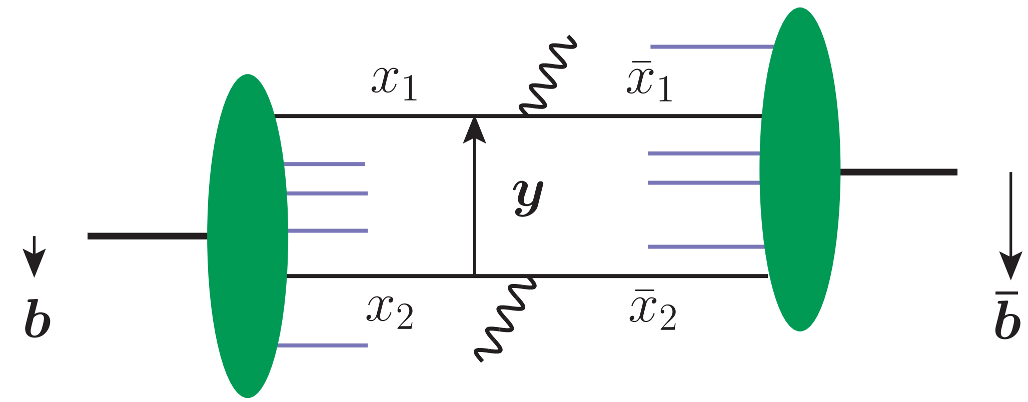
Double scattering process

The loop momentum comes from the mismatch between the momenta in amplitude and c.c. amplitude in a graph.



In coordinate representation: Fourier conjugate variable

In collinear approximation it is the distance between the active partons in the scattering



$$\sigma = \int dx_1 dx_2 dx_1' dx_2' d^2y d^2b d^2\bar{b}\ \tilde{D}_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; y, b)\hat{\sigma}^{f_1 f_1'}(\hat{s}_1, Q_1^2)\hat{\sigma}^{f_2 f_2'}(\hat{s}_2, Q_2^2)\tilde{D}_2^{f_1' f_2'}(x_1', x_2', Q_1^2, Q_2^2; y, \bar{b})$$

Evolution equations for single PDFs

DGLAP evolution equation for single PDF:

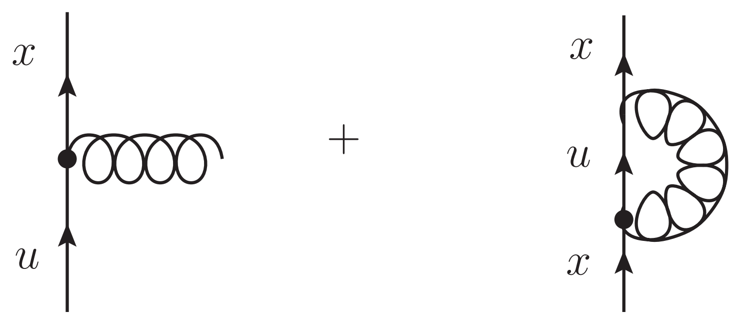
$$\boxed{\partial_t D_f(x,t) = \sum_{f'} \int_0^1 du\, \mathcal{K}_{ff'}(x,u,t)\, D_{f'}(u,t)}$$

Evolution variable:

$$t = \ln Q^2/Q_0^2$$

Real and virtual parts of the kernel:

$$\mathcal{K}_{ff'}(x,u,t) = \mathcal{K}^R_{ff'}(x,u,t) - \delta(u-x)\,\delta_{ff'}\,\mathcal{K}^V_f(x,t)$$



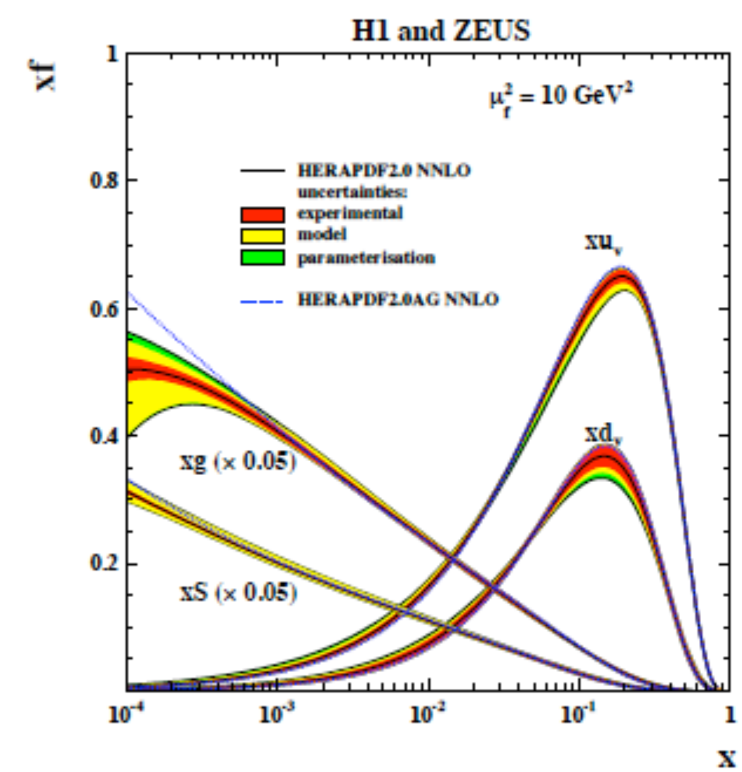
Real emission kernel:

$$\mathcal{K}^R_{ff'}(x,u,t) = \frac{1}{u} P_{ff'}(\frac{x}{u},t)\,\theta(u-x)$$

Splitting functions:

$$P_{ff'}(z,t) = \frac{\alpha_s(t)}{2\pi} P^{(0)}_{ff'}(z) + \frac{\alpha_s^2(t)}{(2\pi)^2} P^{(1)}_{ff'}(z) + \ldots.$$

Fits and extraction

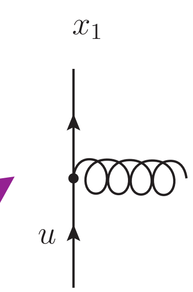


Flexible initial conditions, constrained by the momentum and sum rule only.

Evolution equations for double PDFs

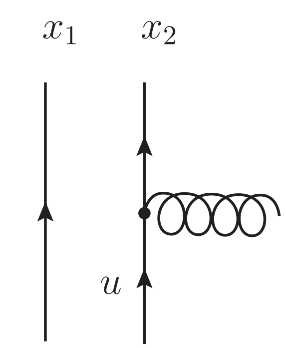
Evolution equations for dPDFs are known for the case of equal scales and momentum transfer zero: $D_2^{f_1 f_2}(x_1, x_2; Q^2, Q^2; \Delta = 0)$ (integrated over the transverse coordinate space)

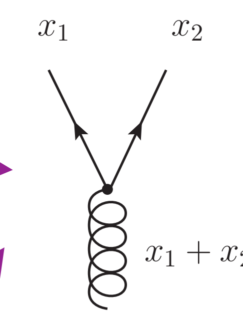
In (leading logarithmic approximation) they correspond to the inclusive probability of finding two partons in a hadron with longitudinal momenta x_1 and x_2



DGLAP evolution equation for double PDF:

$$\begin{aligned}
\partial_t D_{f_1 f_2}(x_1, x_2, t) &= \sum_{f'} \int_0^{1-x_2} du\, \mathcal{K}_{f_1 f'}(x_1, u, t)\, D_{f' f_2}(u, x_2, t) \\
&+ \sum_{f'} \int_0^{1-x_1} du\, \mathcal{K}_{f_2 f'}(x_2, u, t)\, D_{f_1 f'}(x_1, u, t) \\
&+ \sum_{f'} \mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t)\, D_{f'}(x_1 + x_2, t)
\end{aligned}$$





Konishi, Ukawa, Veneziano; Snigirev, Zinovev, Shelest

Non-homogeneous term
Splitting term of one parton into two:

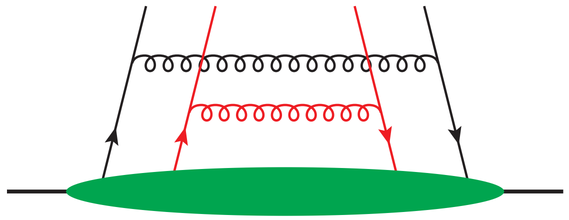
$$\mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t) = \frac{\alpha_s(t)}{2\pi} \frac{1}{x_1 + x_2} P^{(0)}_{f' f_1}\left(\frac{x_1}{x_1 + x_2}\right)$$

Evolution equations for double PDFs

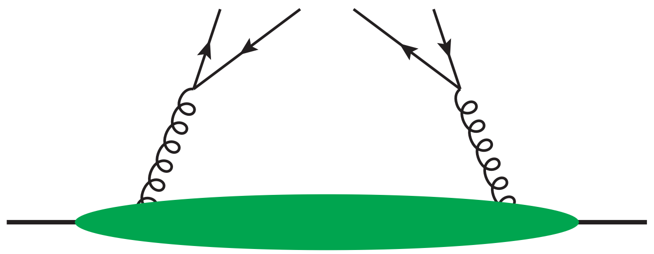
DGLAP evolution equation for double PDF:

$$\partial_t D_{f_1 f_2}(x_1, x_2, t) = \sum_{f'} \int_0^{1-x_2} du\, \mathcal{K}_{f_1 f'}(x_1, u, t)\, D_{f' f_2}(u, x_2, t)$$
$$+ \sum_{f'} \int_0^{1-x_1} du\, \mathcal{K}_{f_2 f'}(x_2, u, t)\, D_{f_1 f'}(x_1, u, t)$$
$$+ \sum_{f'} \mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t)\, D_{f'}(x_1 + x_2, t),$$

Homogeneous term



Non-homogeneous (splitting) term



DGLAP evolution equation for single PDF:

$$\partial_t D_f(x, t) = \sum_{f'} \int_0^1 du\, \mathcal{K}_{f f'}(x, u, t)\, D_{f'}(u, t)$$

Need to be solved together with suitable initial conditions.

Sum rules for single and double PDFs

Momentum sum rule for single PDFs

$$\sum_f \int_0^1 dx\, x D_f(x,t) = 1$$

Quark number sum rule for single PDFs

$$\int_0^1 dx\, \{D_{q_i}(x,t) - D_{\bar{q}_i}(x,t)\} = N_i$$

Momentum sum rule for double PDFs

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 \frac{D_{f_1 f_2}(x_1,x_2,t)}{D_{f_2}(x_2,t)} = 1 - x_2$$

Conditional probability to find the parton f_1 with the momentum fraction x_1 while keeping fixed the second parton f_2 with momentum x_2 .

Valence quark number sum rule for double PDFs

$$\int_0^{1-x_2} dx_1 \{D_{q_i f_2}(x_1,x_2,t) - D_{\bar{q}_i f_2}(x_1,x_2,t)\}$$
$$= \begin{cases} N_i\, D_{f_2}(x_2,t) & \text{for } f_2 \neq q_i, \bar{q}_i \\ (N_i - 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = q_i \\ (N_i + 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = \bar{q}_i \end{cases}$$

If sum rules hold for initial conditions they will hold for higher scales after the evolution.

How to consistently impose the initial conditions for sPDF and dPDF with sum rules?

Problem of initial conditions in dPDFs

Usually simplifying assumption is taken:

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2)$$

Factorizable ansatz, could work well for rather small x but is inconsistent with sum rules.

Improvement with correlating factor:

Gaunt, Stirling

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2) \frac{(1 - x_1 - x_2)^2}{(1 - x_1)^{2+n_1} (1 - x_2)^{2+n_2}}$$

Takes into account some correlation but still does not obey sum rules exactly.

Initial conditions: Dirichlet distribution

Consider Beta distribution and gluons only (for now)

$$D(x) = N_1 \, x^{-\alpha} \, (1-x)^{\beta}$$

Mellin transform:

$$\tilde{D}(n) = \int_0^1 dx x^{n-1} D(x)$$

Momentum sum rule in Mellin space:

$$\tilde{D}(2) = 1$$

$$\tilde{D}(n) \;=\; \frac{1}{B(2-\alpha, 1+\beta)} \int_0^1 dx \, x^{n-1} x^{-\alpha} (1-x)^{\beta} = \frac{B(n-\alpha, \beta+1)}{B(2-\alpha, \beta+1)}$$

Take the ansatz for double distribution in the form of the Dirichlet distribution:

$$D(x_1, x_2) = N_2 \, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}$$

Double Mellin transform:

$$\tilde{D}(n_1, n_2) = \int_0^1 dx_1 x_1^{n_1 - 1} \int_0^1 dx_2 x_2^{n_2 - 1} D(x_1, x_2) \longrightarrow \tilde{D}(n_1, n_2) = N_2 \frac{\Gamma(n_1 - \tilde{\alpha})\Gamma(n_2 - \tilde{\alpha})\Gamma(1 + \tilde{\beta})}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta} - 2\tilde{\alpha})}$$

Initial conditions: relating the parameters

The momentum sum rule for dPDFs in Mellin space

LHS: Double PDFs in Mellin space →

$$\boxed{\begin{aligned}\tilde{D}(n_1,2) &= \tilde{D}(n_1) - \tilde{D}(n_1+1)\\ \tilde{D}(2,n_2) &= \tilde{D}(n_2) - \tilde{D}(n_2+1)\end{aligned}}$$

← RHS: Single PDFs in Mellin space

RHS:

$$\tilde{D}(n_1) - \tilde{D}(n_1+1) = \frac{1}{B(2-\alpha,\beta+1)}\big(B(n_1-\alpha,\beta+1) - B(n_1+1-\alpha,\beta+1)\big) = \frac{1}{B(2-\alpha,\beta+1)}\frac{\Gamma(n_1-\alpha)\Gamma(2+\beta)}{\Gamma(2+\beta+n_1-\alpha)}$$

Where the following property of Beta function was used:

$$B(a,b) = B(a+1,b) + B(a,b+1)$$

LHS:

$$\tilde{D}(n_1,2) = N_2\frac{\Gamma(n_1-\tilde{\alpha})\Gamma(2-\tilde{\alpha})\Gamma(1+\tilde{\beta})}{\Gamma(n_1+3+\tilde{\beta}-2\tilde{\alpha})}$$

Comparing the functional form of both sides we see that the equality can be satisfied if

$$\boxed{\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1}$$

and

$$N_2 = \frac{1}{B(2-\alpha,\alpha+\beta)B(2-\alpha,\beta+1)}$$

Initial conditions

If the single distribution is given by a Beta distribution

$$D(x) = N_1 \, x^{-\alpha} \, (1-x)^{\beta}$$

There is a unique solution in terms of the Dirichlet distribution for the double parton density:

$$D(x_1, x_2) = N_2 \, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}$$

With powers of the dPDF being related to the powers of sPDF

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1$$

Normalization for dPDF in this particular case is uniquely determined.

Small x powers for single and double PDFs are the same.

The large x power of the correlating factor in dPDF is related to the sum of large and small x powers of the single distribution.

Initial conditions: expansion

Realistic parametrizations are however more complicated than a single Beta distribution.

Example MSTW2008 gluon PDF: $$xD_1^g\left(x,Q^2\right) = N_1 x^{-\delta_g}\,(1-x)^{\eta_g}\left(1+\epsilon_g\sqrt{x}+\gamma_g x\right),$$

However, this parametrization is sum of Beta distributions of the form:

$$\boxed{D(x) = N_1 \sum_{k=1}^{K} a_k x^{-\alpha_k}\,(1-x)^{\beta_k}}$$

Assuming that the dPDF is the sum of Dirichlet distributions:

$$\boxed{D(x_1,x_2) = N_2 \sum_{k=1}^{K} c_k x_1^{-\tilde{\alpha}_k} x_2^{-\tilde{\alpha}_k}\,(1-x_1-x_2)^{\tilde{\beta}_k}}$$

Performing the same analysis as before (for single channel) one obtains the conditions for each k:

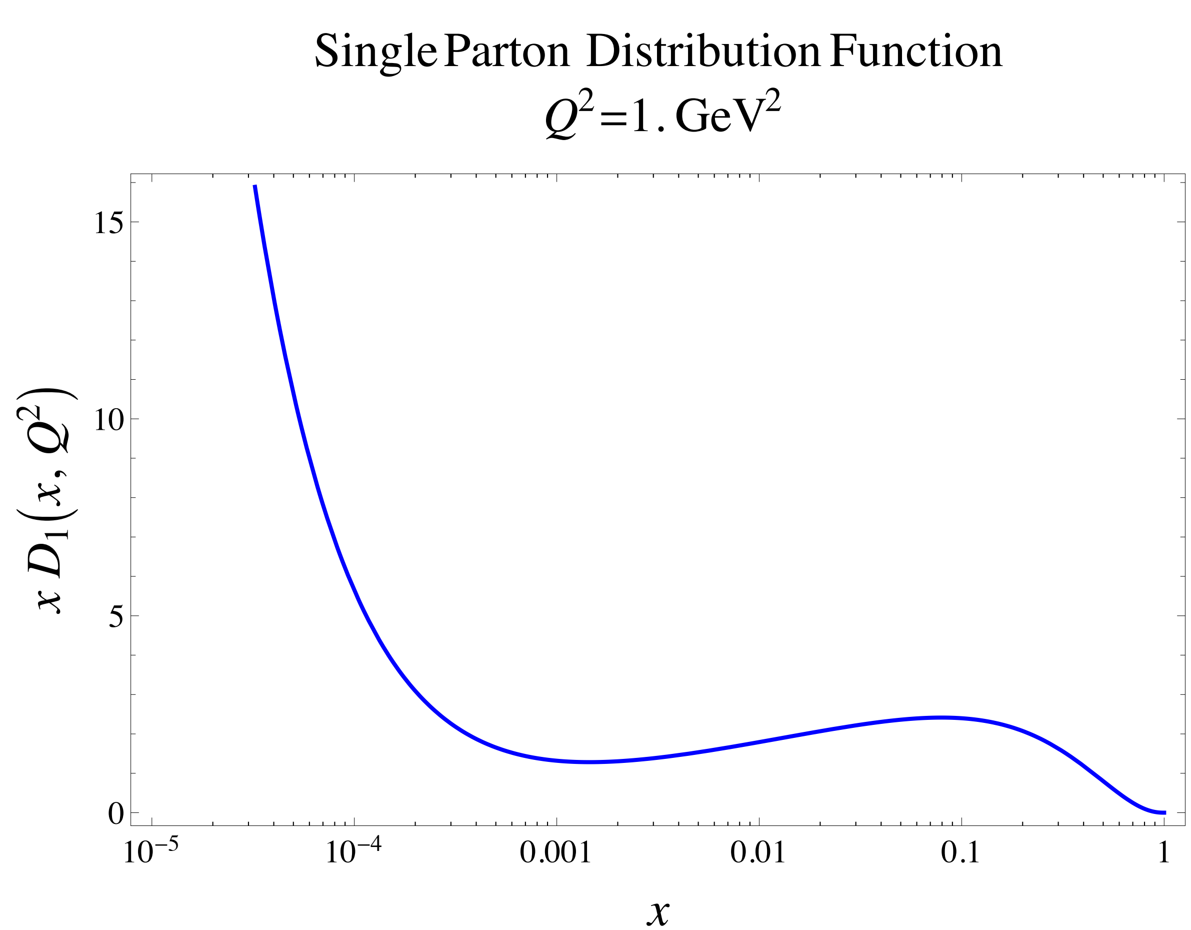
$$\boxed{\tilde{\alpha}_k = \alpha_k} \qquad \boxed{\tilde{\beta}_k = \beta_k - 1 + \alpha_k}$$

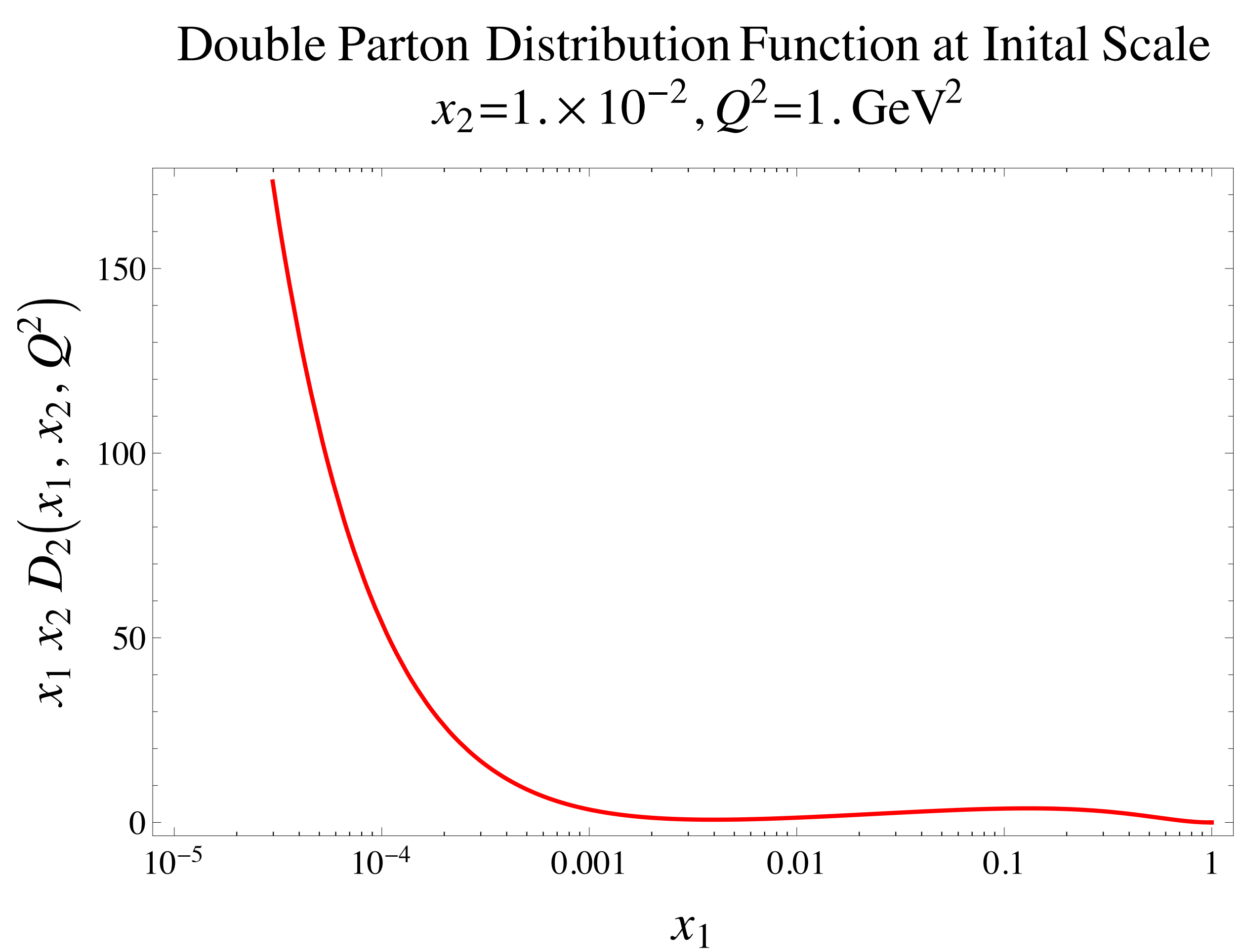
The normalizations:

$$c_k = a_k \frac{B(\alpha_1+\beta_1, 2-\alpha_1)}{B(\beta_k+\alpha_k, 2-\alpha_k)} \qquad N_2 = N_1 \frac{1}{B(\alpha_1+\beta_1, 2-\alpha_1)}$$

Initial conditions for dPDFs

- Use this algorithm, expansion in terms of Beta and Dirichlet distributions, to construct dPDF from MSTW2008 gluon.
- Single channel (gluons) only.
- Using different normalization for the LO MSTW2008 gluon.



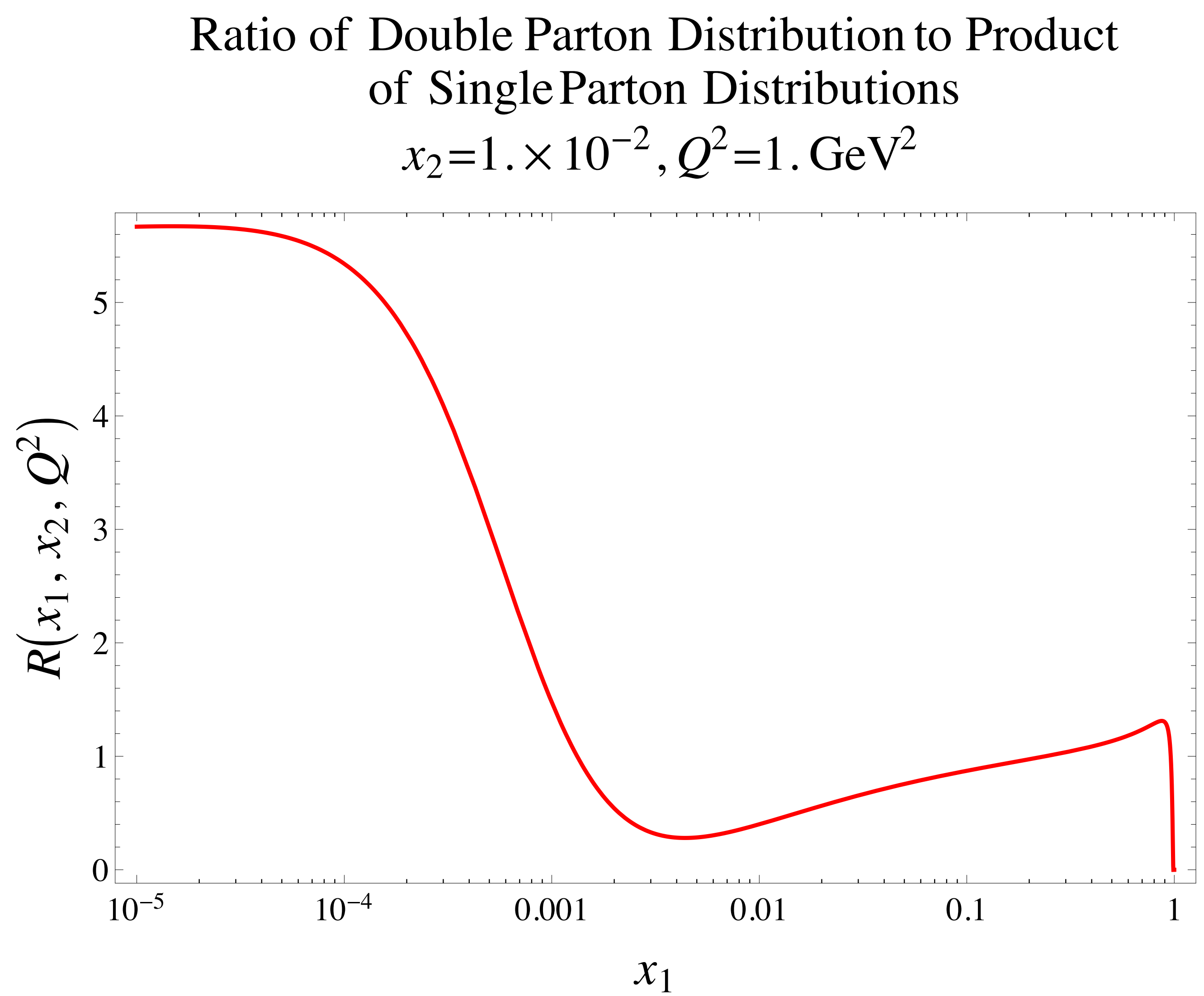


Initial conditions for dPDFs: ratios

Ratio of double distribution to product of single distributions:

$$R^{gg}\left(x_1, x_2, Q^2\right) = \frac{D_2^{gg}\left(x_1, x_2, Q^2\right)}{D_1^g\left(x_1, Q^2\right) D_1^g\left(x_2, Q^2\right)}$$

- Measure of the correlations at the initial scale.
- For this parametrization the correlations are very significant.
- Ratio different from unity over wide range of x.
- Factorization of powers at small x but different normalization.
- In principle can extend to quarks, requires some constraints put onto the form of the single PDFs.

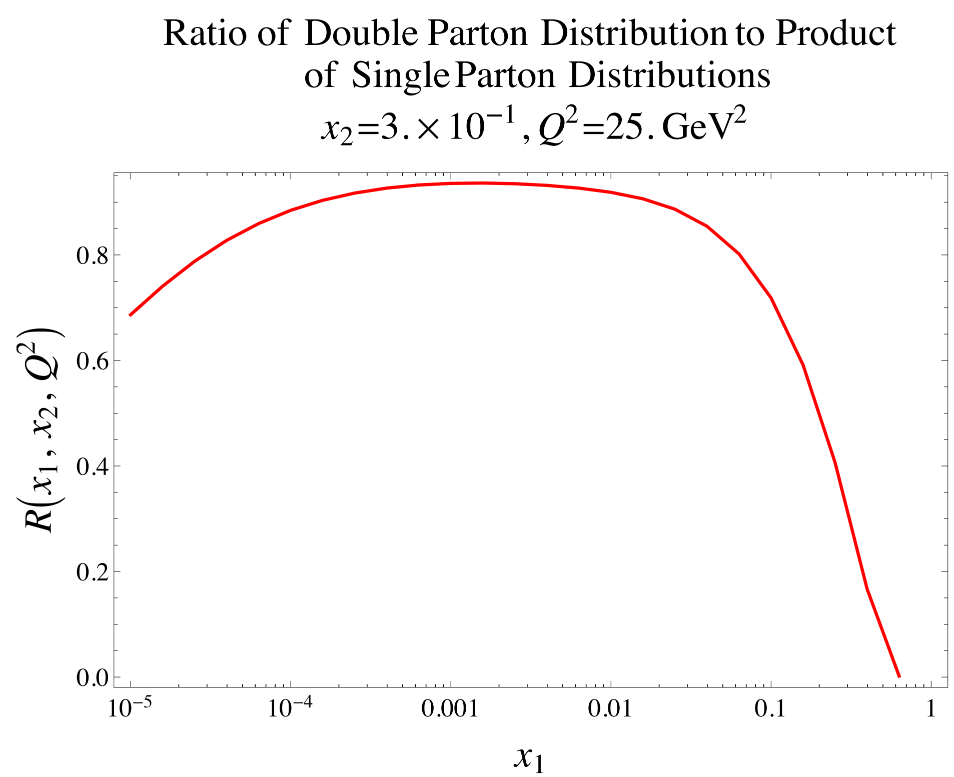


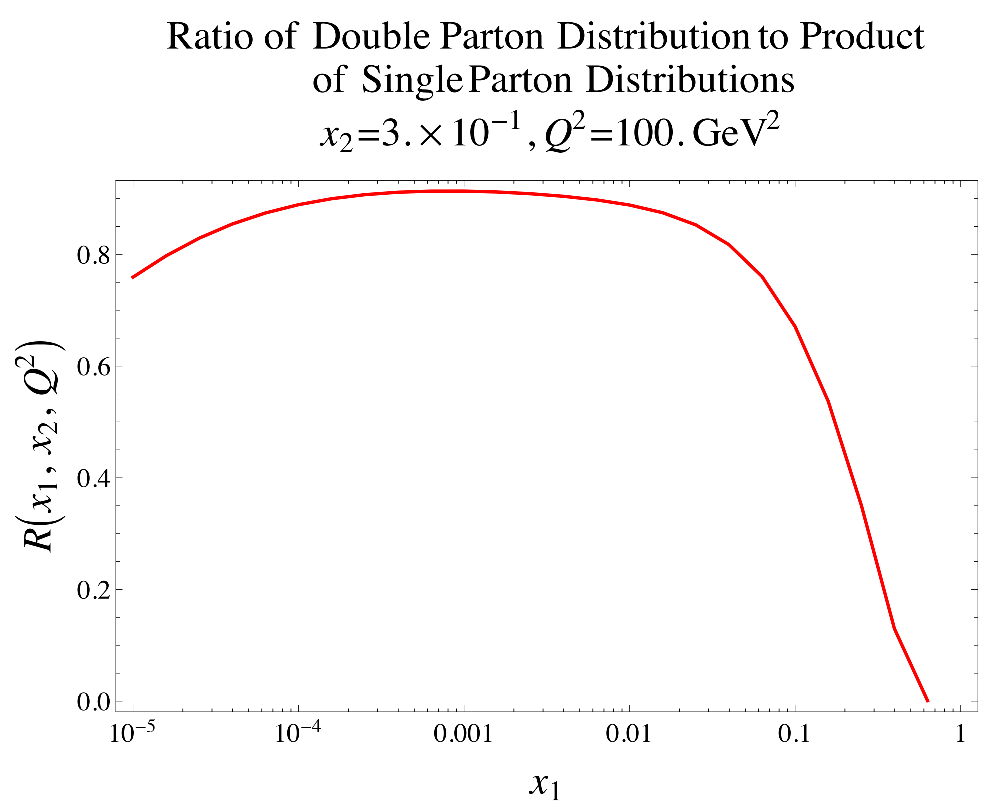
Evolution of single and double PDFs

Evolve the dPDFs and sPDFs using DGLAP equations;

$$D_1^f(x, Q_0) \to D_1^f(x, Q)$$

$$D_2^{f_1 f_2}(x_1, x_2, Q_0) \to D_2^{f_1 f_2}(x_1, x_2, Q)$$





Correlation washed out by evolution except for large x.

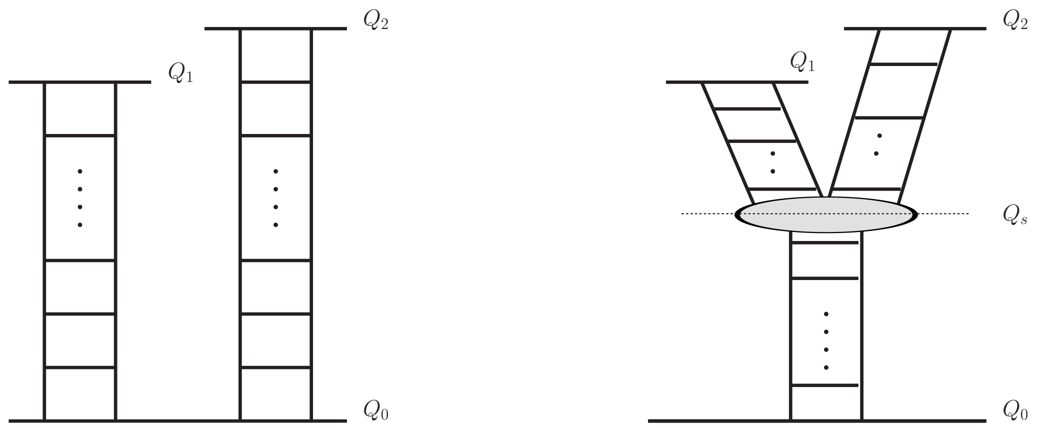
More on Factorization of dPDFs into sPDFs

How good is factorization of the dPDFs into product of sPDFs

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2) \quad ?$$

The solution to the evolution equations (generally for unequal scales) can be written as a sum of homogenous and inhomogenous terms

$$D_{f_1 f_2} = D^{(h)}_{f_1 f_2} + D^{(nh)}_{f_1 f_2}$$



$$D_{f_1 f_2} = \sum_{f',f''} \left\{ \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} E_{f_1 f'}\left(\frac{x_1}{z_1}, Q_1, Q_0\right) E_{f_2 f''}\left(\frac{x_2}{z_2}, Q_2, Q_0\right) \times D_{f' f''}(z_1, z_2, Q_0, Q_0) \right.$$

$$\left. + \int_{Q_0^2}^{Q^2_{\min}} \frac{dQ_s^2}{Q_s^2} \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} E_{f_1 f'}\left(\frac{x_1}{z_1}, Q_1, Q_s\right) E_{f_2 f''}\left(\frac{x_2}{z_2}, Q_2, Q_s\right) D^{(sp)}_{f' f''}(z_1, z_2, Q_s) \right\}$$

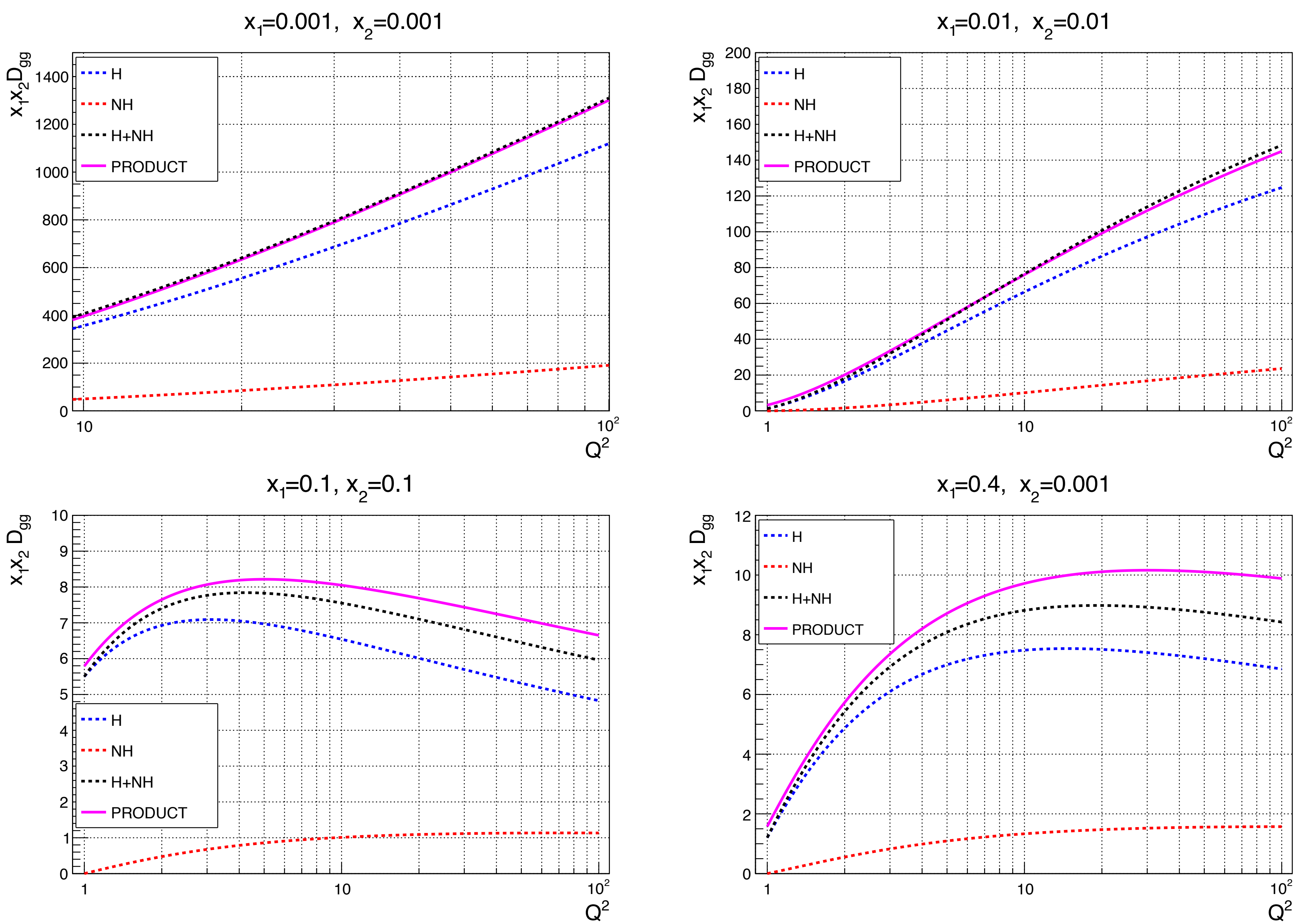
Green's functions for DGLAP evolution:

$$D_a(x,\mu) = \sum_b \int_x^1 \frac{dz}{z} E_{ab}\left(\frac{x}{z}, \mu, \mu_0\right) D_b(z, \mu_0)$$

Splitting/non-homogenous term

$$D^{(sp)}_{a'a''}(x_1, x_2, Q_s) = \frac{\alpha_s(Q_s)}{2\pi} \sum_a \frac{D_a(x_1+x_2, Q_s)}{x_1+x_2} P_{a\to a'a''}\left(\frac{x_1}{x_1+x_2}\right)$$

More on Factorization of dPDFs into sPDFs



Factorization works well for small x and large Q, provided momentum sum rule is implemented in the initial conditions.

Factorization holds for the sum of homogeneous and inhomogeneous terms.

Unintegrated DPDFs

What about the transverse momentum dependence of the DPDFs?

Possible formulation:

Small x Color Glass Condensate formalism: higher Wilson line correlators

TMD formulation *Buffing, Diehl, Kasemets*

Simple practical approach:

- *Kimber - Martin - Ryskin* approach to the unintegrated parton densities.
- Includes transverse momentum dependence in the parton densities.
- Practical approach for the phenomenology, using integrated densities, convoluted with the Sudakov form factors

Unintegrated PDFs

Martin, Kimber, Ryskin

DGLAP evolution for single PDF

$$\frac{\partial D_a(x,\mu)}{\partial \ln \mu^2} = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,\mu)\, D_{a'}\left(\frac{x}{z},\mu\right) - D_a(x,\mu) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,\mu)$$

real virtual

after integrating out the virtual part

$$D_a(x,Q) = T_a(Q,Q_0,)D_a(x,Q_0) + \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} f_a(x,k_\perp,Q)$$

where the "unintegrated density":

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,k_\perp)\, D_{a'}\left(\frac{x}{z},k_\perp\right)$$

or

$$f_a(x,k_\perp,Q) = \frac{\partial}{\partial \ln k_\perp^2} \left[T_a(Q,k_\perp) D_a(x,k_\perp)\right]$$

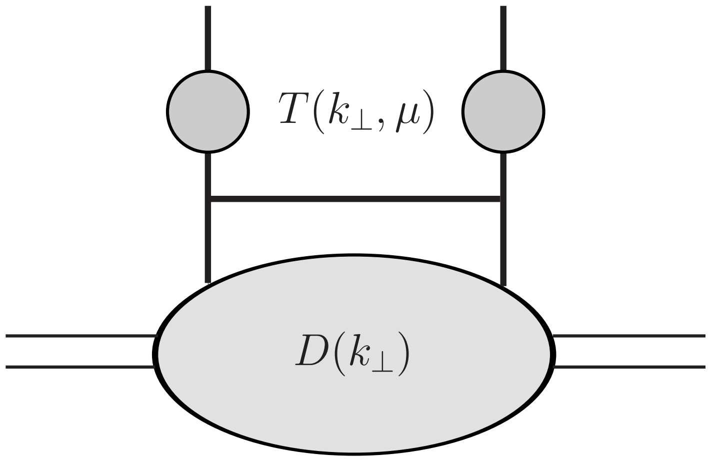
with Sudakov formfactor

$$T_a(Q,k_\perp) = \exp\left\{ - \int_{k_\perp^2}^{Q^2} \frac{dp_\perp^2}{p_\perp^2} \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,p_\perp) \right\}$$

$$T_a(Q,k_\perp) \simeq 1, \quad Q \sim k_\perp \qquad T_a(Q,k_\perp) \simeq 0, \quad Q \gg k_\perp$$

Unintegrated PDFs

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,k_\perp)\, D_{a'}\left(\frac{x}{z},k_\perp\right)$$



Dependence on two scales obtained in the last step of the evolution

Need to specify the cutoff :

<u>DGLAP ordering:</u>

$$\Delta = \frac{k_\perp}{Q}$$

<u>CCFM angular ordering:</u>

$$\Delta = \frac{k_\perp}{k_\perp + Q}$$

$$\Theta(\theta - \theta') \;\; \Rightarrow \;\; \mu \; > \; zk_t/(1-z) \qquad\qquad z_{max} \; = \; \frac{\mu}{\mu + k_t}$$

Larger phase space for emissions, tail in transverse momentum extends to $k_\perp > Q$

Extending the KMR framework to DPDFs

Use parton-to-parton evolution function:

(n is Mellin variable conjugated to x)

$$\tilde{D}_a(n,\mu) = \sum_b \tilde{E}_{ab}(n,\mu,\mu_0)\,\tilde{D}_b(n,\mu_0).$$

It evolves sPDF to scale μ from scale μ_0

$$\frac{\partial}{\partial \ln \mu^2}\tilde{E}_{ab}(n,\mu,\mu_0) = \sum_{a'} \tilde{P}_{aa'}(n,\mu)\,\tilde{E}_{a'b}(n,\mu,\mu_0) - \tilde{E}_{ab}(n,\mu,\mu_0)\sum_{a'}\int_0^1 dz z P_{a'a}(z,\mu)$$

initial condition

$\tilde{E}_{ab}(n,\mu_0,\mu_0) = \delta_{ab}.$

Formally integrating out virtual part:

$$\tilde{E}_{ab}(n,Q,Q_0) = T_a(Q,Q_0)\,\delta_{ab} + \int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}T_a(Q,k_\perp)\sum_{a'}\tilde{P}_{aa'}(n,k_\perp)\,\tilde{E}_{a'b}(n,k_\perp,Q_0)$$

Double parton distributions (DGLAP eq):

$$\tilde{D}_{a_1a_2}(n_1,n_2,\mu_1,\mu_2) = \sum_{a',a''}\Big\{\tilde{E}_{a_1a'}(n_1,\mu_1,\mu_0)\,\tilde{E}_{a_2a''}(n_2,\mu_2,\mu_0)\,\tilde{D}_{a'a''}(n_1,n_2,\mu_0,\mu_0)$$

homogenous term

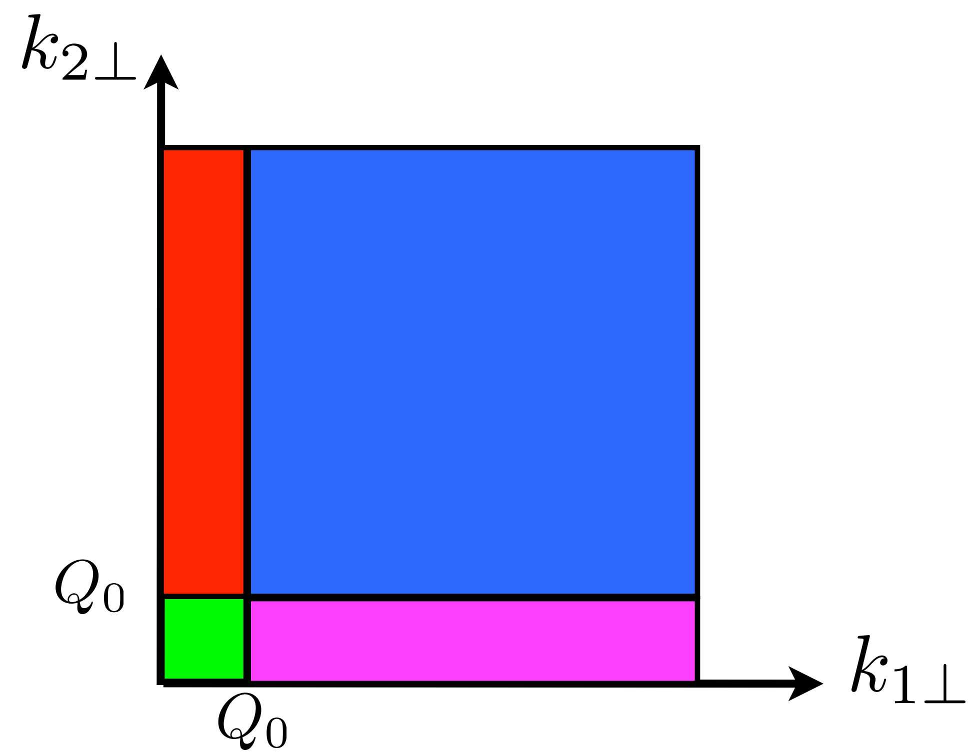
$$+\int_{\mu_0^2}^{\mu_{min}^2}\frac{d\mu_s^2}{\mu_s^2}\tilde{E}_{a_1a'}(n_1,\mu_1,\mu_s)\,\tilde{E}_{a_2a''}(n_2,\mu_2,\mu_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,\mu_s)\Big\}$$

inhomogenous term

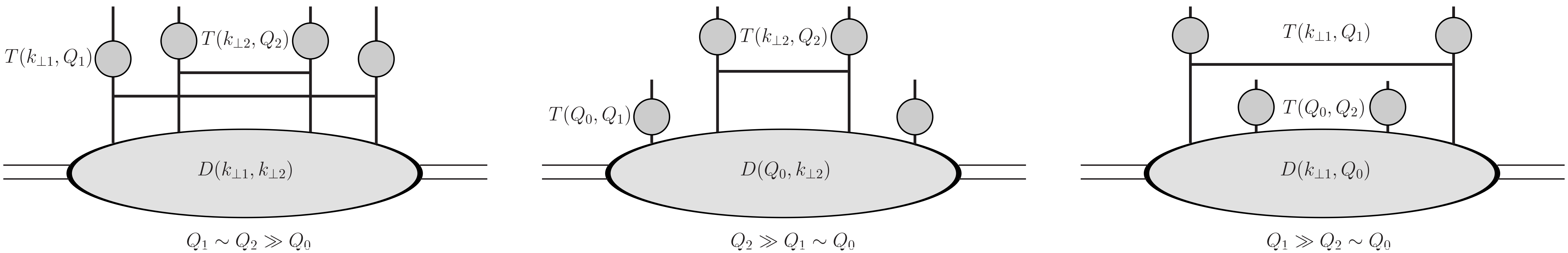
Homogeneous part of DPDF evolution

$$
\begin{aligned}
\tilde{D}^{(h)}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = {} & T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,Q_0)\,\tilde{D}_{a_1a_2}(n_1,n_2,Q_0,Q_0) \\
& + \int_{Q_0^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\left\{T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,k_{2\perp})\sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp})\left[\sum_{a''}\tilde{E}_{ba''}(n_2,k_{2\perp},Q_0)\tilde{D}_{a_1a''}(n_1,n_2,Q_0,Q_0)\right]\right\} \\
& + \int_{Q_0^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\left\{T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,Q_0)\sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp})\left[\sum_{a'}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_0)\tilde{D}_{a'a_2}(n_1,n_2,Q_0,Q_0)\right]\right\} \\
& + \int_{Q_0^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\int_{Q_0^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\left\{T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})\sum_{b,c}\tilde{P}_{a_1b}(n_1,k_{1\perp})\,\tilde{P}_{a_2c}(n_2,k_{2\perp})\left[\sum_{a',a''}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_0)\tilde{E}_{ca''}(n_2,k_{2\perp},Q_0)\tilde{D}_{a'a''}(n_1,n_2,Q_0,Q_0)\right]\right\}
\end{aligned}
$$

Four distinct regions of phase space depending on the ordering of scales.



Homogeneous part



$Q_1^2 \sim Q_2^2 \gg Q_0^2$ $\quad$ $k_{1\perp}$ unintegrated $\quad$ $k_{2\perp}$ unintegrated

$$f^{(h)}_{a_1 a_2}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, k_{2\perp}) \sum_{b,c} \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} \int_{\frac{x_2}{1-x_1/z_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_1 b}(z_1, k_{1\perp}) P_{a_2 c}(z_2, k_{2\perp})\, D^{(h)}_{bc}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, k_{1\perp}, k_{2\perp}\right)$$

$Q_1^2 \sim Q_0^2$ and $Q_2^2 \gg Q_0^2$ $\quad$ $k_{1\perp}$ integrated $\quad$ $k_{2\perp}$ unintegrated

$$f1^{(h)}_{a_1, a_2}(x_1, x_2, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, Q_0)\, T_{a_2}(Q_2, k_{2\perp}) \sum_{b} \int_{\frac{x_2}{1-x_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_2 b}(z_2, k_{2\perp})\, D^{(h)}_{a_1 b}\left(x_1, \frac{x_2}{z_2}, Q_0, k_{2\perp}\right)$$

$Q_1^2 \gg Q_0^2$ and $Q_2^2 \sim Q_0^2$ $\quad$ $k_{2\perp}$ integrated $\quad$ $k_{1\perp}$ unintegrated

$$f1^{(h)}_{a_1, a_2}(x_1, x_2, k_{1\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, Q_0) \sum_{b} \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} P_{a_1 b}(z_1, k_{1\perp})\, D^{(h)}_{b a_2}\left(\frac{x_1}{z_1}, x_2, k_{1\perp}, Q_0\right)$$

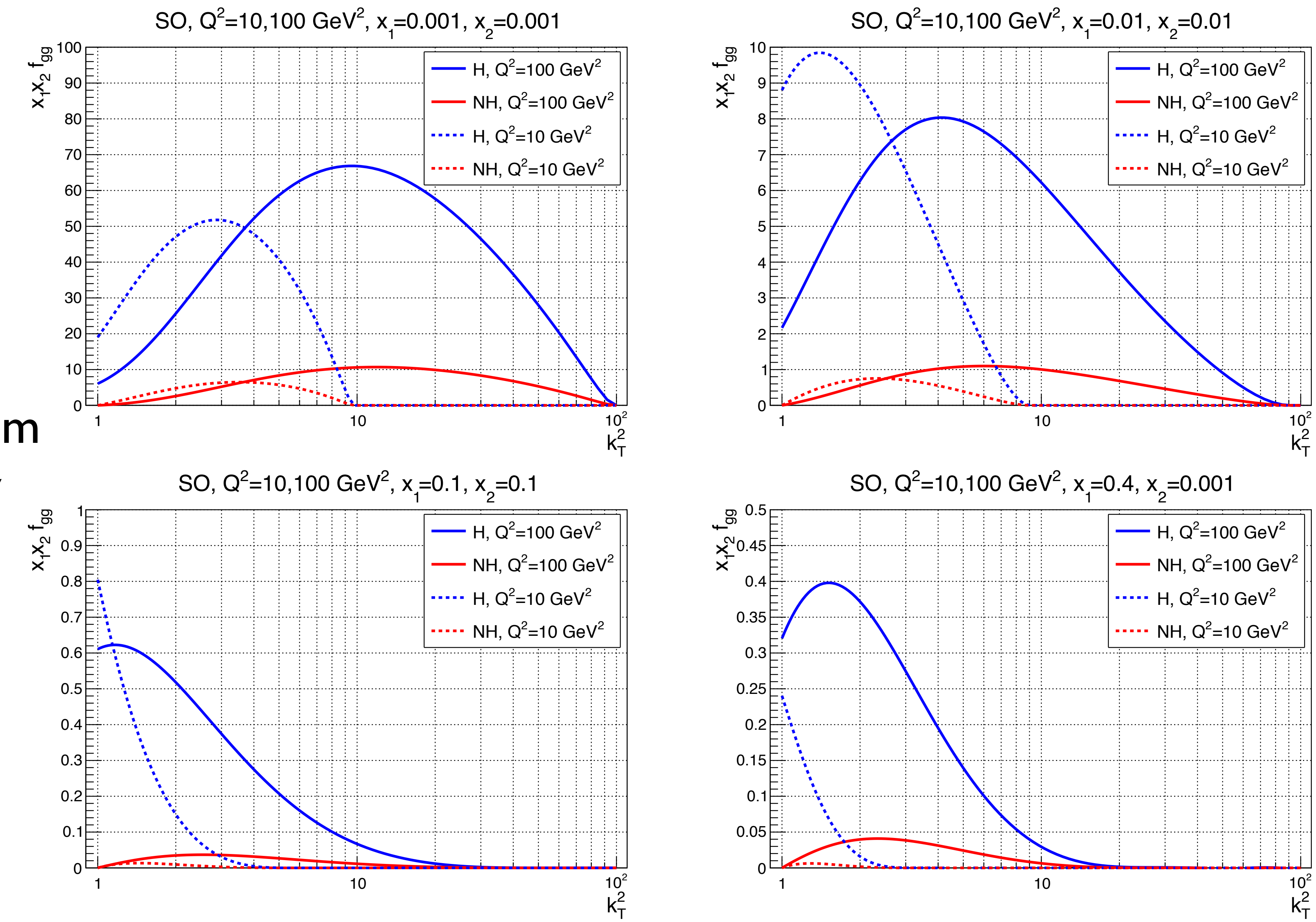
$Q_1^2 \sim Q_2^2 \sim Q_0^2$ $\quad$ non-perturbative region - parametrized by integrated density

Results for unintegrated dPDFs

The same method can be applied to the non-homogeneous term.

$$f_{gg} = f_{gg}^{(h)}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) + f_{gg}^{(nh)}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2)$$

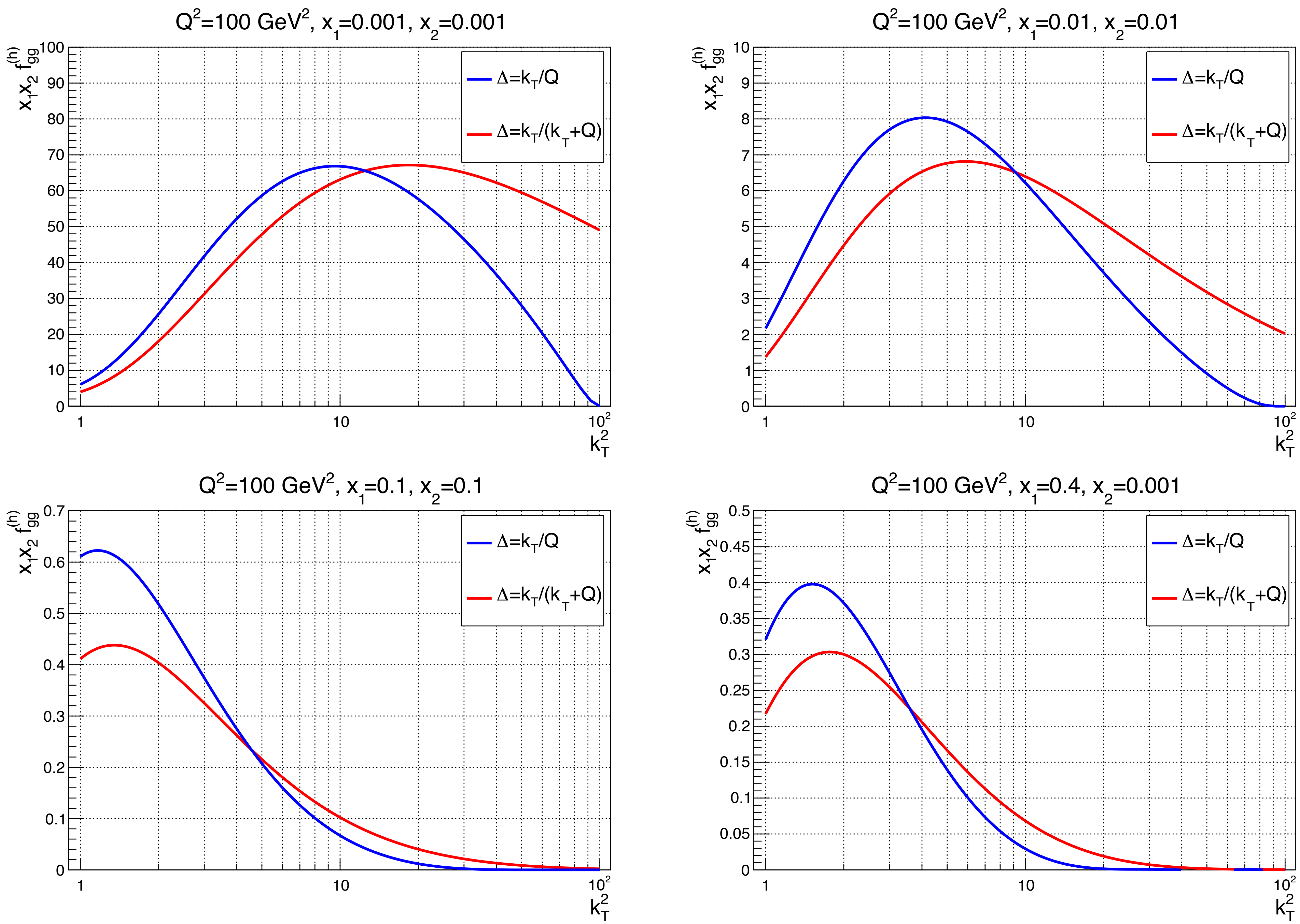
Showing contributions from perturbative region only



Non-homogenous term smaller than the homogenous one(that ratio could depend on the initial condition)

Higher Q, lower x, the maximum shifts to higher transverse momenta

Results for unintegrated dPDFs



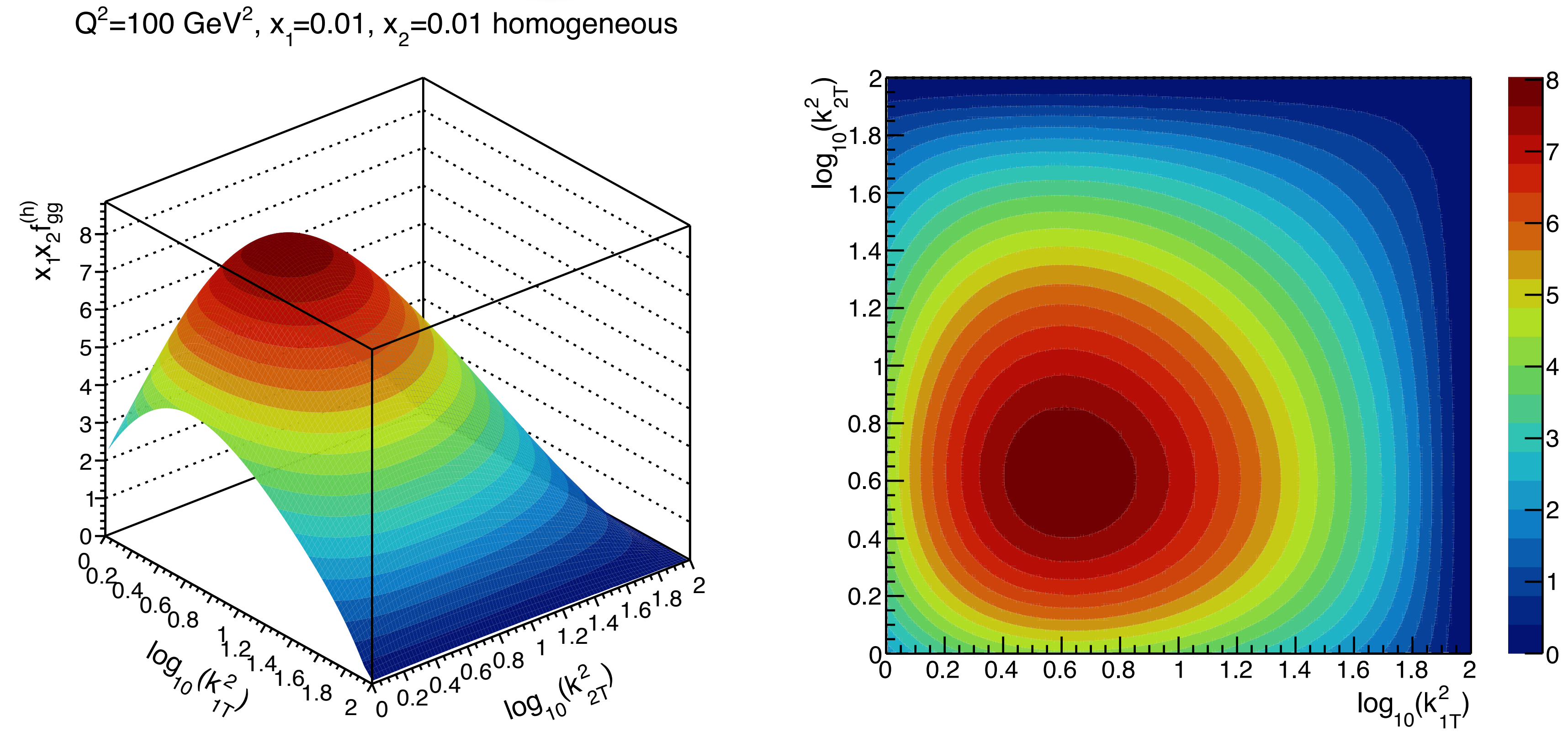
Effect of angular ordering cutoff: larger tails in kT

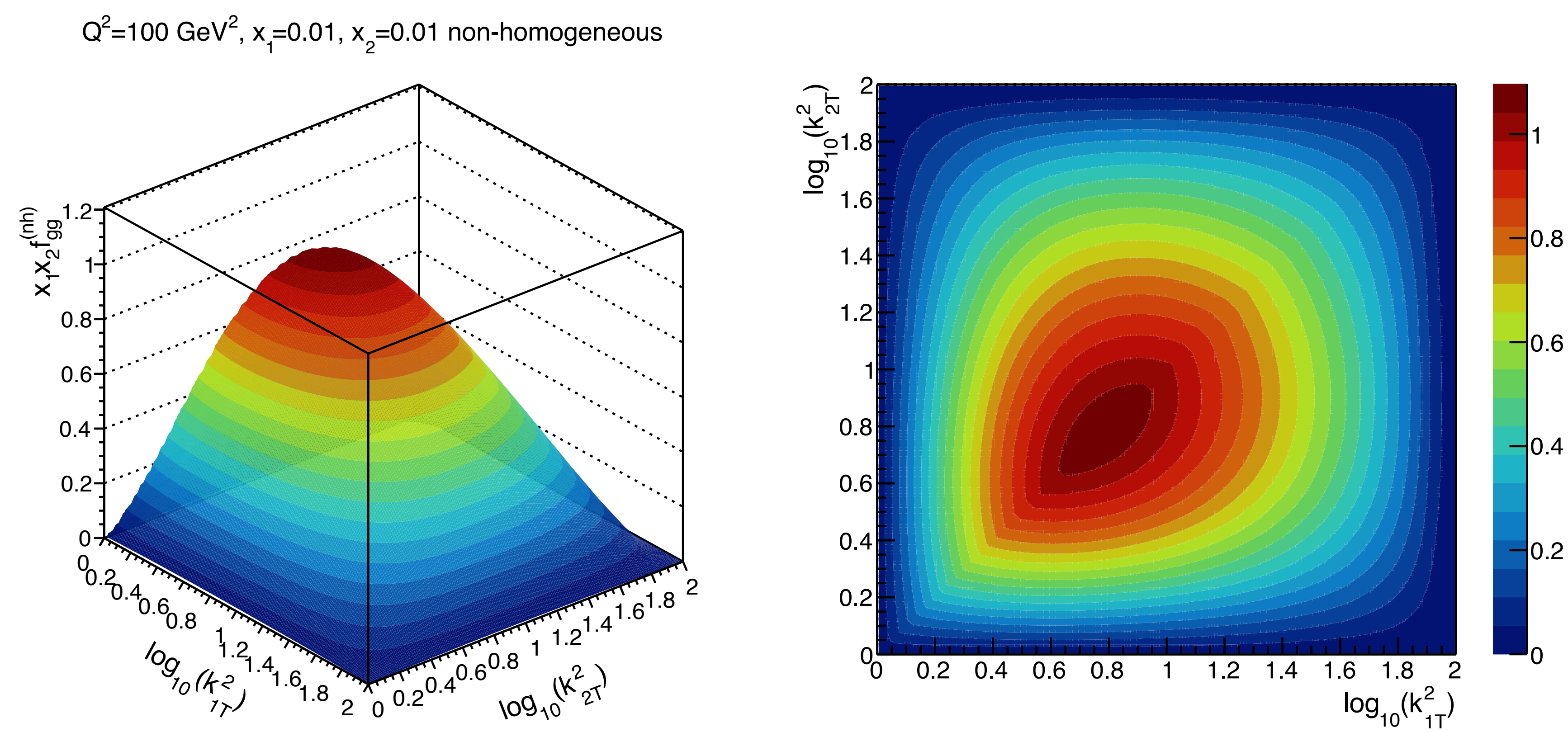
Results for unintegrated dPDFs

Fixed Q, fixed x1,x2

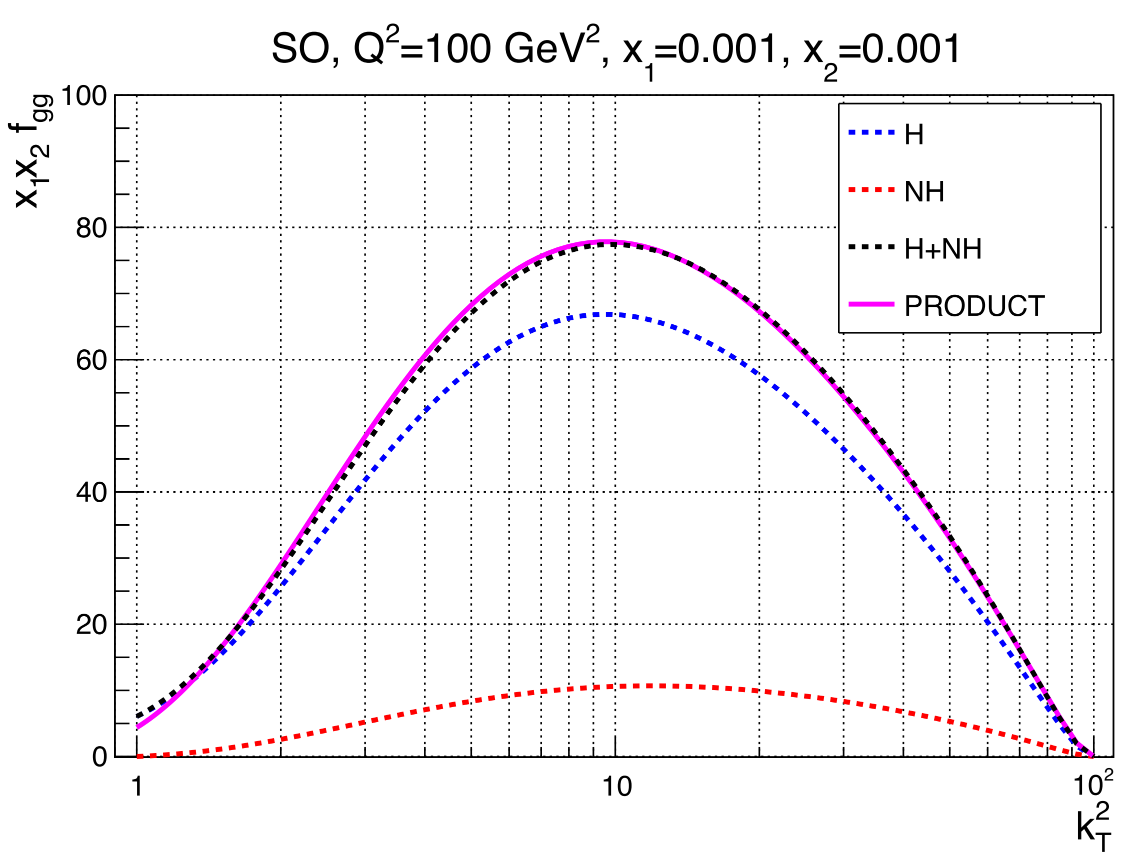
Non-homogeneous term is more perturbative, higher transverse momenta

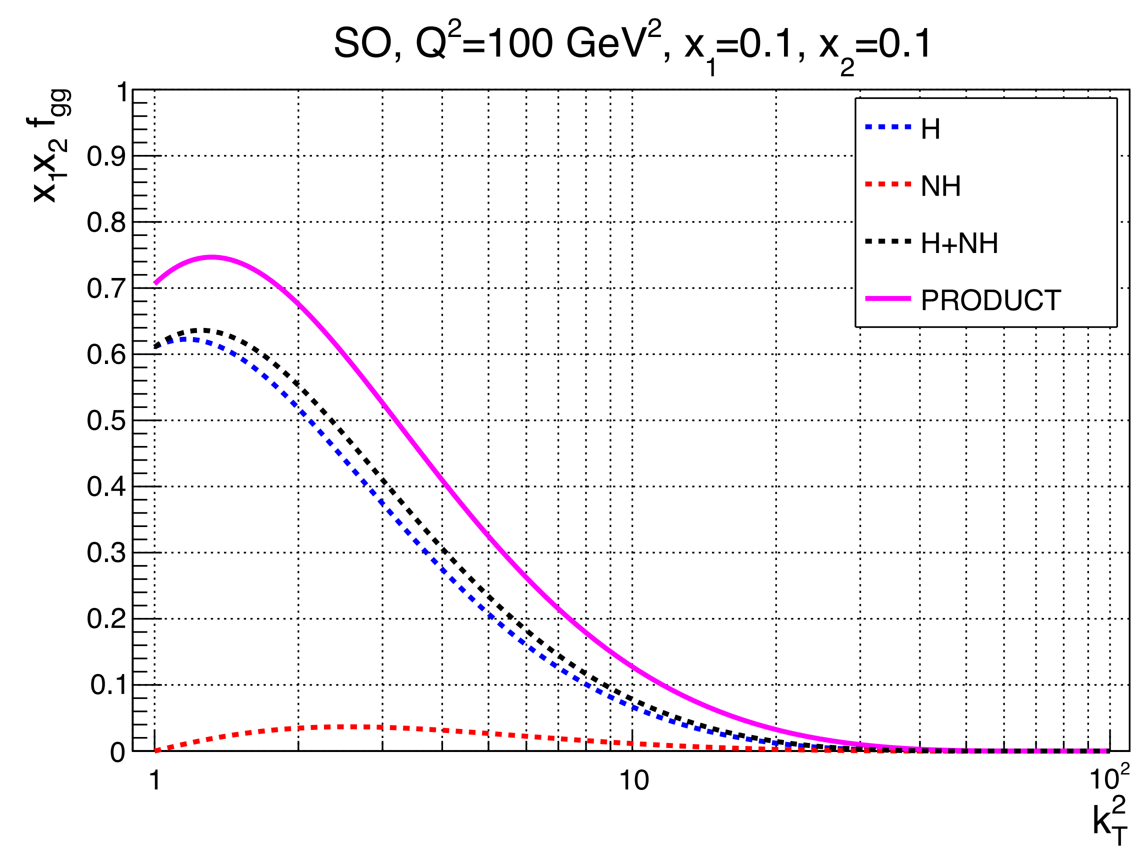
Smaller value for nonhomogeneous term.



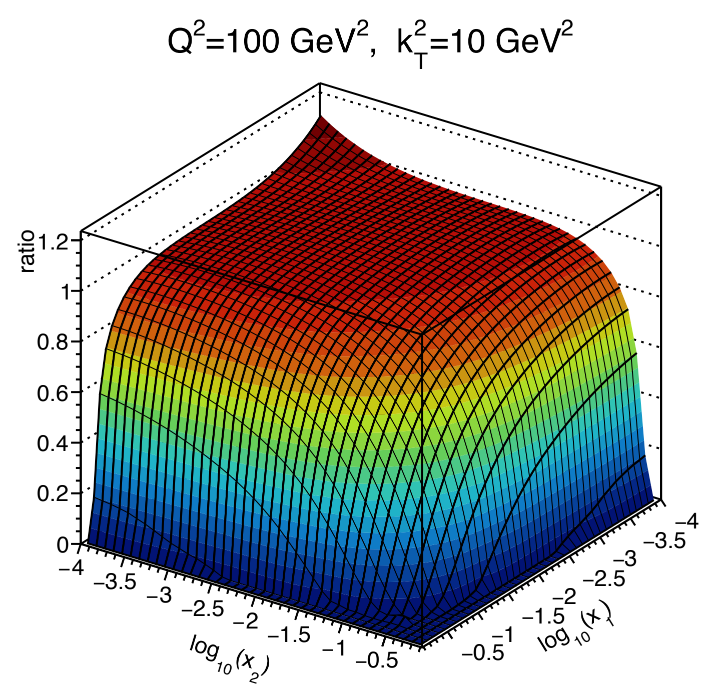


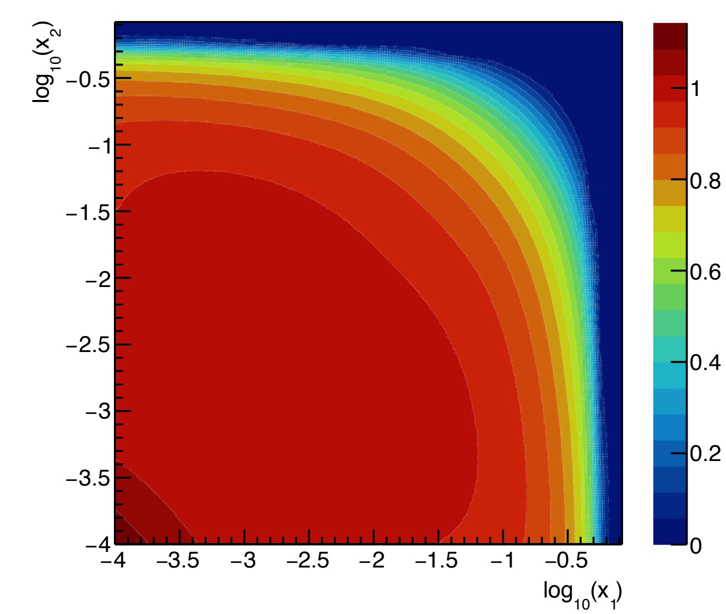
Factorization of unintegrated dPDFs





$$r = \frac{f_{gg}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q, Q)}{f_g(x_1, k_{1\perp}, Q) f_g(x_2, k_{2\perp}, Q)}$$





Summary and outlook

- Summary & outlook 1 - initial conditions:
  - Double integrated PDFs need consistent initial conditions for the evolution.
  - Beta functions for single PDF and Dirichlet distributions for double PDF with suitably matched powers and coefficients are good initial conditions. The momentum sum rule and quark number sum rule are satisfied simultaneously.
  - Extending the formalism: expansion in terms of Dirichlet distributions. First numerical tests with gluons. Sum rules provide relations between the powers at small and large x for single and double parton distributions.
  - In principle one can include quarks into the formalism; some additional constraints needed.
  - Is there any deeper physical meaning to the presented algorithm?

Summary and outlook

- Summary & outlook 2 - transverse momentum dependence:
    - Extended the KMR approach to dPDFs.
    - Expressions include correlations through the integrated dPDFs. Additional correlations enter through the regularization cutoffs.
    - Unintegrated dPDFs: small x , higher kT values. Homogenous term dominates over the non-homogenous one.
    - Numerical simulations indicate very good factorization for large scales and small x for the sum of homogeneous and non-homogenous contributions. That property is contingent upon the momentum sum rule.
    - Include the momentum transfer.

backup

Initial conditions: quarks and gluons

Momentum sum rule with quarks:

$$\sum_{f_1} \tilde{D}_{f_1 f_2}(2, n_2) = \tilde{D}_{f_2}(n_2) - \tilde{D}_{f_2}(n_2+1)$$

Quark number sum rule:

$$\tilde{D}_{q_i f_2}(1, n_2) - \tilde{D}_{\bar{q}_i f_2}(1, n_2) = A_{i f_2} \tilde{D}_{f_2}(n_2)$$

$$A_{i f_2} = N_i - \delta_{f_2 q_i} + \delta_{f_2 \bar{q}_i}$$

Ansatz for dPDF with different flavors:

$$\boxed{D_{f_1 f_2}(x_1, x_2) = N_2\, x_1^{-\tilde{\alpha}^{f_1}}\, x_2^{-\tilde{\alpha}^{f_2}} (1 - x_1 - x_2)^{\tilde{\beta}^{f_1 f_2}}}$$

Ansatz for sPDF :

$$\boxed{D_f(x) = N_1\, x^{-\alpha^f}\, (1-x)^{\beta^f}}$$

- Can perform the same analysis as before.
- Conditions for powers for dPDFs and sPDFs are exactly the same from both momentum and quark sum rules.
- Can satisfy simultaneously both sum rules:

Small x powers are identical:

$$\tilde{\alpha}^{f_2} = \alpha^{f_2}$$

$$\tilde{\alpha}^{f_1} = \alpha^{f_1}$$

Large x powers:

$$\tilde{\beta}^{f_1 f_2} = \beta^{f_2} + \alpha^{f_1} - 1$$

Symmetry with respect to the parton exchange

$$\tilde{\beta}^{f_1 f_2} = \tilde{\beta}^{f_2 f_1}$$

Implies the correlation of powers in sPDFs:

$$\beta^{f_2} + \alpha^{f_1} = \beta^{f_1} + \alpha^{f_2}$$