Fermionic PEPS and Topological Order

Carolin Wille, Oliver Buerschaper, Jens Eisert, arXiv:1609.02574

Entanglement in strongly correlated systems, Benasque 2/17/2017

Message

- It is a good idea to talk about topological order in 2D spin systems in PEPS-language.
- This idea can be extended to fermionic systems.

Outline

- PEPS and topological order in spin lattices
  - PEPS and parent Hamiltonians
  - MPO-injectivity [1]

1) Schuch, Cirac, Perez-Garcia 1001.3807
Buerschaper 1307.7763
Bultinck, Marien, Williamson, Sahinoglu, Haegeman, Verstraete 1511.08090

Outline

- PEPS and topological order in spin lattices
  - PEPS and parent Hamiltonians
  - MPO-injectivity
- Fermionic PEPS[2] and topological order

2) Barthel, Pineda, Eisert Phys. Rev. A 80, 042333 (2009)
Kraus, Schuch, Verstraete, Cirac Phys. Rev. A 81, 052338 (2010)
Gu, Verstraete, Wen 1004.2563

Outline

- PEPS and topological order in spin lattices
  - PEPS and parent Hamiltonians
  - MPO-injectivity
- Fermionic PEPS and topological order
  - Fermionic string nets
  - Fermionic MPO-injectivity[3] via graphical calculus

3) Wille, Buerschaper, Eisert 1609.02574
Williamson, Bultinck, Haegeman, Verstraete 1609.02897

Topological Order in 2D

What is topological order?

- topological ground state degeneracy
- locally indistinguishable ground states
- topological entanglement entropy
- anyon excitations

Topological Order in 2D

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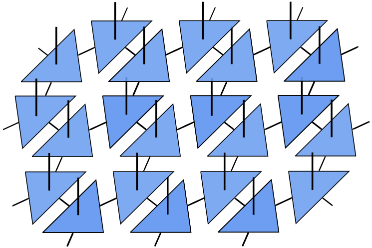
How can we use PEPS to describe topological order?

Setting

$$|\Psi\rangle = \sum_{i_1,\dots,i_N} \mathrm{ttr}\left[A^{i_1}\dots A^{i_N}\right]|i_1,\dots,i_N\rangle$$

- translation invariance
- state $|\Psi\rangle$
- parent Hamiltonian

  $H|\Psi\rangle = 0$
- 2D, gapped, local

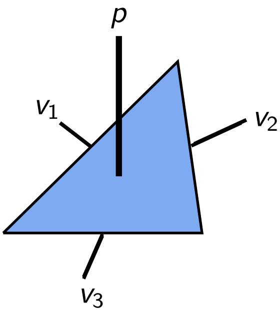


What properties does a **single tensor** A need to have such that the parent Hamiltonian has **topological order**?

Injectivity

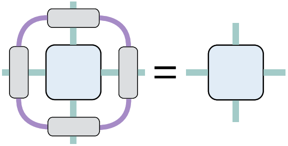
A is a linear map $A : \textit{virtual} \rightarrow \textit{physical}$

- injective
  - *virtual* $\leftrightarrow$ *physical*
  - unique ground state
- non-injective
  - $A\,|v\rangle = A\,|w\rangle = |p\rangle$
  - virtual symmetry
  - possibly topological order



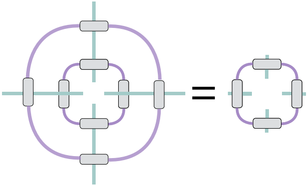
Axioms of MPO-Injectivity

- MPO-symmetry



Axioms of MPO-Injectivity

- MPO-symmetry
- **MPO projector**



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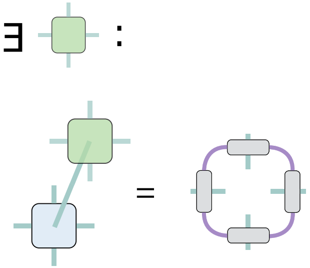
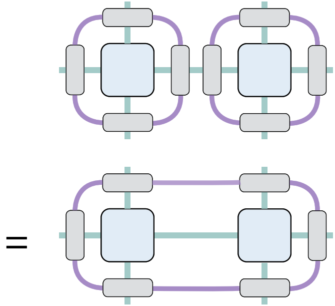
- **MPO-injectivity**



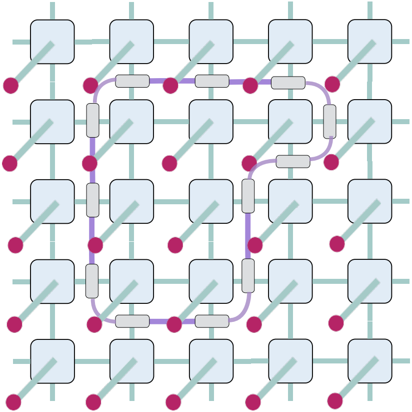
Axioms of MPO-Injectivity

- MPO-symmetry
- MPO projector
- MPO-injectivity
- **Stability under concatenation**



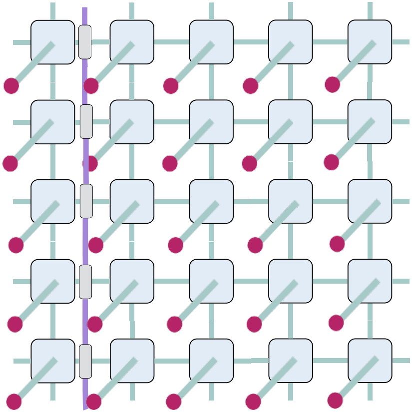
Deformable Loops

trivial loops: vanish



Deformable Loops

non trivial loops: new locally indistinguishable ground state



MPO-injective PEPS formalism

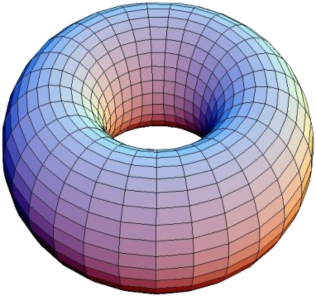
If a PEPS fulfills the axioms of MPO-injectivity, its parent Hamiltonian has topological order.

Examples:

- G-injective PEPS: toric code, quantum double models
- Twisted injectivity : twisted quantum double models
- MPO-injectivity: Levin-Wen string-net models

MPO-Injectivity – Topological Order

- Powerful formalism



MPO-Injectivity – Topological Order

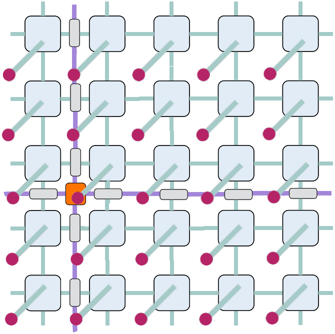
- Powerful formalism
  - Topological correction

$$S(\rho_R) = \alpha L - \gamma$$

$$L = |\partial R|$$

MPO-Injectivity – Topological Order

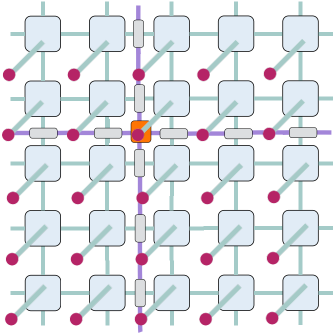
- Powerful formalism
  - Topological correction
  - Ground state space



Bultinck, Marien, Williamson, Sahinoglu, Haegeman, Verstraete 1511.08090

MPO-Injectivity – Topological Order

- Powerful formalism
  - Topological correction
  - **Ground state space**



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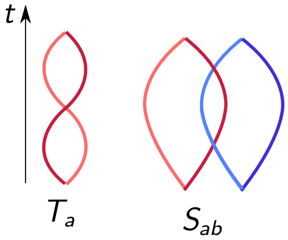
MPO-Injectivity – Topological Order

- Powerful formalism
  - Topological correction
  - Ground state space
  - **Anyonic statistics**

  T- **and** S**-matrices**



Bultinck, Marien, Williamson, Sahinoglu, Haegeman, Verstraete 1511.08090

Kitaev, Annals of Physics 321 (2006):

...the powerful but heavy language of categories and functors (also known as 'abstract nonsense').

Observations

Fusion category for bosons (spins)

- $\leftrightarrow$ Levin-Wen string-nets
- $\leftrightarrow$ consistent triangulization of 3D volumes

  Dijkgraaf-Witten partition function, Tuaerev-Viro state sum invariants
- $\leftrightarrow$ PEPS with MPO-injectivity

Levin, Wen Phys.Rev. B71 (2005) 045110

R. Dijkgraaf, E. Witten, Commun.Math. Phys. (1990) 129: 393

V. G. Turaev and O. Y. Viro, Topology 31, 865 (1992)

R. Koenig, G. Kuperberg, B. W. Reichardt, Annals of Physics 325, 2707-2749 (2010)

O. Buerschaper, Ann. Phys. 351, 447-476 (2014)

Observations

Fusion category for bosons (spins)

$\leftrightarrow$ Levin-Wen string-nets

$\leftrightarrow$ consistent triangulization of 3D volumes

$\leftrightarrow$ PEPS with MPO-injectivity

Superfusion category for Fermions

$\leftrightarrow$ Fermionic string-nets

$\leftrightarrow$ consistent triangulization of (graded) 3D volumes

$\leftrightarrow$ fPEPS with fMPO-injectivity

Gu, Wang, Wen, Phys. Rev. B 91, 125149 (2015), Gu, Wen, Phys. Rev. B 90, 115141 (2014)
Williamson, Bultinck, Haegeman, Verstraete, arXiv:1609.02897,Wille, Buerschaper, Eisert, arXiv:1609.02574
D. Gaiotto, A. Kapustin, arXiv:1505.05856, L. Bhardwaj, D. Gaiotto, A. Kapustin, arXiv:1605.01640

Fermionic MPO formalism from fermionic string-net duality

Levin-Wen string nets

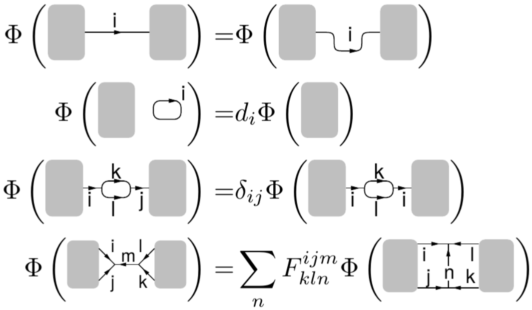
- lattice model (2D)
- commuting projector Hamiltonian

  $H = \sum_v Q_v + \sum_p Q_p$
- anyon excitations
- local degrees of freedom edges: string-types
- string-types: objects of fusion category $\mathcal{F}$
- RG flow: axioms of $\mathcal{F}$

Levin, Wen Phys.Rev. B71 (2005) 045110

Levin-Wen string nets

RG flow

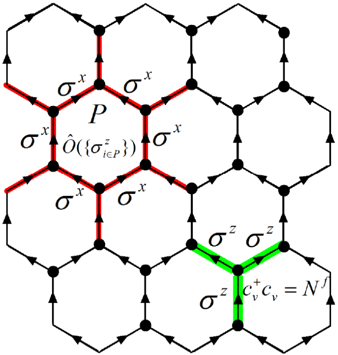


- $\mathcal{F} \rightarrow$ 'string-net' $\rightarrow$ TQFT
- $G \rightarrow$ 'string-net' $\rightarrow D(G)$

Levin, Wen Phys.Rev. B71 (2005) 045110

Fermionic Toric Code

- Edges: spin-1/2
- Vertices: fermions
- $H = \sum_v Q_v + \sum_p Q_p$

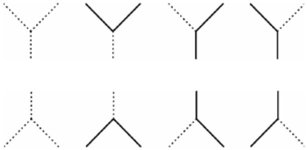


Gu, Wang, Wen Phys. Rev. B 90, 085140

Fermionic Toric Code

- Edges: spin-1/2
- Vertices: fermions
- $H = \sum_v Q_v + \sum_p Q_p$
- $\mathbf{Q_v} = \frac{1}{2}\left(\mathbf{1} + \prod_{\mathbf{i}\in\mathbf{v}} \sigma_{\mathbf{i}}^{\mathbf{z}}\right)$

Vertex projector



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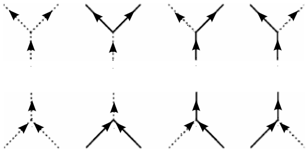

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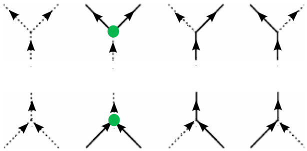

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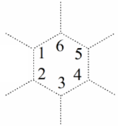
- $\mathbf{Q_p} = \frac{1}{2}\left(\mathbf{1} + \prod_{\mathbf{i} \in \mathbf{p}} \sigma_{\mathbf{i}}^{\mathbf{x}}\right) \hat{\mathbf{F}}_{\mathbf{p}}$

Plaquette projector



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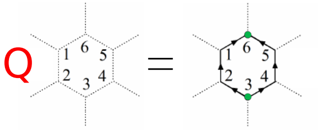

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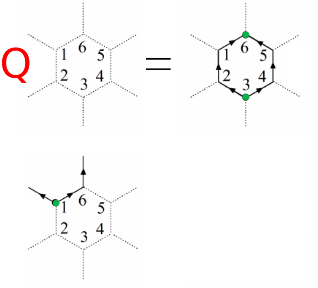

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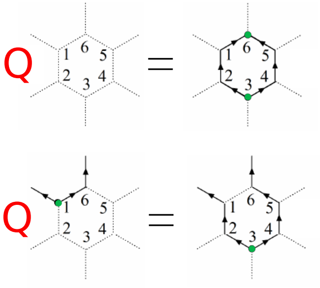

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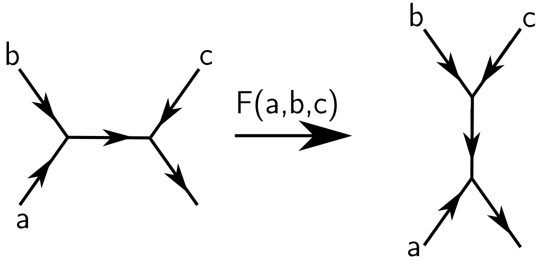
String-net with unique fusion

Renormalization group flow: F-move



Gu, Wang, Wen, Phys. Rev. B 91, 125149 (2015)

Fermionic String-net with unique fusion

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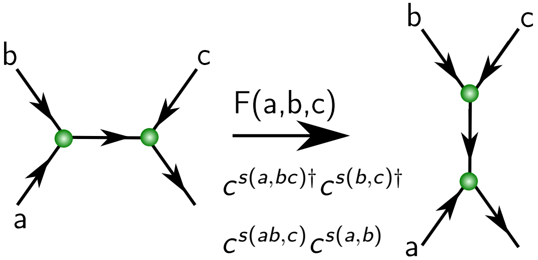

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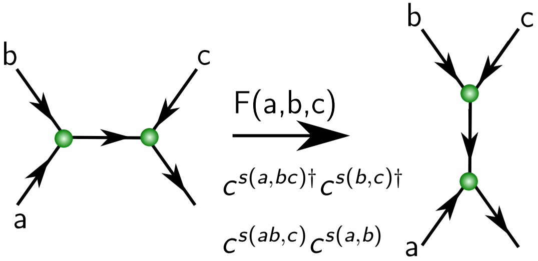
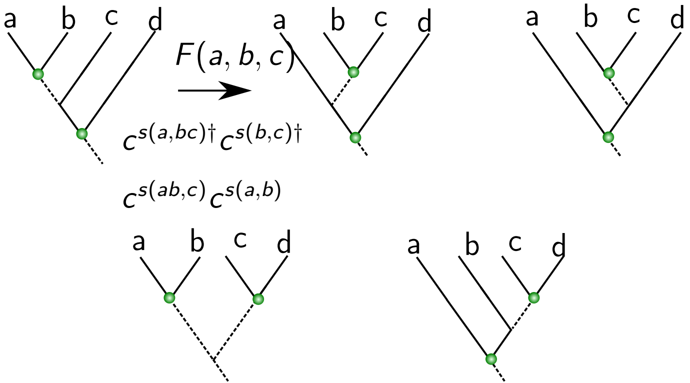


Fermion parity: $s(a,b) + s(ab,c) + s(b,c) + s(a,bc) = 0$

Gu, Wang, Wen, Phys. Rev. B 91, 125149 (2015)

Fermionic pentagon equation



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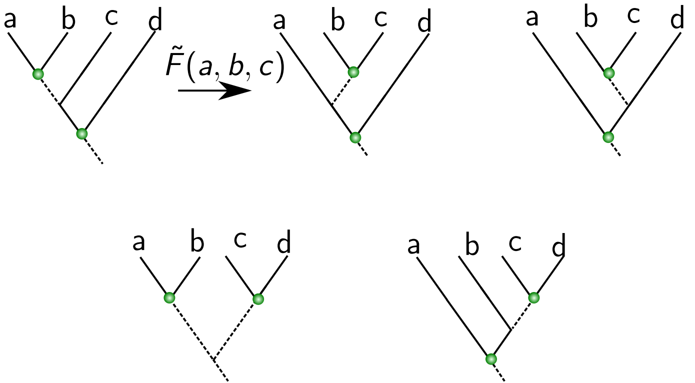

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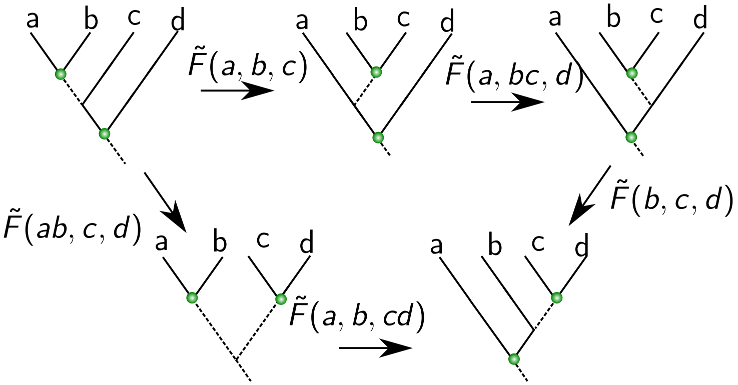

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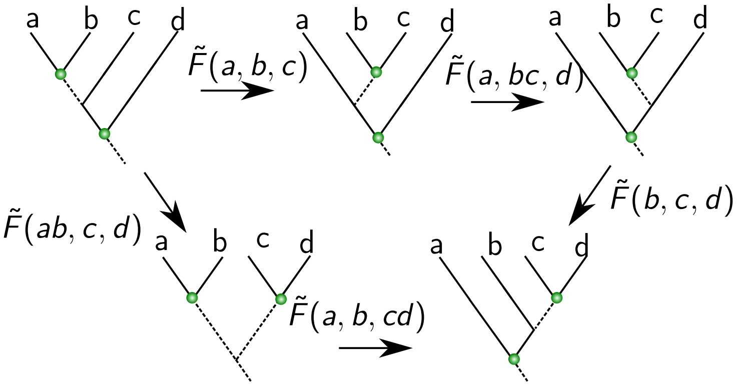
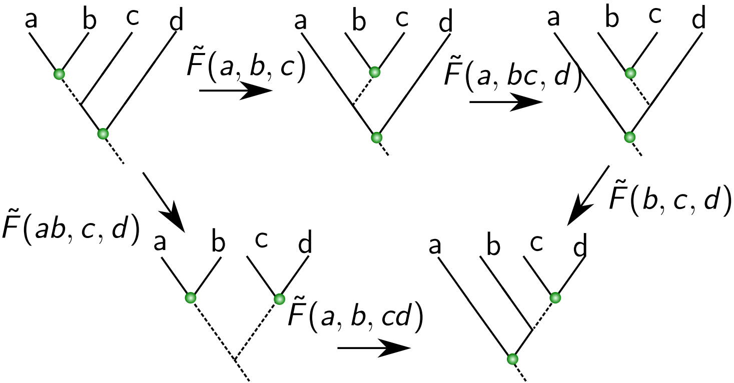


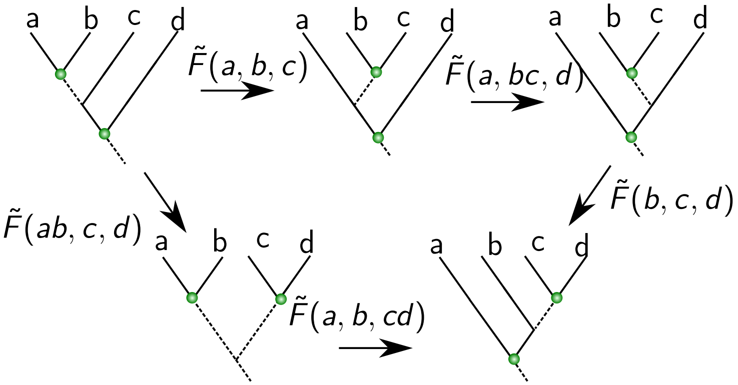
$$\tilde{F}(a,b,c)\tilde{F}(a,bc,d)\tilde{F}(b,c,d) = \tilde{F}(ab,c,d)\tilde{F}(a,b,cd)$$

Fermionic pentagon equation



$$F(a,b,c)F(a,bc,d)F(b,c,d) \; c^{s(c,d)\dagger} c^{s(a,b)} = F(ab,c,d)F(a,b,cd) \; c^{s(a,b)} c^{s(c,d)\dagger}$$

Fermionic pentagon equation



$$F(a,b,c)F(a,bc,d)F(b,c,d) = (-1)^{s(a,b)s(c,d)}F(ab,c,d)F(a,b,cd)$$

Twisted fermionic quantum double models

- $\mathbb{Z}_2$-graded group cohomology: triple (G, s, ω)
  - group G
  - 2-cocyle $s \in \{0, 1\}$: even parity

    $$s(a, b) + s(ab, c) + s(a, bc) + s(b, c) = 0$$
  - graded-3-cocycle ω

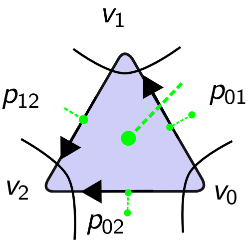
    $$\omega(a, b, c)\omega(a, bc, d)\omega(b, c, d) = (-1)^{s(a,b)s(c,d)}\omega(ab, c, d)\omega(a, b, cd)$$
- Fermionic toric code: simplest triple
  - $G = \mathbb{Z}_2$
  - $s(1, 1) = 1$, $s = 0$ otherwise

Gu, Wang, Wen Phys. Rev. B 91, 125149 (2015)

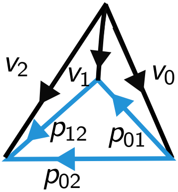
The GS of fermionic twisted quantum doubles are Fermionic PEPS fulfilling the axioms of fermionic MPO-injectivity.

Tensor network – graphical calculus

$$A_+ = \sum_{v_0, v_1, v_2} \omega(v_0, v_0^{-1} v_1, v_1^{-1} v_2)$$
$$\times \theta^{s(v_0^{-1} v_1, v_1^{-1} v_2)} \theta^{s(v_0, v_0^{-1} v_2)}$$
$$\times \bar{\theta}^{s(v_1, v_1^{-1} v_2)} \bar{\theta}^{s(v_0, v_0^{-1} v_1)}$$
$$\times |p_{01} p_{12} p_{02}\rangle \langle v_0 v_1 v_2|$$



- $p_{ij} = v_i v_j^{-1}$
- fermions on faces



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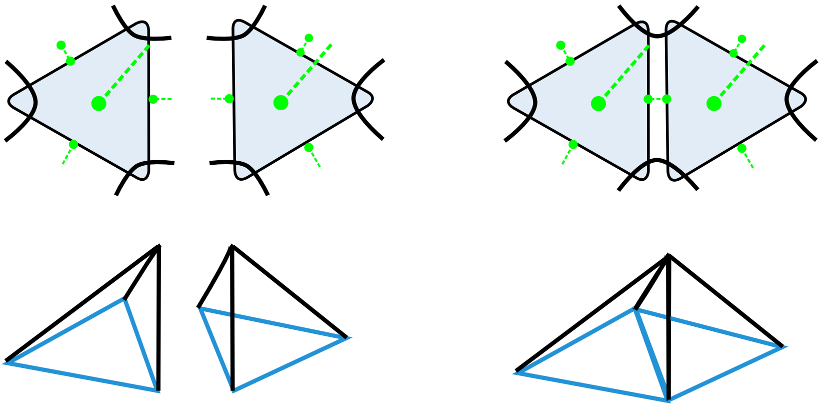
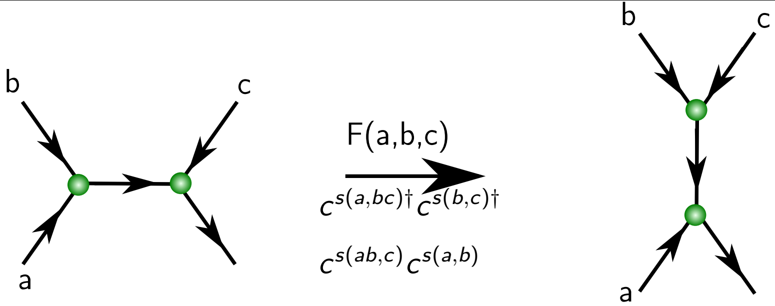
Contraction



Fermionic PEPS for Fermionic string nets



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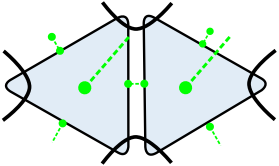
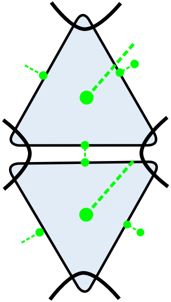


F(a,b,c)

$$c^{s(a,bc)\dagger}\, c^{s(b,c)\dagger}$$

$$c^{s(ab,c)}\, c^{s(a,b)}$$



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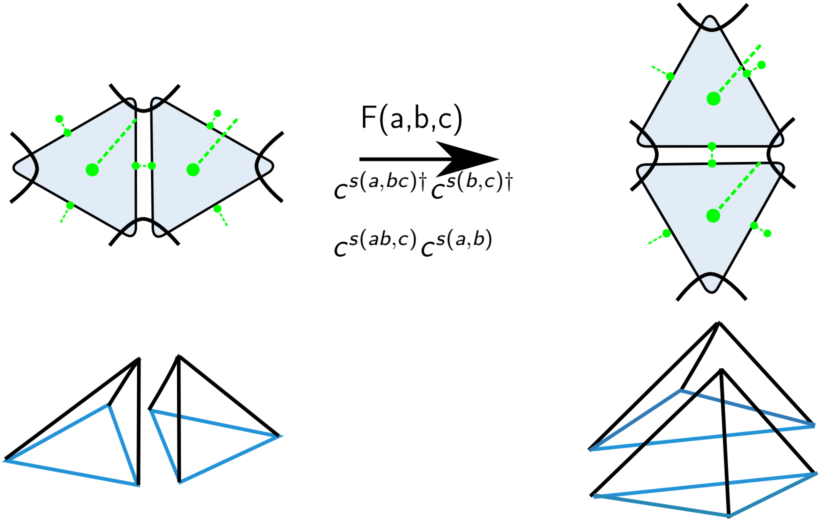

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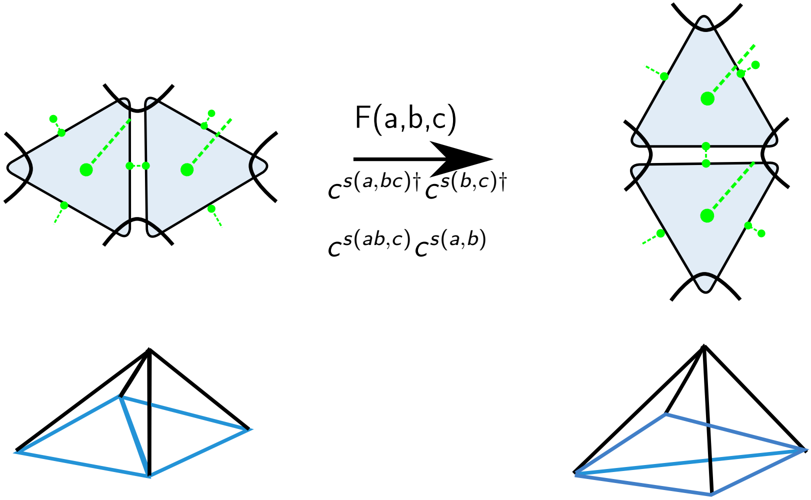

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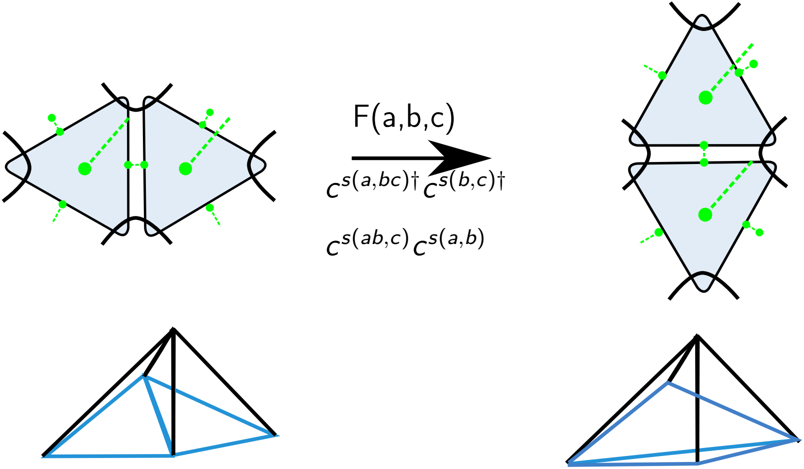
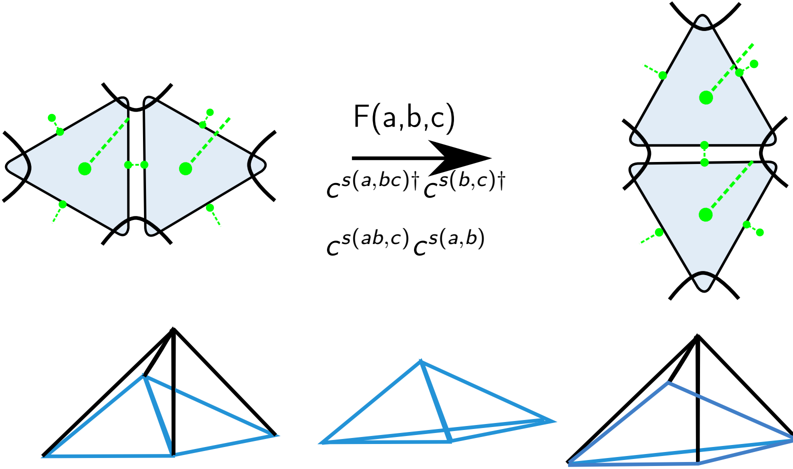
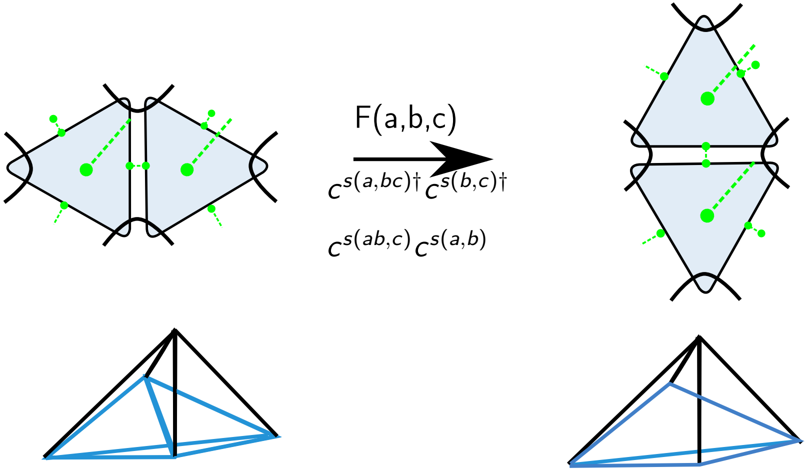


Fermionic PEPS for Fermionic string nets

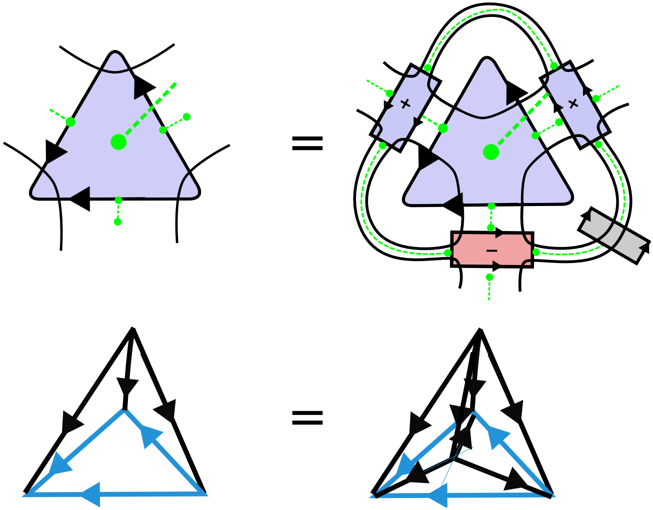


Fermionic PEPS for Fermionic string nets

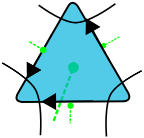
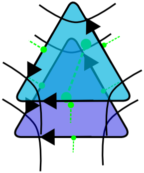
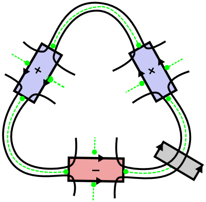


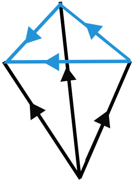
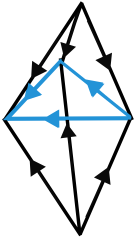
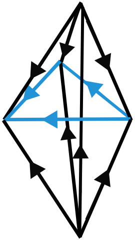
Axioms of fMPO-injectivity follow from consistency of retriangularization of 'volumes' via graphical calculus.

1-4 Pachner move: fMPO-symmetry



2-3 Pachner move: fMPO-injectivity

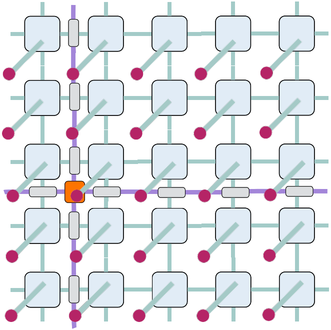
$\exists$  :  = 

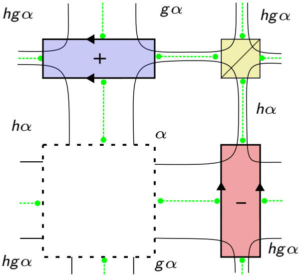
$\exists$  :  = 

The GS of fermionic twisted quantum doubles are Fermionic PEPS fulfilling the axioms of fermionic MPO-injectivity.

Application

Ground state degeneracy from fMPO





Ground state degeneracy

c_s^ω-regular pair conjugacy classes $\mathcal{C}(g,h)$

A pair conjugacy class $\mathcal{C}(g,h)$ is called c-regular if for all elements of the centralizer $k \in \mathcal{Z}(g,h)$ we have

$$c_g(h,k) = c_g(k,h) \qquad \text{and} \qquad s(g,h) = s(h,g) \tag{1}$$

with

$$c_g(h,k) = \frac{\omega(g,h,k)\omega(h,k,g)}{\omega(h,g,k)} \ . \tag{2}$$

Future research

- Formalism
  - S- and T-matrices
  - superfusion category theory
  - spin structures
- Models
  - (Para)fermionic string-nets
  - Majorana Dimer model[1]

[1] Ware, Son, Cheng, Mishmash, Alicea, Bauer Phys. Rev. B 94, 115127 (2016)

Thank you for your attention.