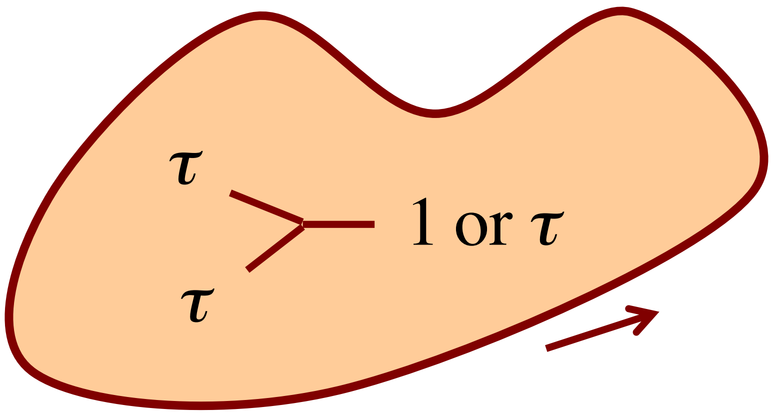
Non-Abelian anyons in 2+1 D



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Non-Abelian anyons in 2+1 D

Main themes of this talk

- simplified axiomatic approach to TQFT
- building a catalogue?
- simple-current constructions of anyon wavefunctions

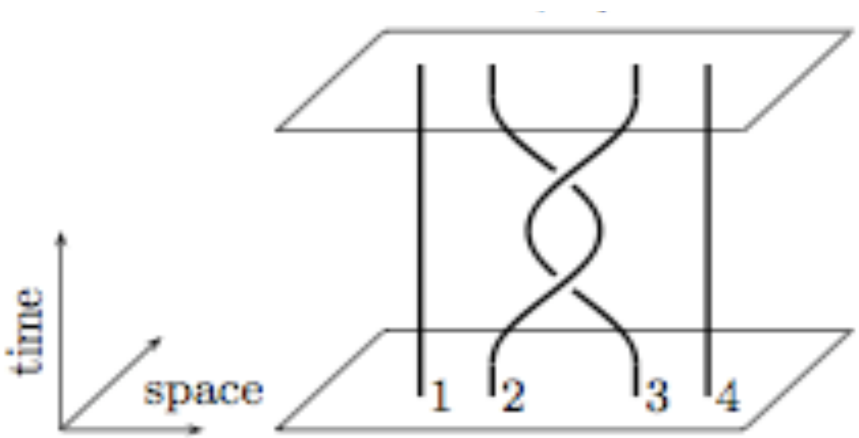
Wen 2015

KjS-Wen 2015

Topological phases

quantum phases where time-evolution of many-particle wavefunction is completely determined by the topology of the particle world-lines → TQFT

$$\left|\psi_{\text{final}}\right\rangle$$

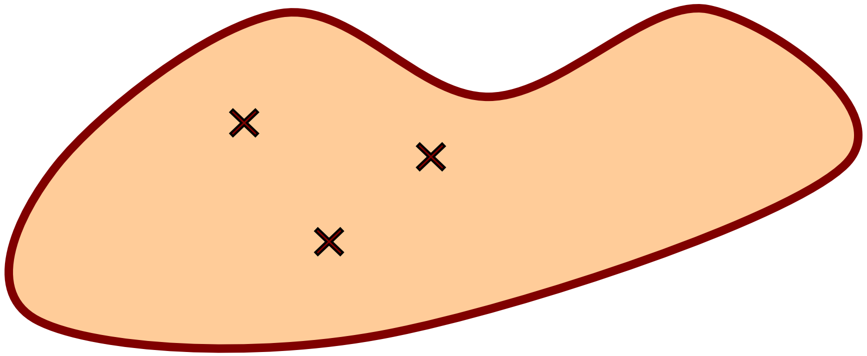


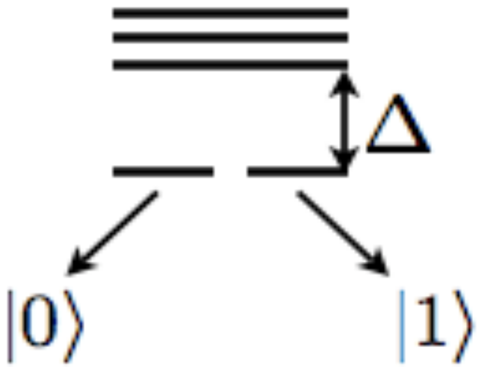
$$\left|\psi_{\text{initial}}\right\rangle$$

Non-Abelian topological phases

Topological quantum register

many-body quantum states where presence of bulk defects gives rise to degenerate groundstates, protected by gap Δ

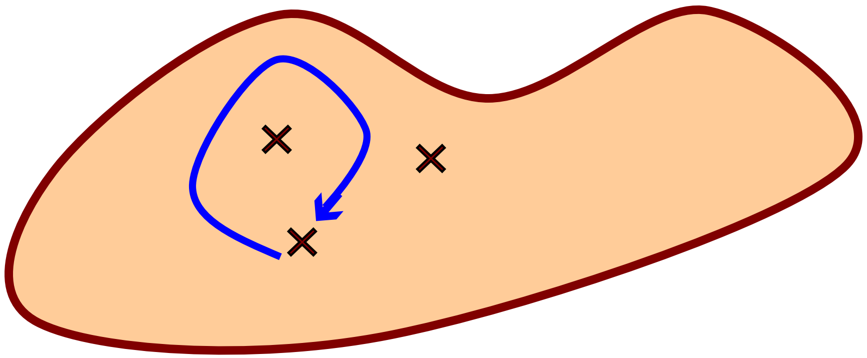


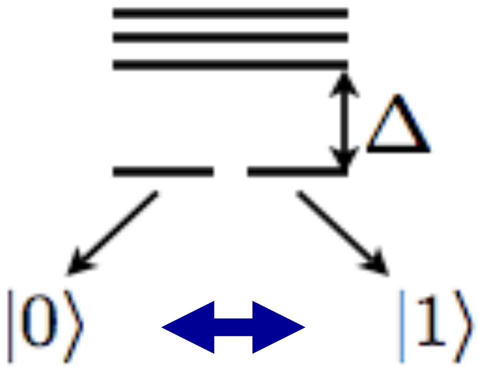


Non-Abelian topological phases

Braiding - topological quantum gates

braiding one defect around another leads to a mixing of the degenerate groundstates

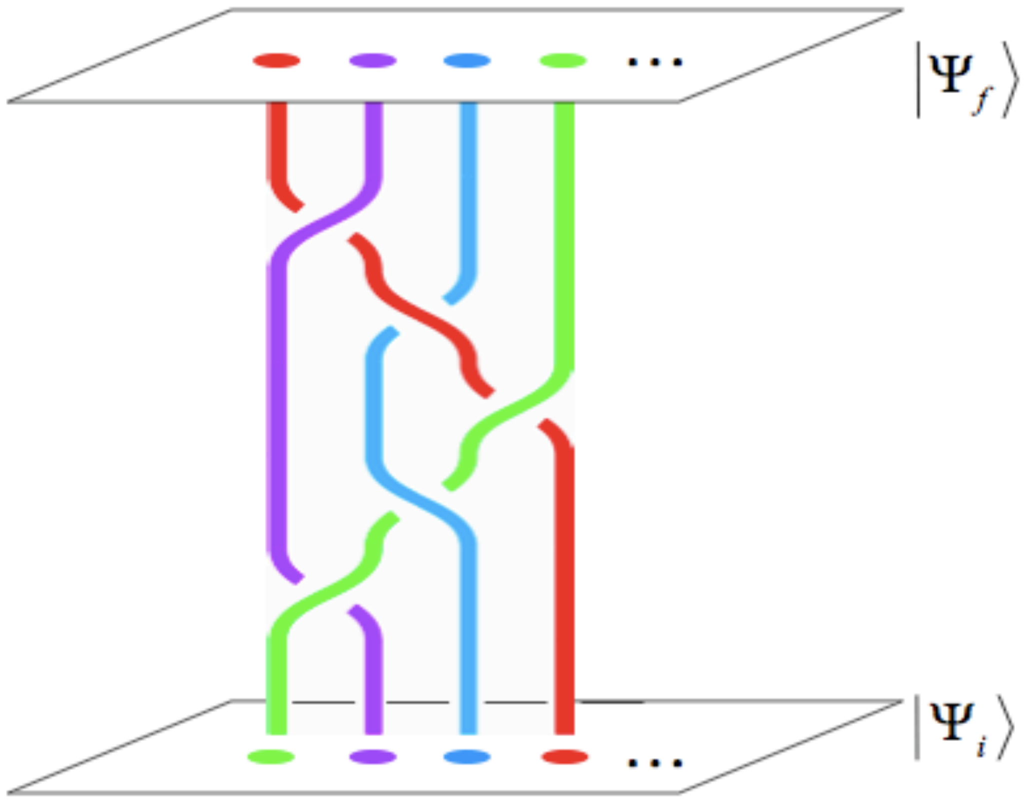




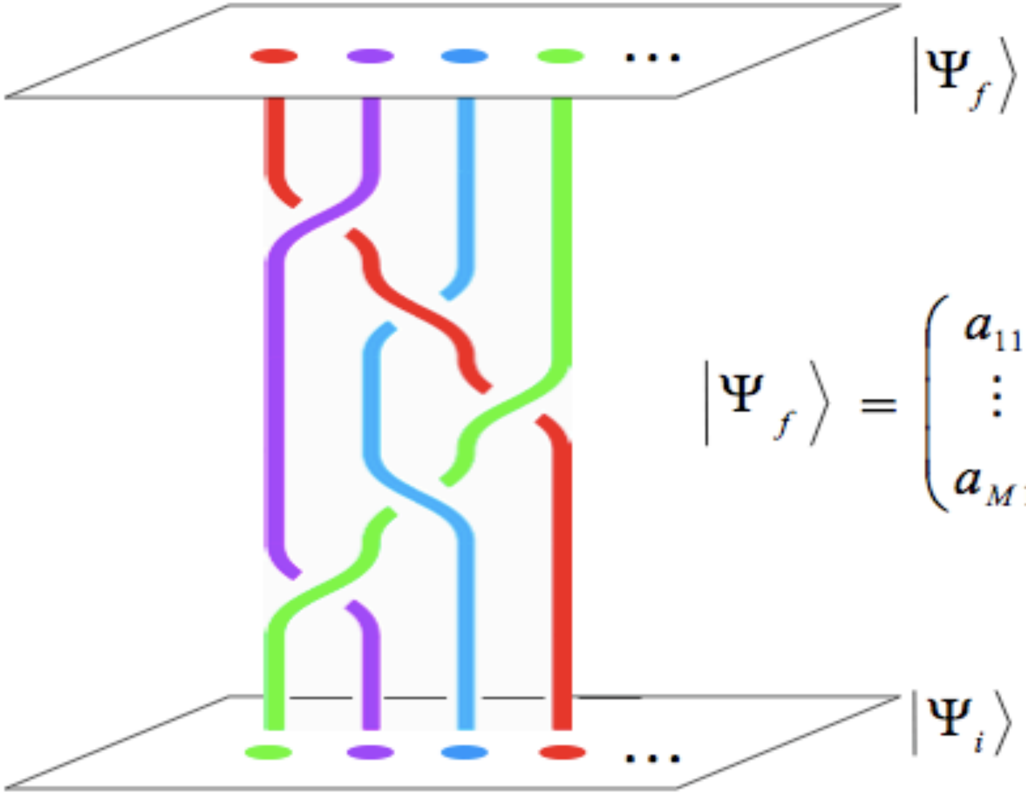
Non-Abelian braiding

figures: Nick Bonesteel

Non-Abelian braiding

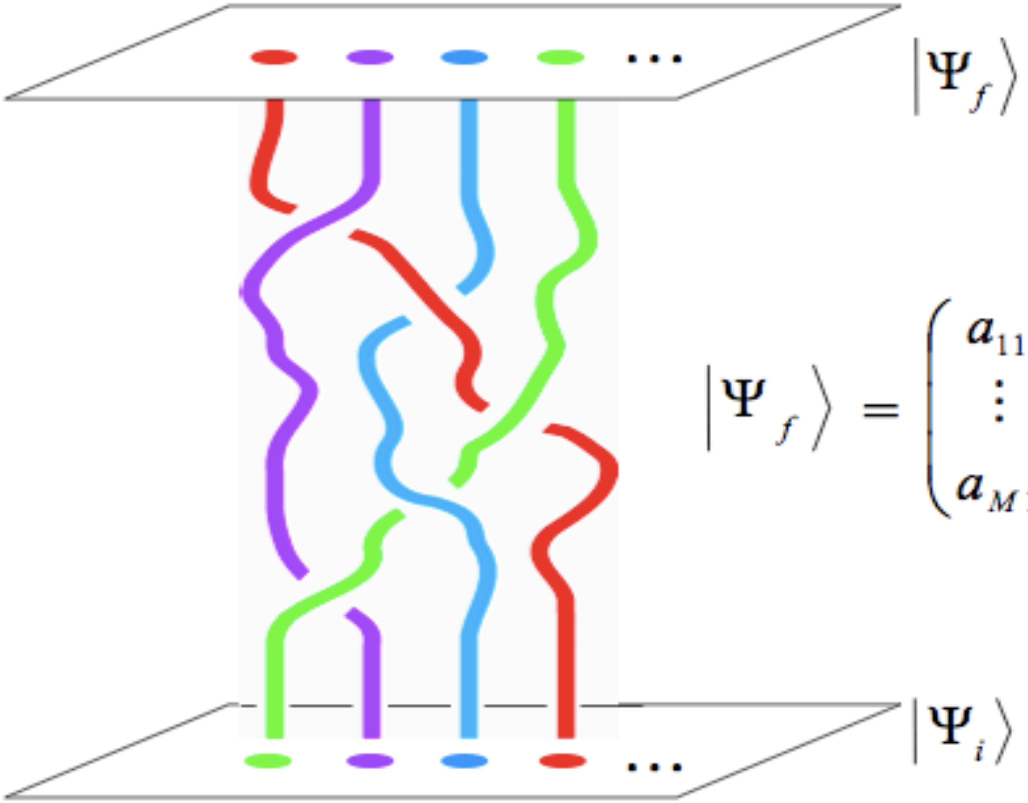


Non-Abelian braiding



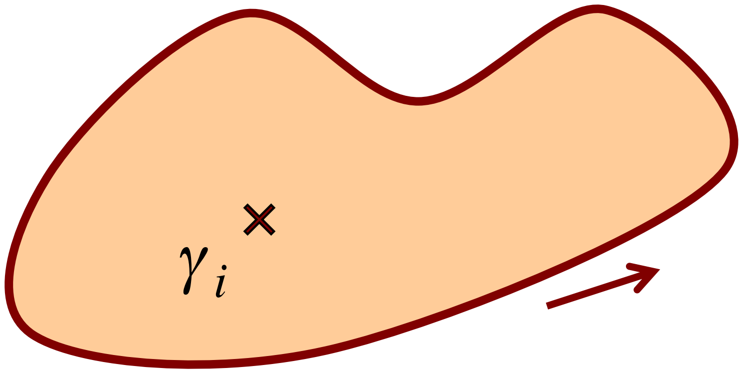
$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Non-Abelian braiding



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

iconic example: Ising anyons



defect in appropriate quantum liquid supports Majorana zero-mode γ_i

degeneracy from fusion

the presence of n Majorana zero-modes leads to a degeneracy $d_n = 2^{n/2-1}$

! $n=4$ Majorana zero-modes define a qubit **!**

Ising anyons

braiding

With n=4 Majoranas,
denote states of the qubit as

$$|0_{12}0_{34}\rangle, \quad |1_{12}1_{34}\rangle$$

Then braiding 2 around 3
represented by 2x2 matrix

$$U_{2\leftrightarrow 3} \propto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

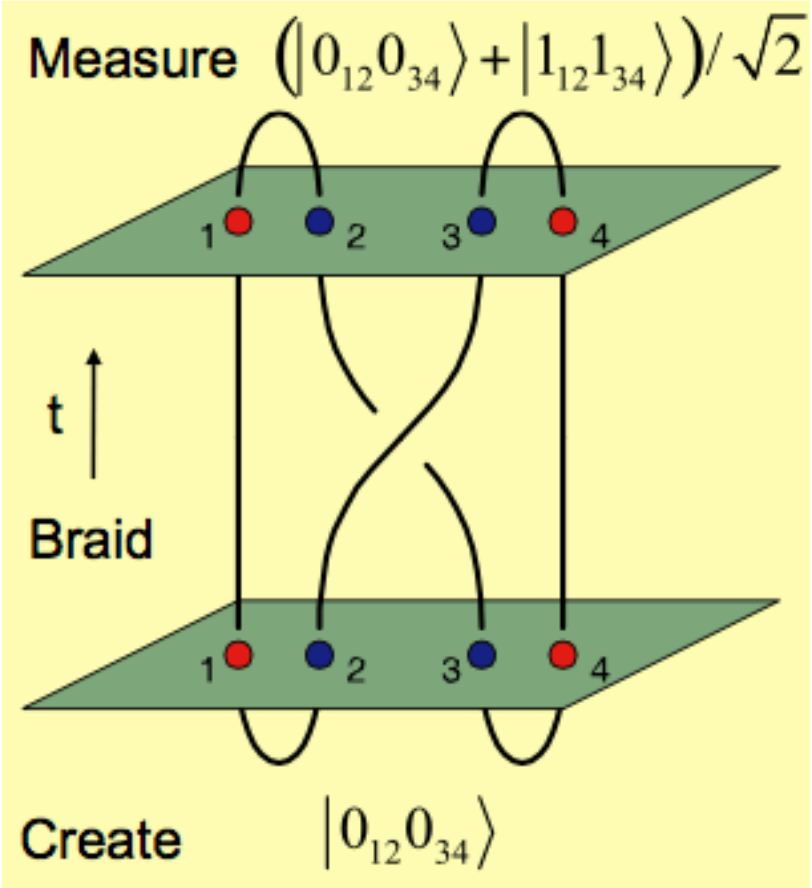


figure: C. Kane

Non-Abelian anyons in 2+1 D

Systematic analysis: TQFT

unitary modular braided fusion categories
also known as **modular tensor categories (MTC)**

Q: can we go beyond Ising anyons? ✓

Q: if so, can we be universal for TQC? ✓

Q: can we make a full catalogue? **?**

brute force solution of MTC axioms (ever since Verlinde) has led to table through $N=4$ particle types

Rowell-Stone-Wang 2009

Non-Abelian anyons in 2+1 D

Fractional quantum Hall states and simple-current constructions of MTCs

- fqH states provide many examples of nA anyons
- systematic understanding through fqH-CFT connection
- similarly, CFT connection allows for explicit constructions of MTCs [simple-current constructions]

Non-Abelian anyons in 2+1 D

recent progress

- renewed systematic search for $N>4$ MTCs based on \`simplified axioms',

  $N = 5$, $\max[N_k^{ij}] < 4$ (10 solutions)

  $N = 6$, $\max[N_k^{ij}] < 3$ (50 solutions)

  $N = 7$, $\max[N_k^{ij}] < 2$ (24 solutions)

Wang 2010, Wen 2015

- systematics of simple-current constructions; verification of all $N=5$ - $N=7$ solutions

KjS-Wen 2015

FQH states and simple-current constructions of MTCs

(bosonic) quantum Hall wavefunction

$$\Psi_{\text{Laughlin}}(z_1,\ldots,z_N;w_1,\ldots,w_n) =$$

$$\prod_{i<j}(z_i - z_j)^2 \prod_{i,j}(z_i - w_j) \prod_{i<j}(w_i - w_j)^{1/2}$$

quasi-holes are (Abelian) semions

MTC:

semion (rank-2, Abelian)

also denoted as $2^B{}_1$ or $(A_1,1)$

FQH – CFT correspondence

idea:

- FQH state is condensate of constituent bosons or electrons – represent these by CFT operators $\psi_{\mathrm{e}}(z_i)$

- $\psi_{\mathrm{e}}(z_i)$ should be **bosonic simple currents**
[only single fusion channel with any other field]

- quasi-hole operators selected by [no branch cuts]

$$\phi_{\mathrm{qh}}(w)\psi_{\mathrm{e}}(z_1) = (z-w)^{\mathrm{integer}}\left[\phi_2(w) + \ldots\right]$$

FQH – CFT correspondence

ground state wave function

$$\Psi_{\mathrm{GS}}(z_1,\dots,z_N) \cong \left\langle \psi_{\mathrm{e}}(z_1)\dots\psi_{\mathrm{e}}(z_N)\psi_{\mathrm{background}}(z_\infty)\right\rangle_{\mathrm{CFT}}$$

electron (boson)
condensate operator

neutralizing
background charge

excited state wave function:

$$\Psi_{\mathrm{qh}}(w_1,w_2,\dots;z_1,z_2,\dots) \cong \left\langle \phi_{\mathrm{qh}}(w_1)\phi_{\mathrm{qh}}(w_2)\dots\psi_{\mathrm{e}}(z_1)\psi_{\mathrm{e}}(z_2)\dots\right\rangle_{\mathrm{CFT}}$$

FQH – CFT correspondence

Laughlin wavefunction

$$\psi_e(z) = e^{i\sqrt{2}\varphi}(z)$$

CFT: free boson, *c=1*

$$\phi_{\mathrm{qh}}(w) = e^{\frac{i}{\sqrt{2}}\varphi}(z)$$

Moore-Read wavefunction

$$\psi_e(z) = \psi\, e^{i\varphi}(z)$$

CFT: boson times Ising, *c=3/2*

$$\phi_{\mathrm{qh}}(w) = \sigma e^{\frac{i}{2}\varphi}(z)$$

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathrm{Pf}\,\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)$$

Moore-Read 1990

FQH/MTC – CFT correspondence – general

- CFT data:
  rational CFT **plus collection of simple currents**
- **simple-current correlators**
  define ground state wavefunctions
  → vacuum for the TQFT
- **simple-current primaries**
  represent excitations over the ground state
  → particle types of the TQFT

MTCs from simple-current algebra

- basic building blocks are WZW models based on

$$(X_l, k) \qquad\qquad [\ X_l \text{ Lie algebra, } k \text{ level}\ ]$$

- further constructions through

conjugation: $\quad N_c^B \to \left[N_c^B\right]^* = N_{-c}^B$

stacking: $\quad N_c^B, N_{c'}^{\prime B} \to \left[NN'\right]_{c+c'}^B = N_c^B \otimes N_{c'}^{\prime B}$

simple-current reductions: $\quad N_c^B \to N_c^{\prime B} \quad$ with $\quad N' < N$

MTCs from simple-current algebra

Example: constructing an MTC with two particle types and Fibonacci fusion rules

Start from *(A₁, k=3)* , times a *U(1)* factor.
4 x 2 primaries:

$$[\Phi_l \quad l = 0,\ldots,3 \quad s_l = 0, \frac{3}{20}, \frac{2}{5}, \frac{3}{4}] \quad \text{times} \quad [1, e^{\frac{i}{\sqrt{2}}\varphi} \quad s = 0, \frac{1}{4}]$$

Simple-current algebra extended with $\Phi_3 e^{\frac{i}{\sqrt{2}}\varphi}$
has single primary $\Phi_1 e^{\frac{i}{\sqrt{2}}\varphi}$ of scaling dimension $s = \frac{2}{5}$

$$2^{\mathrm{B}}_{14/5}: \quad (A_1, 3)_{1/2} = [(A_1, 3) \otimes U(1)_1]_{1/4}$$

Catalogue of bosonic MTCs, I

Brute force search up to $N=4$ particle types gives 35 bosonic MTCs

We label them as N^B_c and list

- quantum dimensions d_i, $i=1, \ldots, N$
- scaling dimensions s_i, $i=1, \ldots, N$
- type from simple-current construction

$\boxed{N = 2}$

4 distinct topological orders $N_c^{\ B}$

B: Bosonic
c : central charge mod 8

s_i : scaling dimensions

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type	N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
1_0^B	0	1	0						
2_1^B	0.5	$1, 1$	$0, \frac{1}{4}$	$U(1)_1$, $(A_1, 1)$	2_{-1}^B	0.5	$1, 1$	$0, -\frac{1}{4}$	$(E_7, 1)$
$2_{14/5}^B$	0.9276	$1, \zeta_3^1$	$0, \frac{2}{5}$	$(G_2, 1)$, $(A_1, 3)_{\frac{1}{2}}$	$2_{-14/5}^B$	0.9276	$1, \zeta_3^1$	$0, -\frac{2}{5}$	$(F_4, 1)$, $(A_2, 2)_{\frac{1}{3}}$

d_i : quantum dimensions

$$\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$$

$\boxed{N = 2}$

4 distinct topological orders $N_c^{\,B}$

B: Bosonic
c : central charge mod 8

s_i : scaling dimensions

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type	N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
1_0^B	0	1	0						
2_1^B	0.5	1, 1	$0, \frac{1}{4}$	$U(1)_1$, $(A_1, 1)$	2_{-1}^B	0.5	1, 1	$0, -\frac{1}{4}$	$(E_7, 1)$
$2_{14/5}^B$	0.9276	$1, \zeta_3^1$	$0, \frac{2}{5}$	$(G_2, 1)$, $(A_1, 3)_{\frac{1}{2}}$	$2_{-14/5}^B$	0.9276	$1, \zeta_3^1$	$0, -\frac{2}{5}$	$(F_4, 1)$, $(A_2, 2)_{\frac{1}{3}}$

d_i : quantum dimensions

Abelian order (bosonic Laughlin state)

$$\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$$

$$\boxed{N = 2}$$

4 distinct topological orders N_c^B

B: Bosonic
c : central charge mod 8

s_i : scaling dimensions

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type	N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
1_0^B	0	1	0						
2_1^B	0.5	1, 1	$0, \frac{1}{4}$	$U(1)_1$, $(A_1, 1)$	2_{-1}^B	0.5	1, 1	$0, -\frac{1}{4}$	$(E_7, 1)$
$2_{14/5}^B$	0.9276	$1, \zeta_3^1$	$0, \frac{2}{5}$	$(G_2, 1)$, $(A_1, 3)_{\frac{1}{2}}$	$2_{-14/5}^B$	0.9276	$1, \zeta_3^1$	$0, -\frac{2}{5}$	$(F_4, 1)$, $(A_2, 2)_{\frac{1}{3}}$

d_i : quantum dimensions

Fibonacci anyons

$$\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$$

$N = 3$

12 distinct topological orders N_c^B

Ising anyons

3_2^B	0.7924	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	$(A_2, 1), (A_1, 4)_{\frac{1}{4}}$	3_{-2}^B	0.7924	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	$(E_6, 1)$
$3_{1/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$	$(B_8, 1)$	$3_{-1/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	$(B_7, 1), (E_8, 2)$
$3_{3/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	$(A_1, 2)$	$3_{-3/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	$(B_6, 1)$
$3_{5/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	$(B_2, 1)$	$3_{-5/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	$(B_5, 1)$
$3_{7/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$	$(B_3, 1)$	$3_{-7/2}^B$	1	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	$(B_4, 1)$
$3_{8/7}^B$	1.6082	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$	$(A_1, 5)_{\frac{1}{2}}$	$3_{-8/7}^B$	1.6082	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	$(A_4, 2)_{\frac{1}{5}}$

$N = 3$

12 distinct topological orders N_c^B

3^B_2	0.7924	$1,1,1$	$0,\frac{1}{3},\frac{1}{3}$	$(A_2,1),(A_1,4)_{\frac{1}{4}}$	3^B_{-2}	0.7924	$1,1,1$	$0,-\frac{1}{3},-\frac{1}{3}$	$(E_6,1)$
$3^B_{1/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{1}{16}$	$(B_8,1)$	$3^B_{-1/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{1}{16}$	$(B_7,1),(E_8,2)$
$3^B_{3/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{3}{16}$	$(A_1,2)$	$3^B_{-3/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{3}{16}$	$(B_6,1)$
$3^B_{5/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{5}{16}$	$(B_2,1)$	$3^B_{-5/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{5}{16}$	$(B_5,1)$
$3^B_{7/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{7}{16}$	$(B_3,1)$	$3^B_{-7/2}$	1	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{7}{16}$	$(B_4,1)$
$3^B_{8/7}$	1.6082	$1,\zeta_5^1,\zeta_5^2$	$0,-\frac{1}{7},\frac{2}{7}$	$(A_1,5)_{\frac{1}{2}}$	$3^B_{-8/7}$	1.6082	$1,\zeta_5^1,\zeta_5^2$	$0,\frac{1}{7},-\frac{2}{7}$	$(A_4,2)_{\frac{1}{5}}$

family of eight related orders of type $3^B_{l+1/2}$ → **symmetry** T_8

$$T_8: \quad N_c^B \rightarrow \left[N_c^B \otimes 4_1^B \right]_{1/4}$$

$U(1)_2$ - generator T_8

\`double semion' – generator T_2

$N = 4$

18 distinct topological orders N_c^B

$4^{B,a}_0$	1	$1,1,1,1$	$0,0,0,\frac{1}{2}$	$(D_8,1)$	$4^{B,b}_0$	1	$1,1,1,1$	$0,0,\frac{1}{4},-\frac{1}{4}$	
4^B_1	1	$1,1,1,1$	$0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$	$U(1)_2$	4^B_{-1}	1	$1,1,1,1$	$0,-\frac{1}{8},-\frac{1}{8},\frac{1}{2}$	$(D_7,1)$
4^B_2	1	$1,1,1,1$	$0,\frac{1}{4},\frac{1}{4},\frac{1}{2}$		4^B_{-2}	1	$1,1,1,1$	$0,-\frac{1}{4},-\frac{1}{4},\frac{1}{2}$	$(D_6,1)$
4^B_3	1	$1,1,1,1$	$0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$	$(A_3,1)$	4^B_{-3}	1	$1,1,1,1$	$0,-\frac{3}{8},-\frac{3}{8},\frac{1}{2}$	$(D_5,1)$
4^B_4	1	$1,1,1,1$	$0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$(D_4,1)$, $(A_2,3)_{\frac{1}{9}}$	$4^{B,c}_0$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,\frac{2}{5},-\frac{2}{5},0$	
$4^B_{9/5}$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{4},\frac{3}{20},\frac{2}{5}$	$(A_1,3)$	$4^B_{-9/5}$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{4},-\frac{3}{20},-\frac{2}{5}$	
$4^B_{19/5}$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{4},-\frac{7}{20},\frac{2}{5}$		$4^B_{-19/5}$	1.4276	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{4},\frac{7}{20},-\frac{2}{5}$	$(C_3,1)$
$4^B_{12/5}$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,-\frac{2}{5},-\frac{2}{5},\frac{1}{5}$	$(A_1,8)_{\frac{1}{4}}$	$4^B_{-12/5}$	1.8552	$1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_3^1$	$0,\frac{2}{5},\frac{2}{5},-\frac{1}{5}$	
$4^B_{10/3}$	2.1328	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,\frac{1}{3},\frac{2}{9},-\frac{1}{3}$	$(A_1,7)_{\frac{1}{2}}$	$4^B_{-10/3}$	2.1328	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,-\frac{1}{3},-\frac{2}{9},\frac{1}{3}$	$(G_2,2)$, $(A_6,2)_{\frac{1}{7}}$

in blue: orders obtained by **stacking** two $N = 2$ orders

Catalogue of bosonic MTCs, II

simplified axioms Wang 2010, Wen 2015

- given in terms terms of $(N_k^{ij}\, , s_i\, , c)$
- no explicit mention of F-tensor ;
  no pentagon/hexagon identities
- no a priori guarantee that the axioms are strong enough to guarantee a bona fide MTC
- **but: all solutions found (through N=7) confirmed through explicit simple-current construction**

Simplified axioms

1. N_k^{ij} are non-negative integers that satisfy

$$N_k^{ij} = N_k^{ji}, \quad N_j^{1i} = \delta_{ij}, \quad \sum_{k=1}^{N} N_1^{ik} N_1^{kj} = \delta_{ij},$$

$$\sum_{m=1}^{n} N_m^{ij} N_l^{mk} = \sum_{n=1}^{n} N_l^{in} N_n^{jk} \text{ or } N_k N_i = N_i N_k \tag{21}$$

where $i, j, \cdots = 1, 2, \cdots, n$, and the matrix N_i is given by $(N_i)_{kj} = N_k^{ij}$. In fact N_1^{ij} defines a charge conjugation $i \to \bar{i}$:

$$N_1^{ij} = \delta_{\bar{i}j}. \tag{22}$$

We also refer n as the rank of the corresponding topological order.

Simplified axioms

2. N_k^{ij} and s_i satisfy[97]

$$\sum_r V_{ijkl}^r s_r = 0 \bmod 1 \tag{23}$$

where

$$\begin{aligned} V_{ijkl}^r = {} & N_r^{ij} N_{\bar r}^{kl} + N_r^{il} N_{\bar r}^{jk} + N_r^{ik} N_{\bar r}^{jl} \\ & - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N_m^{ij} N_{\bar m}^{kl} \end{aligned} \tag{24}$$

They also satisfy (see Theorem 3.1.19 in Ref. 98)

$$\begin{aligned} & \mathrm{e}^{2\pi\,\mathrm{i}\,\sum_j s_j M_{ij}} = \mathrm{e}^{2\pi\,\mathrm{i}\,s_i \frac{4}{3} \sum_j M_{ij}} \\ & \text{where } M_{ij} = 2N_j^{i\bar i} N_i^{ij} + N_j^{ii} N_i^{j\bar i}. \end{aligned} \tag{25}$$

Simplified axioms

3. Let d_i be the largest eigenvalue of the matrix N_i.
   Let

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N_k^{ij} \mathrm{e}^{2\pi\,\mathrm{i}\,(s_i + s_j - s_k)} d_k. \tag{27}$$

   Then, S is unitary and satisfies[92]

$$S_{11} > 0, \quad N_k^{ij} = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}}. \tag{28}$$

Simplified axioms

4. Let

$$T_{ij} = \mathrm{e}^{2\pi\,\mathrm{i}\,s_i}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{c}{24}}\delta_{ij}. \qquad (29)$$

Then

$$(ST)^3 = S^2 = C, \quad C^2 = 1. \qquad (30)$$

In fact $C_{ij} = N_1^{ij}$.

Simplified axioms

5. Let

$$\nu_i = \frac{1}{D^2} \sum_{jk} N_i^{jk} d_j d_k \, \mathrm{e}^{4\pi\,\mathrm{i}\,(s_j - s_k)}. \qquad (31)$$

Then[94,95] $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$.

$N = 5$

10 distinct topological orders N_c^B with $\max[N_k^{ij}] < 4$

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
5_0^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	
5_4^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	$(A_4, 1)$
$5_2^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	$(A_1, 4)$, $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$
$5_2^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	$[5_2^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{-2}^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	$(C_4, 1)$, $(A_3, 2)_{\frac{1}{2}}$
$5_{-2}^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$	$[5_{-2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$	$(F_4, 2)$, $(A_1, 9)_{\frac{1}{2}}$
$5_{-16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	$(E_8, 3)$, $(A_8, 2)_{\frac{1}{9}}$
$5_{18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$	$(A_1, 12)_{\frac{1}{4}}$, $(A_2, 4)_{\frac{1}{3}}$
$5_{-18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	$(A_3, 3)_{\frac{1}{4}}$

$N = 5$

10 distinct topological orders N_c^B with $\max[N_k^{ij}] < 4$

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
5_0^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	
5_4^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	$(A_4, 1)$
$5_2^{B,a}$	1.7924	1		$(A_1, 4)$, $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$
$5_2^{B,b}$	1.7924	1		$[5_2^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{-2}^{B,a}$	1.7924	1		$(C_4, 1)$, $(A_3, 2)_{\frac{1}{2}}$
$5_{-2}^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$	$[5_{-2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$	$(F_4, 2)$, $(A_1, 9)_{\frac{1}{2}}$
$5_{-16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	$(E_8, 3)$, $(A_8, 2)_{\frac{1}{9}}$
$5_{18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$	$(A_1, 12)_{\frac{1}{4}}$, $(A_2, 4)_{\frac{1}{3}}$
$5_{-18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	$(A_3, 3)_{\frac{1}{4}}$

this order corresponds to the affine Lie algebra $A_4 = SU(5)$ at level $k=1$

$N = 5$

10 distinct topological orders N_c^B
with $\max[N_k^{ij}] < 4$

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
5_0^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	
5_4^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	$(A_4, 1)$
$5_2^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	$(A_1, 4)$, $(U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$
$5_2^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	$[5_2^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{-2}^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	$(C_4, 1)$, $(A_3, 2)_{\frac{1}{2}}$
$5_{-2}^{B,b}$	1.792[illegible]	[illegible]	… $-\frac{1}{3}$	$[5_{-2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{16/11}^B$	2.557[illegible]	[illegible]	… $-\frac{5}{11}$	$(F_4, 2)$, $(A_1, 9)_{\frac{1}{2}}$
$5_{-16/11}^B$	2.557[illegible]	[illegible]	… $\frac{1}{11}, \frac{5}{11}$	$(E_8, 3)$, $(A_8, 2)_{\frac{1}{9}}$
$5_{18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$	$\boxed{(A_1, 12)_{\frac{1}{4}}}$ $(A_2, 4)_{\frac{1}{3}}$
$5_{-18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	$(A_3, 3)_{\frac{1}{4}}$

example where number
of particle types is reduced
from *4l+1* to *l+2*

$N = 5$

10 distinct topological orders N_c^B with $\max[N_k^{ij}] < 4$

N_c^B	S_{top}	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
5_0^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	
5_4^B	1.1609	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	$(A_4, 1)$
$5_2^{B,a}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	$(A_1, 4), (U(1)_3/\mathbb{Z}_2)_{\frac{1}{2}}$
$5_2^{B,b}$	1.7924	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	$\boxed{[5_2^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}}$
$5_{-2}^{B,a}$	1.7924		$-\frac{1}{3}$	$(C_4, 1), (A_3, 2)_{\frac{1}{2}}$
$5_{-2}^{B,b}$	1.7924		$-\frac{1}{3}$	$[5_{-2}^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$5_{16/11}^B$	2.5573	1	$-\frac{5}{11}$	$(F_4, 2), (A_1, 9)_{\frac{1}{2}}$
$5_{-16/11}^B$	2.5573	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	$(E_8, 3), (A_8, 2)_{\frac{1}{9}}$
$5_{18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$	$(A_1, 12)_{\frac{1}{4}}, (A_2, 4)_{\frac{1}{3}}$
$5_{-18/7}^B$	2.5716	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	$(A_3, 3)_{\frac{1}{4}}$

operation T_2 : combines stacking with order $4_0^{B,b}$ with 4-fold reduction

$N = 6$

$\max[N_k^{ij}] < 3$

50 distinct topological orders N_c^B

N_c^B	S_{top}	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	$N_c^B \otimes \widetilde{N}_{\tilde{c}}^B$	type
6_1^B	1.2924	6	$1,1,1,1,1,1$	$0, \frac{1}{12}, \frac{1}{12}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$	$2_{-1}^B \otimes 3_2^B$	$U(1)_3$
6_{-1}^B	1.2924	6	$1,1,1,1,1,1$	$0, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}$	$2_1^B \otimes 3_{-2}^B$	
6_3^B	1.2924	6	$1,1,1,1,1,1$	$0, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, -\frac{5}{12}, -\frac{5}{12}$	$2_1^B \otimes 3_2^B$	
6_{-3}^B	1.2924	6	$1,1,1,1,1,1$	$0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}, \frac{5}{12}, \frac{5}{12}$	$2_{-1}^B \otimes 3_{-2}^B$	$(A_5, 1)$
$6_{1/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, \frac{3}{16}$	$2_1^B \otimes 3_{-1/2}^B$	
$6_{-1/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, -\frac{3}{16}$	$2_1^B \otimes 3_{-3/2}^B$	
$6_{\frac{3}{2}}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, \frac{5}{16}$	$2_1^B \otimes 3_{1/2}^B$	
$6_{-3/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, -\frac{5}{16}$	$2_1^B \otimes 3_{-5/2}^B$	
$6_{5/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{3}{16}, \frac{7}{16}$	$2_1^B \otimes 3_{3/2}^B$	
$6_{-5/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{3}{16}, -\frac{7}{16}$	$2_1^B \otimes 3_{-7/2}^B$	
$6_{7/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{5}{16}, -\frac{7}{16}$	$2_1^B \otimes 3_{5/2}^B$	
$6_{-7/2}^B$	1.5	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{5}{16}, \frac{7}{16}$	$2_1^B \otimes 3_{7/2}^B$	
$6_{4/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{15}, \frac{1}{15}, \frac{2}{5}$	$2_{14/5}^B \otimes 3_{-2}^B$	
$6_{-4/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{1}{15}, -\frac{1}{15}, -\frac{2}{5}$	$2_{-14/5}^B \otimes 3_2^B$	
$6_{16/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{4}{15}, \frac{4}{15}, -\frac{2}{5}$	$2_{-14/5}^B \otimes 3_{-2}^B$	$(A_2, 2)$
$6_{-16/5}^B$	1.7200	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{4}{15}, -\frac{4}{15}, \frac{2}{5}$	$2_{\frac{14}{5}}^B \otimes 3_2^B$	
$6_{-27/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{23}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{5/2}^B$	$(E_7, 2)$
$6_{-17/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{13}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{7/2}^B$	
$6_{-7/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{3}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-7/2}^B$	
$6_{3/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{7}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-5/2}^B$	
$6_{13/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{17}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-3/2}^B$	
$6_{23/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{27}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{-1/2}^B$	
$6_{33/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{37}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{1/2}^B$	
$6_{-37/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{33}{80}$	$2_{\frac{14}{5}}^B \otimes 3_{3/2}^B$	
$6_{27/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{23}{80}$	$2_{-14/5}^B \otimes 3_{-5/2}^B$	
$6_{17/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{13}{80}$	$2_{-14/5}^B \otimes 3_{-7/2}^B$	
$6_{7/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{3}{80}$	$2_{-14/5}^B \otimes 3_{7/2}^B$	
$6_{-3/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{7}{80}$	$2_{-14/5}^B \otimes 3_{5/2}^B$	
$6_{-13/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{17}{80}$	$2_{-14/5}^B \otimes 3_{3/2}^B$	
$6_{-23/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, \frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{27}{80}$	$2_{-14/5}^B \otimes 3_{1/2}^B$	
$6_{-33/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{37}{80}$	$2_{-14/5}^B \otimes 3_{-1/2}^B$	
$6_{37/10}^B$	1.9276	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1$	$0, \frac{1}{2}, -\frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{33}{80}$	$2_{-14/5}^B \otimes 3_{-3/2}^B$	
$6_{1/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, -\frac{1}{4}, -\frac{1}{7}, -\frac{11}{28}, \frac{1}{28}, \frac{2}{7}$	$2_{-1}^B \otimes 3_{8/7}^B$	
$6_{-1/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, \frac{1}{4}, \frac{1}{7}, \frac{11}{28}, -\frac{1}{28}, -\frac{2}{7}$	$2_1^B \otimes 3_{-8/7}^B$	$(C_5, 1)$
$6_{15/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, \frac{1}{4}, \frac{3}{28}, -\frac{1}{7}, \frac{2}{7}, -\frac{13}{28}$	$2_1^B \otimes 3_{\frac{8}{7}}^B$	$(A_1, 5)$
$6_{-15/7}^B$	2.1082	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0, -\frac{1}{4}, -\frac{3}{28}, \frac{1}{7}, -\frac{2}{7}, \frac{13}{28}$	$2_{-1}^B \otimes 3_{-8/7}^B$	
$6_0^{B,a}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, 0, \frac{1}{2}$		$(D_5,2)_{\frac{1}{2}}$, $(U(1)_5/\mathbb{Z}_2)_{\frac{1}{2}}$
$6_0^{B,b}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, \frac{1}{4}, -\frac{1}{4}$		$[6_0^{B,a} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$6_4^{B,b}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, \frac{1}{4}, -\frac{1}{4}$		$(B_2, 2)$
$6_4^{B,a}$	2.1609	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, 0, \frac{1}{2}$		$[6_4^{B,b} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$6_{58/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, \frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{16}{35}, \frac{4}{35}$	$2_{\frac{14}{5}}^B \otimes 3_{-8/7}^B$	
$6_{-58/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, -\frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{16}{35}, -\frac{4}{35}$	$2_{-14/5}^B \otimes 3_{8/7}^B$	
$6_{138/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, \frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{9}{35}, -\frac{11}{35}$	$2_{\frac{14}{5}}^B \otimes 3_{8/7}^B$	
$6_{-138/35}^B$	2.5359	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0, -\frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{9}{35}, \frac{11}{35}$	$2_{-14/5}^B \otimes 3_{-8/7}^B$	
$6_{46/13}^B$	2.9132	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0, \frac{4}{13}, \frac{2}{13}, -\frac{6}{13}, \frac{6}{13}, -\frac{1}{13}$		$(A_1, 11)_{\frac{1}{2}}$
$6_{-46/13}^B$	2.9132	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0, -\frac{4}{13}, -\frac{2}{13}, \frac{6}{13}, -\frac{6}{13}, \frac{1}{13}$		$(A_{10}, 2)_{\frac{1}{11}}$
$6_{8/3}^B$	3.1107	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_{16}^2,\zeta_{16}^4,\zeta_{16}^6$	$0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, -\frac{1}{3}$		$(A_1, 16)_{\frac{1}{4}}$
$6_{-8/3}^B$	3.1107	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_{16}^2,\zeta_{16}^4,\zeta_{16}^6$	$0, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{3}, \frac{1}{3}$		$(A_2, 6)_{\frac{1}{9}}$
6_2^B	3.3263	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, -\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}, 0, \frac{1}{3}$		
6_{-2}^B	3.3263	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}, 0, -\frac{1}{3}$		$(G_2, 3)$

$N = 7$

$\max[N_k^{ij}] < 2$

24 distinct topological orders N_c^B

N_c^B	S_{top}	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	type
$7_2^{B,a}$	1.4036	7	$1, 1, 1, 1, 1, 1, 1$	$0, \frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, -\frac{3}{7}, -\frac{3}{7}$	
$7_{-2}^{B,a}$	1.4036	7	$1, 1, 1, 1, 1, 1, 1$	$0, -\frac{1}{7}, -\frac{1}{7}, -\frac{2}{7}, -\frac{2}{7}, \frac{3}{7}, \frac{3}{7}$	$(A_6, 1)$
$7_{9/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{3}{32}, \frac{3}{32}, \frac{1}{4}, -\frac{1}{4}, \frac{15}{32}$	$(A_1, 6)$
$7_{13/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{7}{32}, \frac{7}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{13}{32}$	$[7_{9/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-15/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{11}{32}, \frac{11}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{9}{32}$	$[7_{13/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-11/4}^{B}$	2.3857	27.313	[illegible]	[illegible] $\frac{1}{4}, -\frac{5}{32}$	$[7_{-15/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-7/4}^{B}$	2.3857	27.313	[illegible]	[illegible] $-\frac{1}{4}, -\frac{1}{32}$	$[7_{-11/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-3/4}^{B}$	2.3857	27.313	[illegible]	[illegible] $\frac{1}{4}, \frac{3}{32}$	$[7_{-7/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{1/4}^{B}$	2.3857	27.313	[illegible]	[illegible] $-\frac{1}{4}, \frac{7}{32}$	$[7_{-3/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{5/4}^{B}$	2.3857	27.313	[illegible]	[illegible] $-\frac{1}{4}, \frac{11}{32}$	$[7_{1/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{7/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{13}{32}, \frac{13}{32}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{32}$	$(C_6, 1)$
$7_{11/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, -\frac{15}{32}, -\frac{15}{32}, \frac{1}{4}, -\frac{1}{4}, \frac{5}{32}$	$[7_{7/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{15/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, -\frac{11}{32}, -\frac{11}{32}, \frac{1}{4}, -\frac{1}{4}, \frac{9}{32}$	$[7_{11/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-13/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, -\frac{7}{32}, -\frac{7}{32}, \frac{1}{4}, -\frac{1}{4}, \frac{13}{32}$	$[7_{15/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-9/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, -\frac{3}{32}, -\frac{3}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{15}{32}$	$[7_{-13/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-5/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{1}{32}, \frac{1}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{11}{32}$	$[7_{-9/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{-1/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{5}{32}, \frac{5}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{7}{32}$	$[7_{-5/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_{3/4}^{B}$	2.3857	27.313	$1, 1, \zeta_6^1, \zeta_6^1, \zeta_6^2, \zeta_6^2, \zeta_6^3$	$0, \frac{1}{2}, \frac{9}{32}, \frac{9}{32}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{32}$	$[7_{-1/4}^B \otimes 4_1^B]_{\frac{1}{4}}$
$7_2^{B,b}$	2.4036	28	$1, 1, 2, 2, 2, \sqrt{7}, \sqrt{7}$	$0, 0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}, \frac{1}{8}, -\frac{3}{8}$	$(U(1)_7/\mathbb{Z}_2)_{\frac{1}{2}}$
$7_2^{B,c}$	2.4036	28	$1, 1, 2, 2, 2, \sqrt{7}, \sqrt{7}$	$0, 0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}, -\frac{1}{8}, \frac{3}{8}$	$[7_2^{B,b} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$7_{-2}^{B,b}$	2.4036	28	$1, 1, 2, 2, 2, \sqrt{7}, \sqrt{7}$	$0, 0, -\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}, -\frac{1}{8}, \frac{3}{8}$	$(B_3, 2), (D_7, 2)_{\frac{1}{2}}$
$7_{-2}^{B,c}$	2.4036	28	$1, 1, 2, 2, 2, \sqrt{7}, \sqrt{7}$	$0, 0, -\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}, \frac{1}{8}, -\frac{3}{8}$	$[7_{-2}^{B,b} \otimes 4_0^{B,b}]_{\frac{1}{4}}$
$7_{8/5}^{B}$	3.2194	86.750	$1, \zeta_{13}^1, \zeta_{13}^2, \zeta_{13}^3, \zeta_{13}^4, \zeta_{13}^5, \zeta_{13}^6$	$0, -\frac{1}{5}, \frac{2}{15}, 0, \frac{2}{5}, \frac{1}{3}, -\frac{1}{5}$	$(A_1, 13)_{\frac{1}{2}}$
$7_{-8/5}^{B}$	3.2194	86.750	$1, \zeta_{13}^1, \zeta_{13}^2, \zeta_{13}^3, \zeta_{13}^4, \zeta_{13}^5, \zeta_{13}^6$	$0, \frac{1}{5}, -\frac{2}{15}, 0, -\frac{2}{5}, -\frac{1}{3}, \frac{1}{5}$	$(A_{12}, 2)_{\frac{1}{13}}$

operation T_8 : combines stacking with order 4_1^B with 4-fold reduction

T_8

T_2

T_2

Non-Abelian anyons in 2+1 D

main results

- systematic search through $N=7$
- simple-current constructions of anyon wavefunctions

to be further clarified

- status of simplified axiomatic approach
- systematics of simple-current constructions, mathematics behind the T_2 and T_8 orbits