# Breaking the area law


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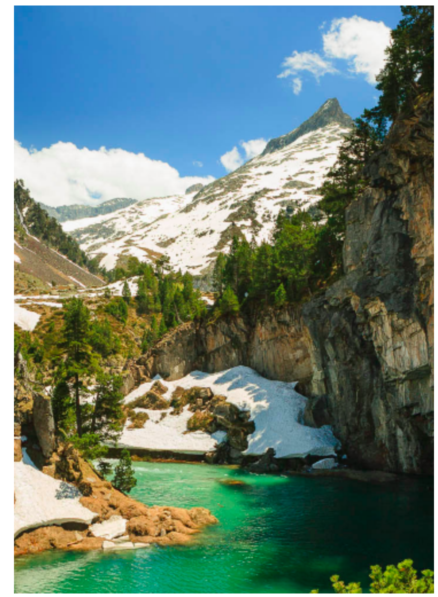
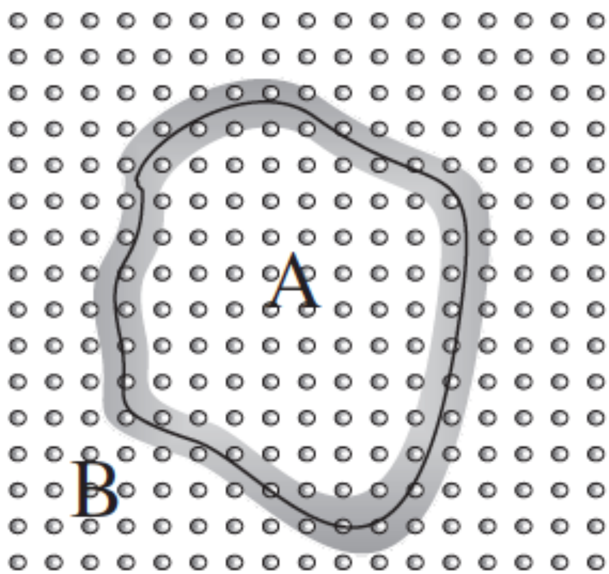


*Workshop "Entanglement in Strongly Correlated Systems"*
*Center of Sciences, Benasque 16th Feb 2017*

# Area law of entanglement entropy

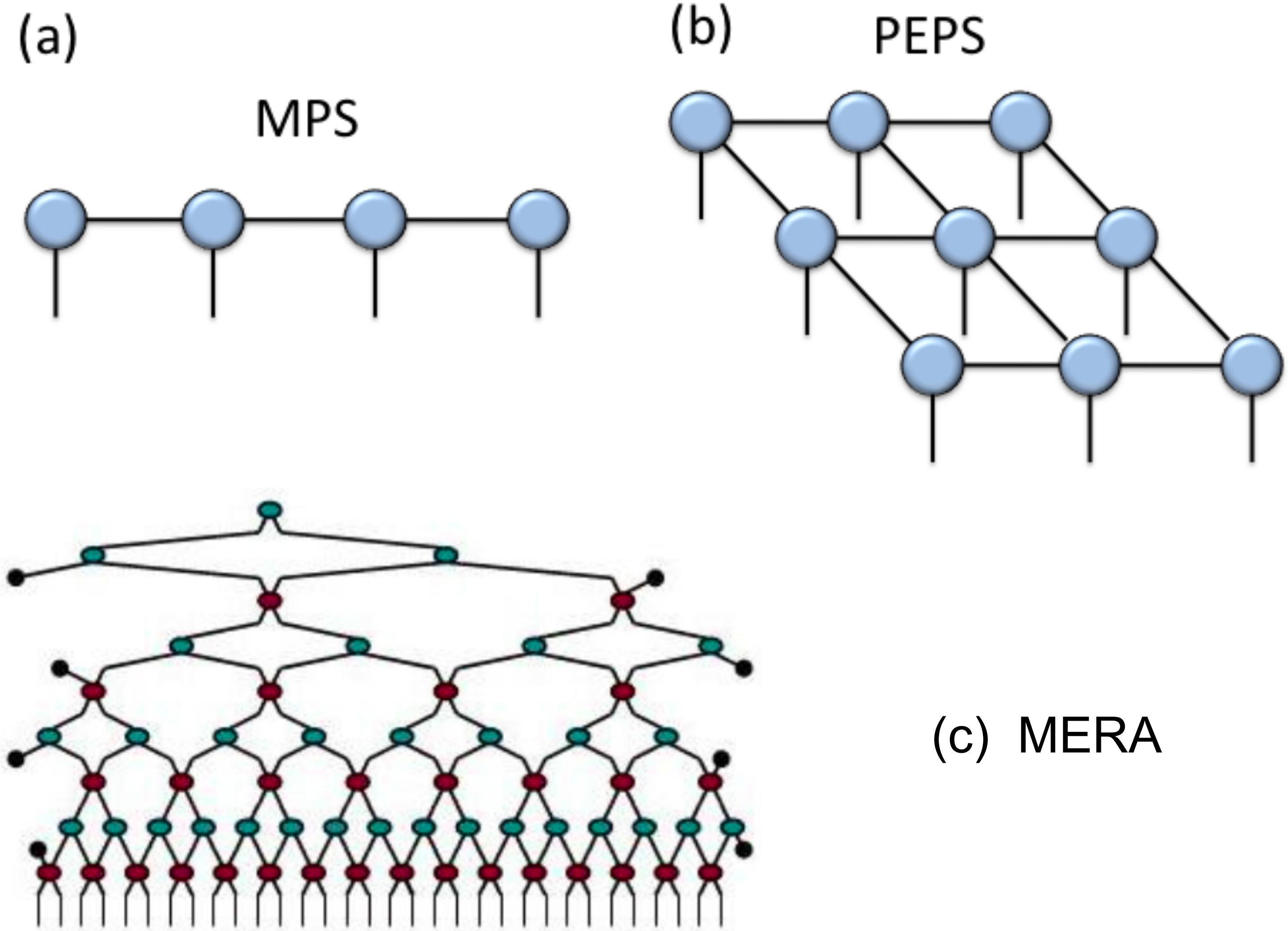
If $|\psi\rangle$ is the ground state of a local Hamiltonian

$$\rho_A = tr_B |\psi\rangle\langle\psi| \qquad S_A = -Tr\,\rho_A \log \rho_A$$




$$S_A \propto |\partial A| = |\partial B|$$

Basis of  Tensor Networks (MPS, PEPS, MERA,..)



Hastings theorem (2007): In 1D $S_A \leq$ constant

Conditions on the Hamiltonian:

-Finite range interactions

-Finite interaction strengths

-Existence of a gap in the spectrum

If satisfied -> Ground State can be well approximated by a MPS

Violations of the area law in 1D require one of the following

- non local interactions

- divergent interactions

- gapless systems

Examples are CFT and quenched disordered systems

$$S_A \propto \log|A|$$

Here we shall show stronger violations

$$S_A \propto |A|^{\chi}, \quad 0 < \chi \leq 1$$

In the Rainbow model and Correlated disorder model

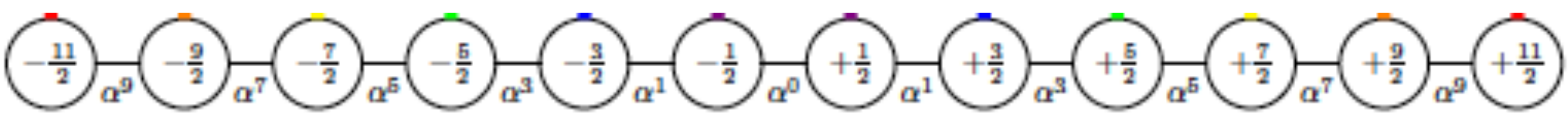
# PART I

# The rainbow model

Inhomogenous free fermion model in an open chain with 2L sites

$$H \equiv -\frac{J_0}{2} c^\dagger_{\frac{1}{2}} c_{-\frac{1}{2}} - \sum_{n=\frac{1}{2}}^{L-\frac{3}{2}} \frac{J_n}{2} \left[ c^\dagger_n c_{n+1} + c^\dagger_{-n} c_{-(n+1)} \right] + \text{h.c.}\,,$$

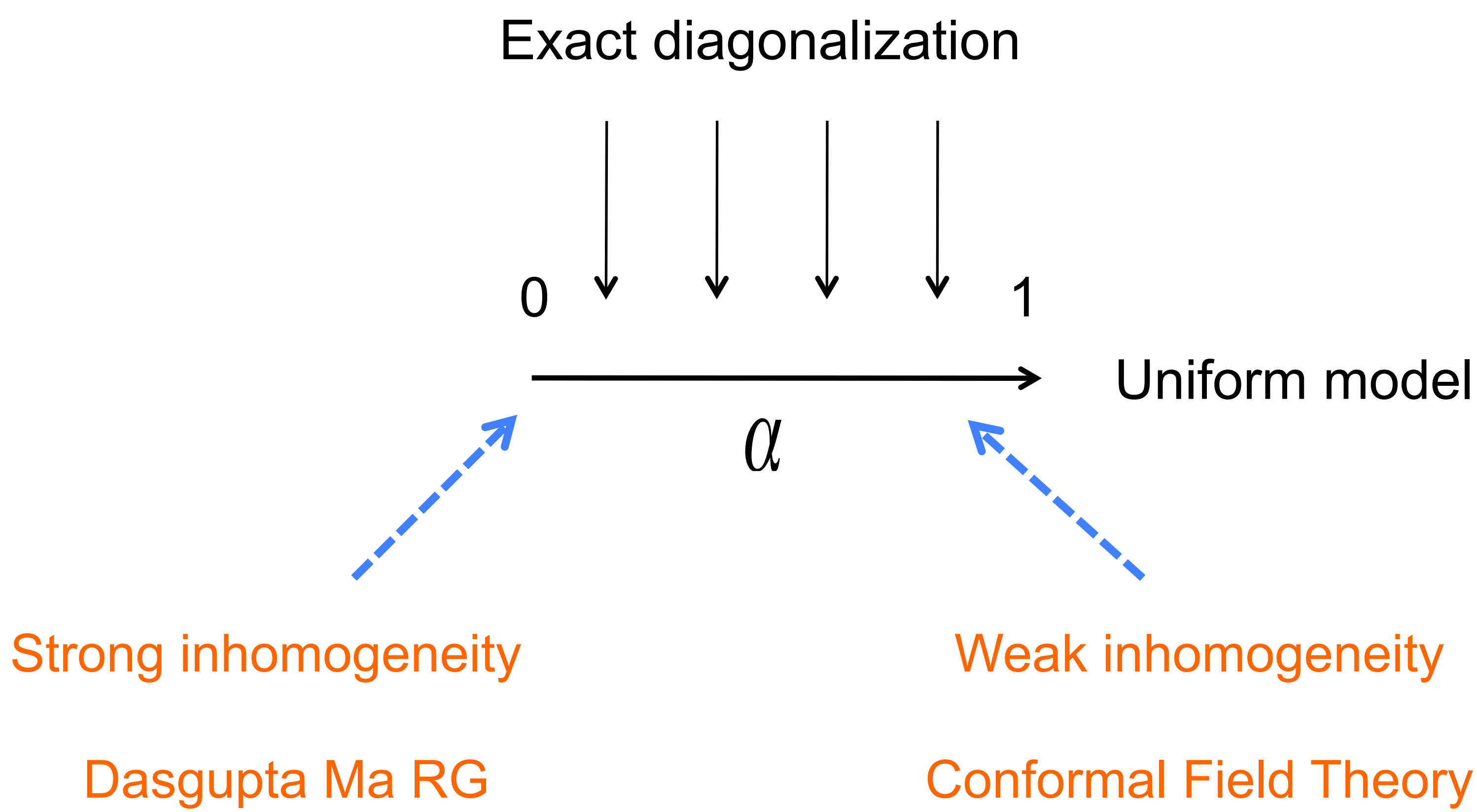
$$\begin{cases} J_0(\alpha) & = & 1, \\ J_n(\alpha) & = & \alpha^{2n},\ n = \frac{1}{2}, \ldots, L - \frac{3}{2}. \end{cases} \qquad 0 < \alpha \leq 1$$



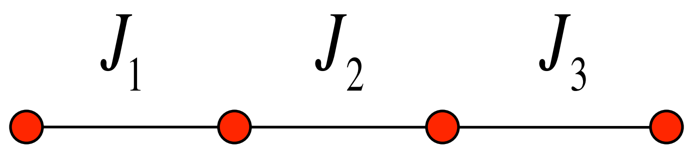

Introduced by Vitigliano, Riera and Latorre (2010)

Jordan-Wigner equivalent to an inhomogenous XX spin chains

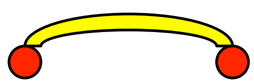
# Methods:


Exact diagonalization
0
1
Uniform model
$\alpha$
Strong inhomogeneity
Dasgupta Ma RG
Weak inhomogeneity
Conformal Field Theory

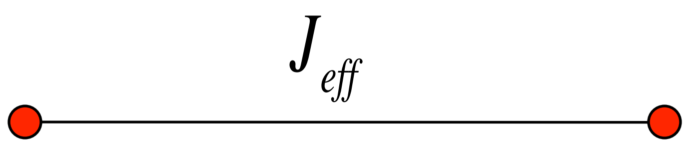
# Dasgupta-Ma RG (1980)




$$J_2 >> J_1, J_3$$



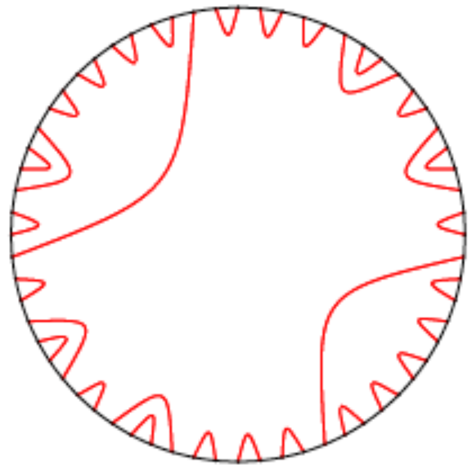
$$|bond\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$




$$J_{eff} = \frac{J_1 J_3}{J_2}$$

This method is asymptoticall exact for quenched disorder (Fisher, …)

Choosing the J's at random -> infinite randomness fixed point



Average entanglement entropy and Renyi entropies (Refael, Moore)

$$\left\langle S_L^{(n)} \right\rangle \approx \frac{\log 2}{3} \log L, \quad \forall n$$

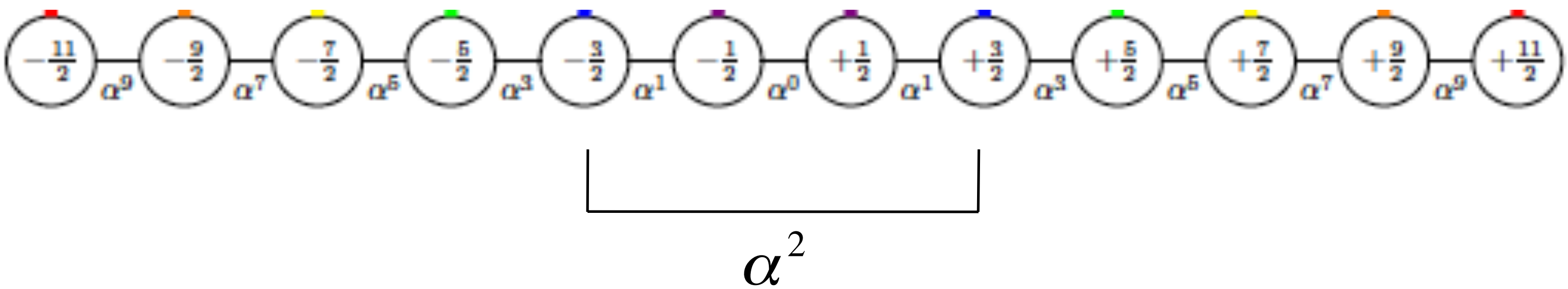
In CFT

$$S_L^{(n)} \approx \frac{c\,(n + 1/n)}{6} \log L$$

Strong inhomogeneity limit $\alpha \to 0^{+}$

The strongest bond is in the middle of the chain

Effective coupling: $$J_{eff} = \frac{J_{1/2}\,J_{1/2}}{J_0} = \alpha^2$$



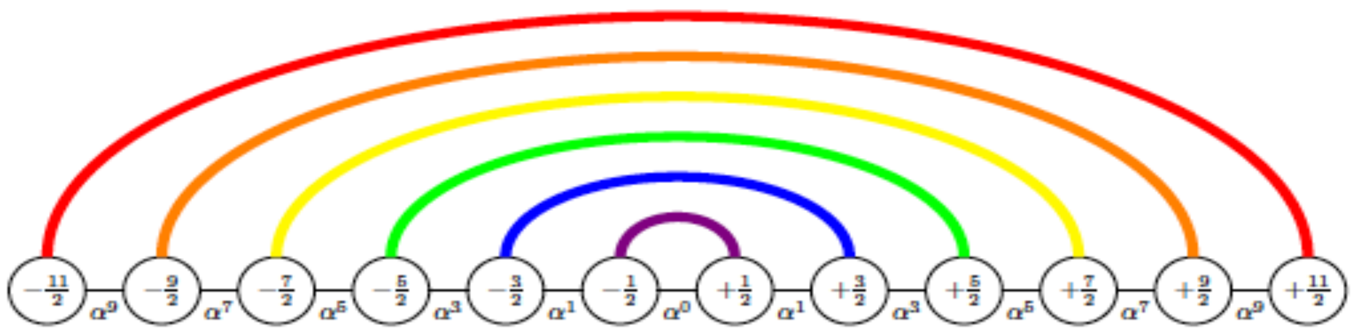

This new bond is again the strongest one because $\alpha^2 > \alpha^3$

Iterate the RG one finds the GS: valence bond state

$$\left|R_L\right\rangle = b_{1/2,-1/2}\, b_{3/2,-3/2} \cdots \left|0\right\rangle$$

$b_{i,j} =$ Bond operator

Ground state = Bell state with L pairs




It is exact in the limit $\alpha \to 0^+$

Density matrix of the rainbow state

$$\rho_B = Tr_{B^c} \left| R_L \right\rangle\left\langle R_L \right|$$

B: a block

$n_B$ number of bonds joining B with the rest of the chain

von Neumann entropy

$$S_B = -tr(\rho_B \log \rho_B) = n_B \log 2$$

Take B to be the half-chain then $n_B = L$

$$\boxed{S_B = L \log 2}$$

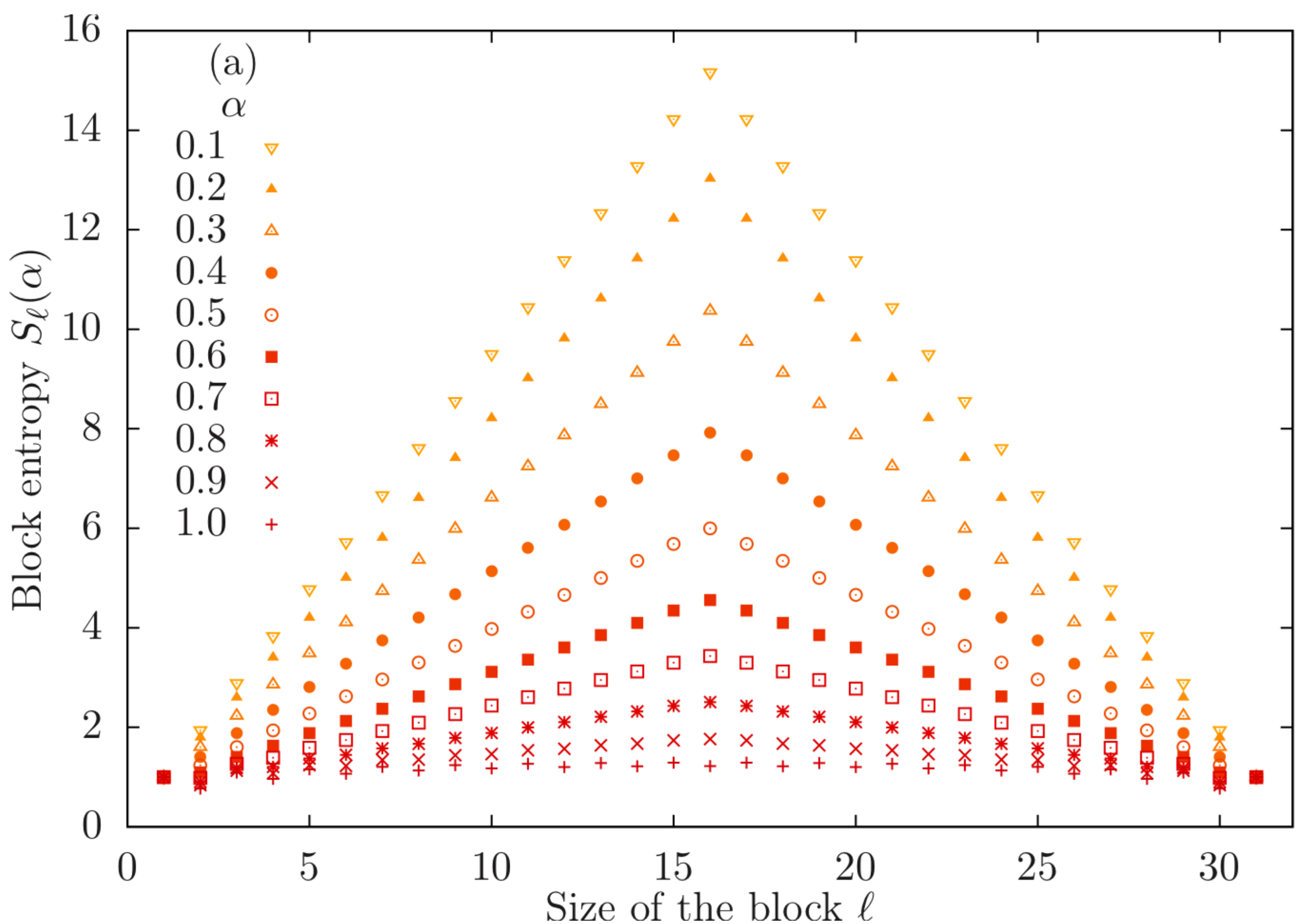
Area law of the entanglement entropy is maximally violated

But Hasting's theorem is satisfied

$$gap \propto \alpha^{2L} \to 0, \quad L \to \infty$$

# Exact diagonalization

System = $B_\ell \times B_{2L-\ell}$ Entanglement entropy of $B_\ell$



For $\alpha$ small and L large there is a violation of the area law that becomes a “volumen” law.

This agrees with the analysis based on the Dasgupta-Ma RG

What about the limit $\alpha \to 1^{-}$ near the critical model?

# Field Theory: continuum limit of the rainbow

Uniform model $\alpha = 1$

Fast-slow separation of degrees of freedom

$$\frac{c_n}{\sqrt{a}} \approx e^{i\pi n/2}\,\psi_L(x) + e^{-i\pi n/2}\,\psi_R(x), \quad x = n\,a$$

$$H \approx H_R + H_L = i\int_{-L}^{L} (\,\overline{\psi}_R\,\partial_x \psi_R - \overline{\psi}_L\,\partial_x \psi_L)$$

Massless Dirac fermion with open boundary conditions

Non uniform model $\boxed{\alpha = e^{-h/2}}$

$$\frac{c_n}{\sqrt{a}} \approx e^{i\pi n/2}\, \psi_L(x) + e^{-i\pi n/2}\, \psi_R(x), \quad x = n\,a \qquad \mathcal{L} = aL.$$

$$H \simeq ia \int_{-\mathcal{L}}^{\mathcal{L}} dx\, e^{-\frac{h|x|}{a}} \left[ \psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L - \frac{h}{2a} \mathrm{sign(x)} (\psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L) \right]$$

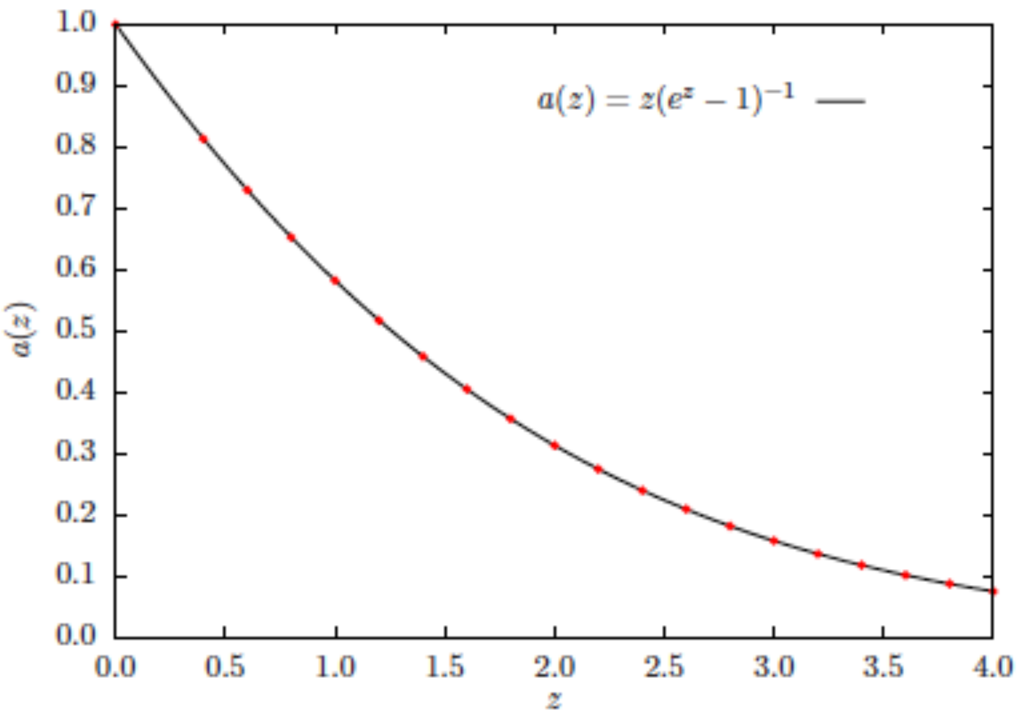
Left and right components are decoupled in the bulk but they mixed at the boundary

$$\psi_R(\pm\mathcal{L}) = \mp i\, \psi_L(\pm\mathcal{L})$$

# Single particle spectrum

$$ie^{-h|x|}\left[\partial_x \mp \frac{h}{2}\,\mathrm{sign(x)}\right]\psi_{R,L}(x) \quad = \quad \pm E\psi_{R,L}(x),$$

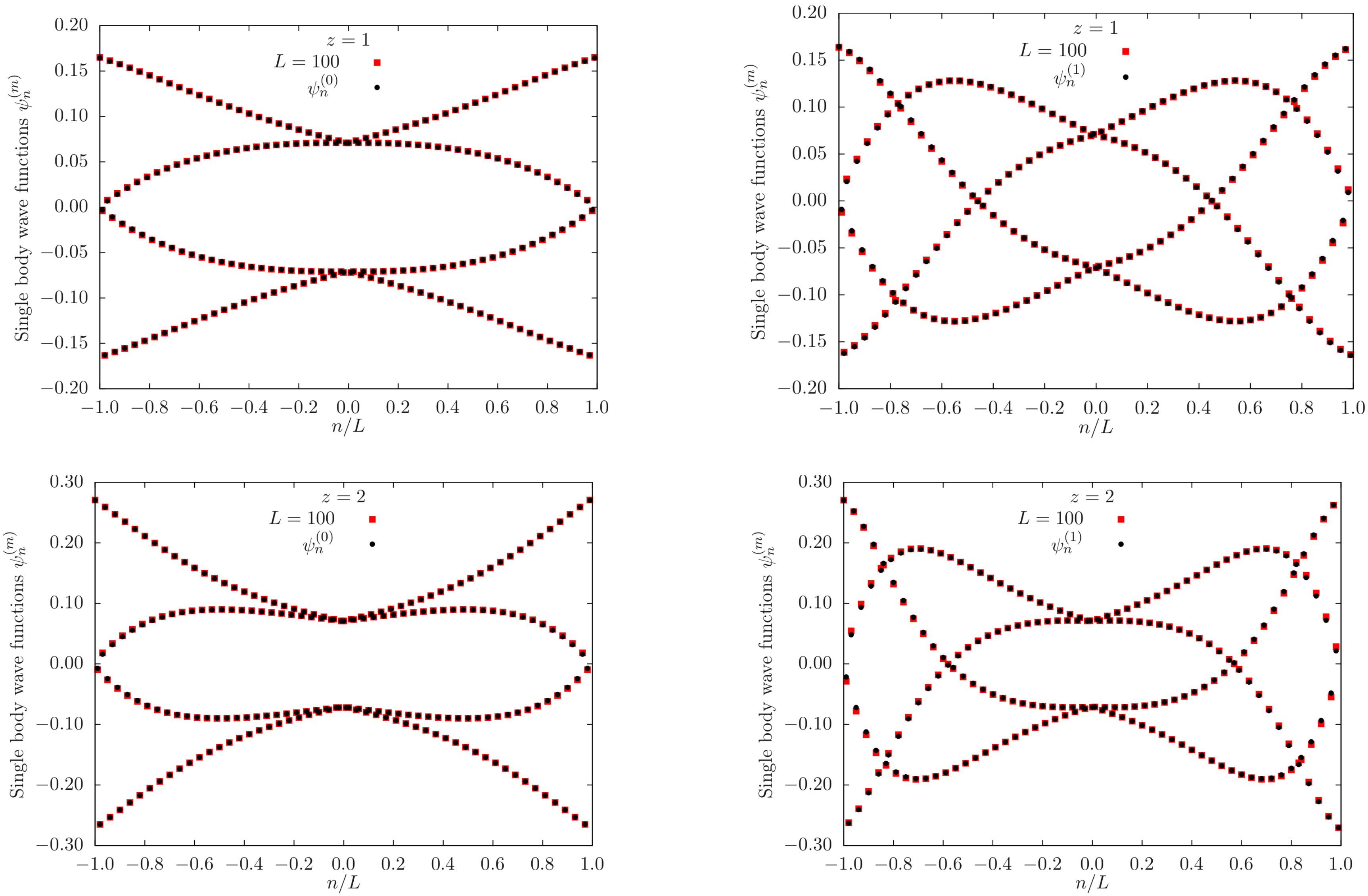
$$E_m \approx a(z)\frac{\pi(m+1/2)}{2L}, \quad m = 0, \pm 1, \cdots$$




$$a(z) = \frac{z}{e^z - 1}, \quad z = hL$$

# Eigenfunctions

$$\psi_n^{(m)} \simeq e^{h|n|/2} \cos\left[\frac{\pi(n-m)}{2} + \mathrm{sign(n)}\frac{\pi(m+1/2)}{2}\frac{e^{h|n|}-1}{e^{hL}-1}\right]$$



## Mapping the rainbow H into the free fermion H

$$\tilde{x} = sign(x)\frac{e^{h|x|}-1}{h}$$

$$H \approx i\int_{-\tilde{L}}^{\tilde{L}} d\tilde{x}\left[\tilde{\psi}_R^* \,\partial_x \tilde{\psi}_R^* - \tilde{\psi}_L^* \,\partial_x \tilde{\psi}_L^*\right]$$

$$\tilde{L} = \frac{e^{hL}-1}{h} \rightarrow L \quad as \quad h \rightarrow 0$$

For x>0 $\tilde{x} = \dfrac{e^{h\,x} - 1}{h}$ is a conformal map

Periodicity in imaginary time $x \rightarrow x + i\,\beta$

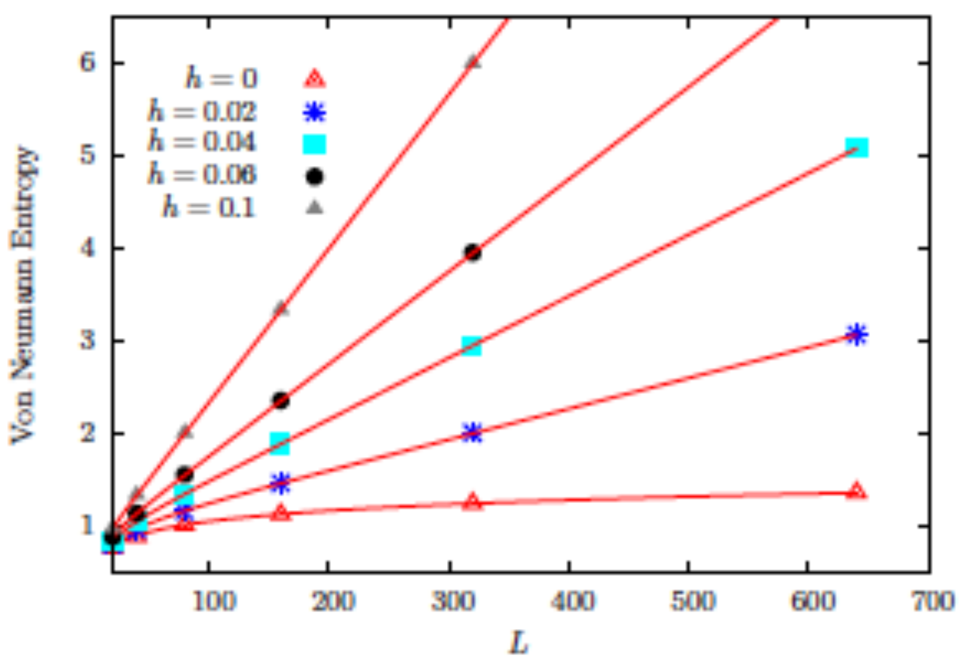
Effective "temperature" $T = \beta^{-1} = \dfrac{h}{2\pi}$

Heuristic trick to get the entanglement entropy of half-open-chain

In a CFT $$S_L^{CFT} \approx \frac{c}{6}\log L + c_1'$$

For the non uniform case the map suggests to replace $L \rightarrow \tilde{L}$

$$S_L^{Rainbow}(h) \approx \frac{c}{6}\log\frac{e^{hL}-1}{h} + c_1'$$



## Volumen law as a thermal effect

$$hL >> 1 \rightarrow S_L^{Rainbow}(h) \approx \frac{c}{6} hL$$

Compare with entropy of an open CFT at finite temperature $T = 1/\beta$

$$S_L^{CFT}(\beta) \approx \frac{\pi c L}{3\beta}, \quad L >> \beta$$

$$S_L^{Rainbow}(h) \approx S_L^{CFT}(T) \rightarrow T_R = \frac{h}{2\pi}$$

The inhomogeneity coupling h can be seen as "temperature"

## Action of the rainbow : a free fermion in curved space-time

$$S \approx \int_{-\infty}^{\infty} dt \int_{-\tilde{L}}^{\tilde{L}} d\tilde{x}\, e^{\sigma} \left[ \tilde{\psi}_R^* \overleftrightarrow{\partial}_{\bar{z}} \tilde{\psi}_R + \tilde{\psi}_L^* \overleftrightarrow{\partial}_z \tilde{\psi}_L \right]$$

Weyl factor

$$z = \tilde{x} + i\,t \in [-\tilde{L}, \tilde{L}] \times R$$

Spacetime metric $\quad ds^2 = e^{\sigma}\, dz\, d\bar{z}, \quad e^{\sigma} = e^{-h|x|}$

$$ds^2 = \frac{d\tilde{x}^2 + dt^2}{(1 + |h\,\tilde{x}|)^2}$$

Poincaré metric with curvature

$$R = -h^2$$

Using recent results by Dubail, Stéphan, Viti and Calabrese we can compute the entanglement entropies of arbitray blocks

$$A = [-L, x], \quad B = [x, L]$$

Half block is x=0

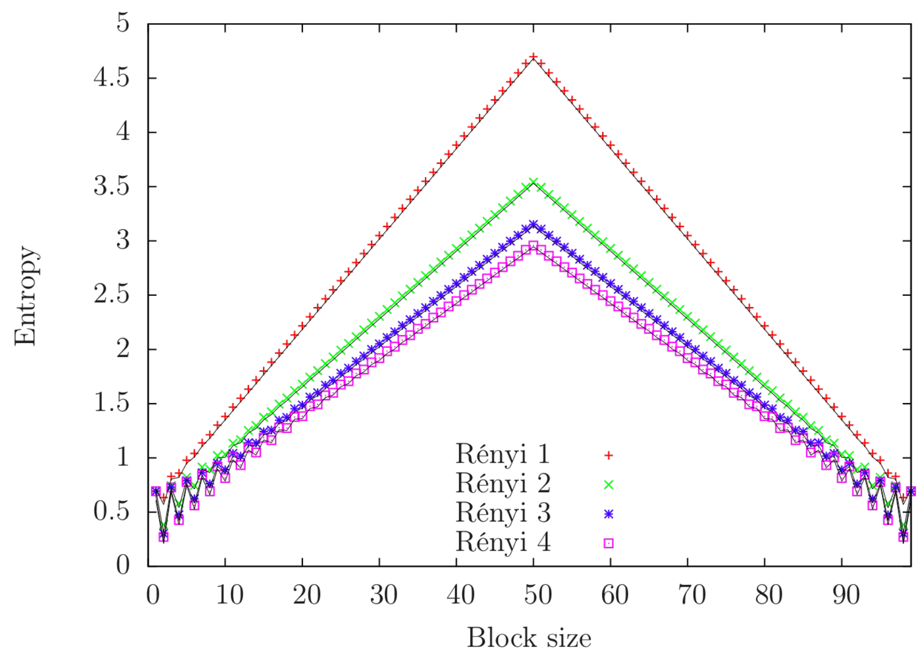
$$S_n = S_n^{CFT} + S_n^{osc}$$

$$S_n^{CFT} = \frac{n+1}{12n} \log Y(x),$$

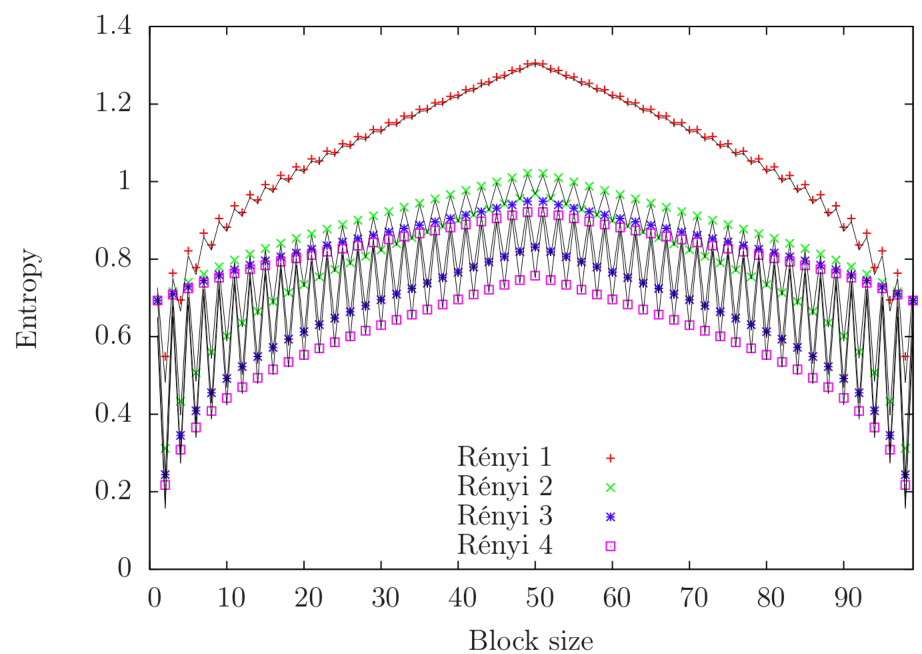
$$S_n^{osc} = f_n (-1)^{x+1} Y(x)^{-1/n} + \frac{E_n}{2}$$

$$Y(x) = 8 e^{-h|x|} \frac{e^{hL} - 1}{\pi h} \cos\left( \frac{\pi}{2} \frac{e^{h|x|} - 1}{e^{hL} - 1} \right)$$

$$2L = 100$$
$$h = 0.5$$




$$2L = 100$$
$$h = 0.05$$



# Entanglement spectrum for the half chain

$$\rho_B = e^{-\beta\, H_{CFT}} \qquad \text{with} \qquad \beta = \frac{2\pi}{h}$$

$$H_{CFT} \approx \frac{\pi}{L} \sum_p p\, b_p^* \, b_p$$

Hamiltonian of a free fermion in a half chain

The density matrix is thermal !!

The GS of the rainbow as a thermofield double

$$|Rainbow\rangle = \sum_n e^{-\frac{\beta}{2} E_n^{CFT}} |n\rangle_L |n\rangle_R$$

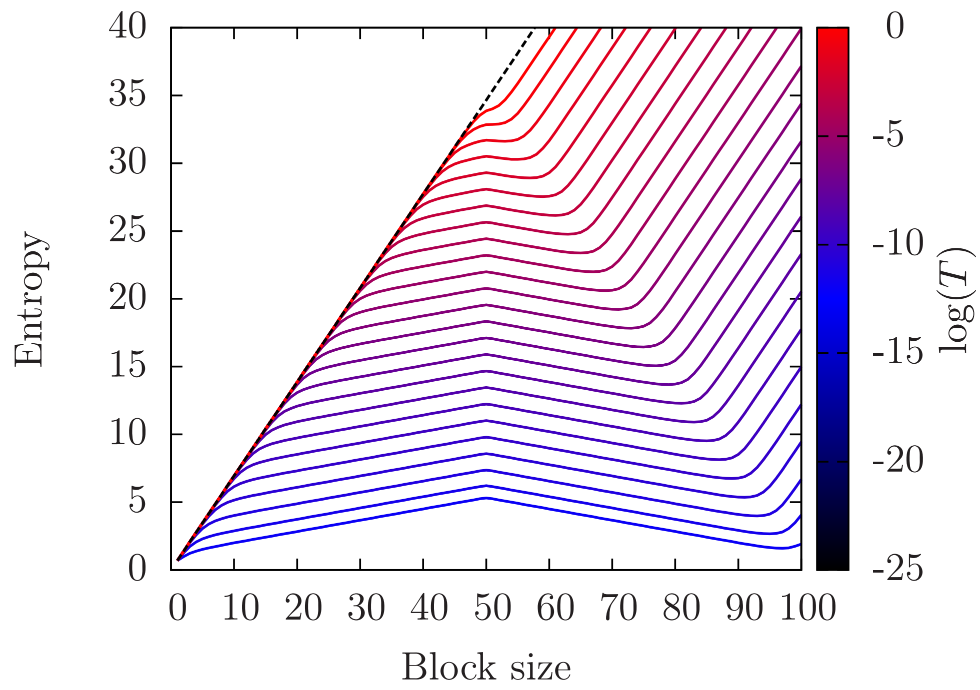
## Similarities between the Rainbow and the Unrhu effect

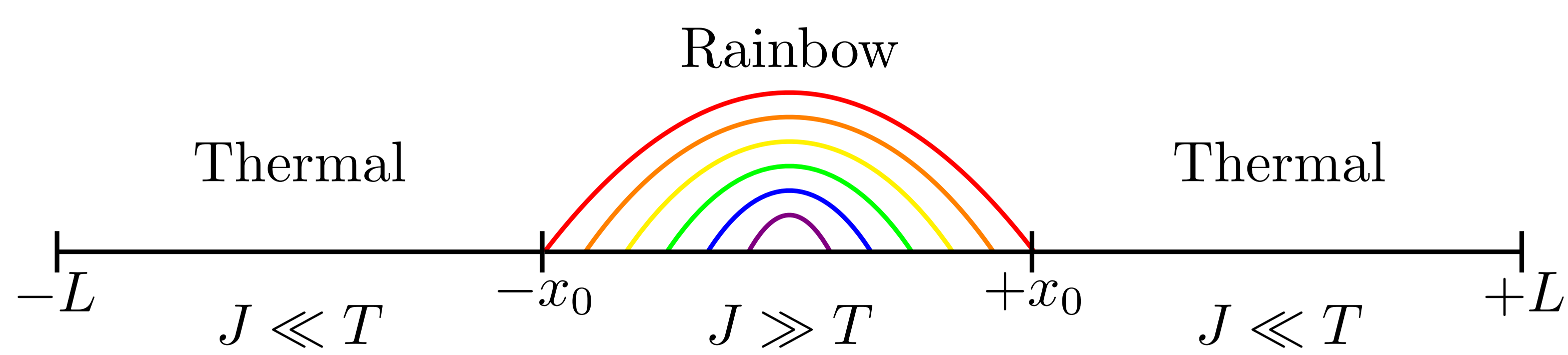
A observer moving in Minkowsky spacetime with acceleration $a$ detects a thermal bath with temperature

$$T_U = \frac{a}{2\pi} \qquad\qquad T_R = \frac{h}{2\pi}$$

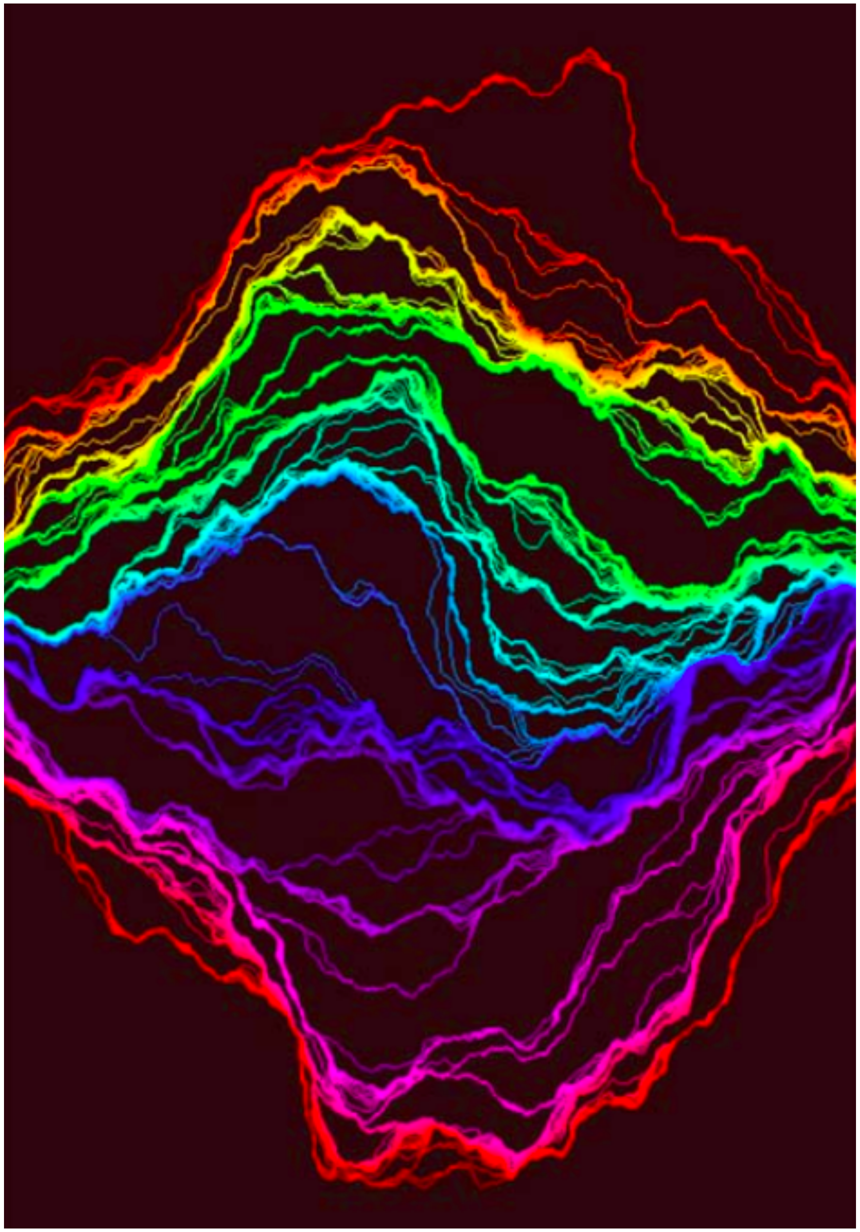
The Minkowski vacuum is a thermofield double in the observer variables

# The rainbow at finite temperature



# PART II

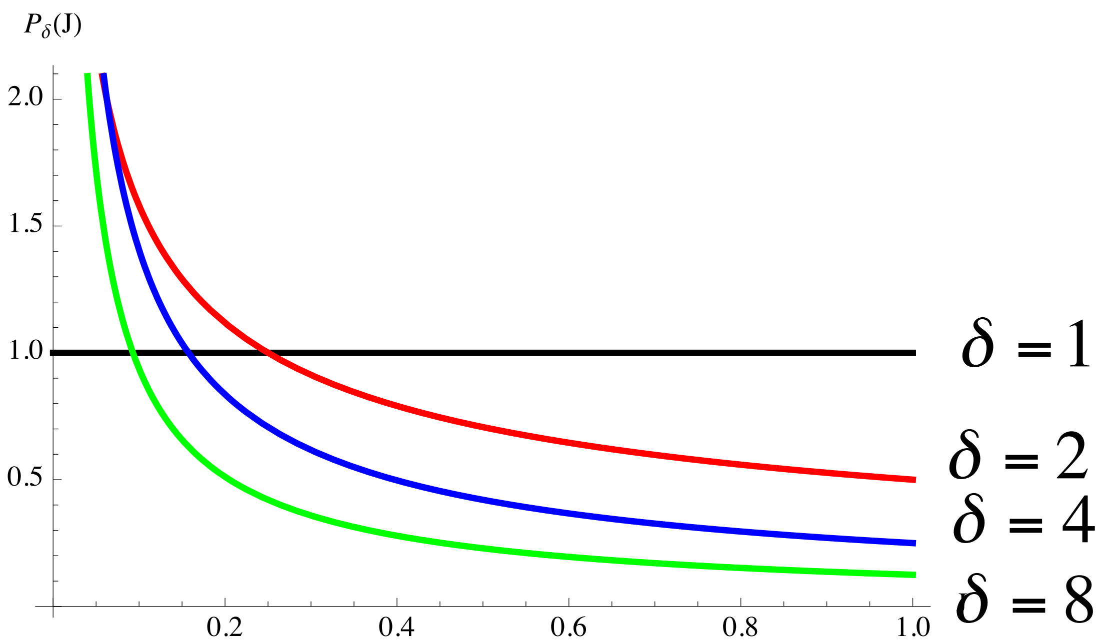


# Random spin chains

$$H = \sum_{i=1}^{L} J_i \left( S_i^x \; S_{i+1}^x + S_i^y \; S_{i+1}^y \right)$$

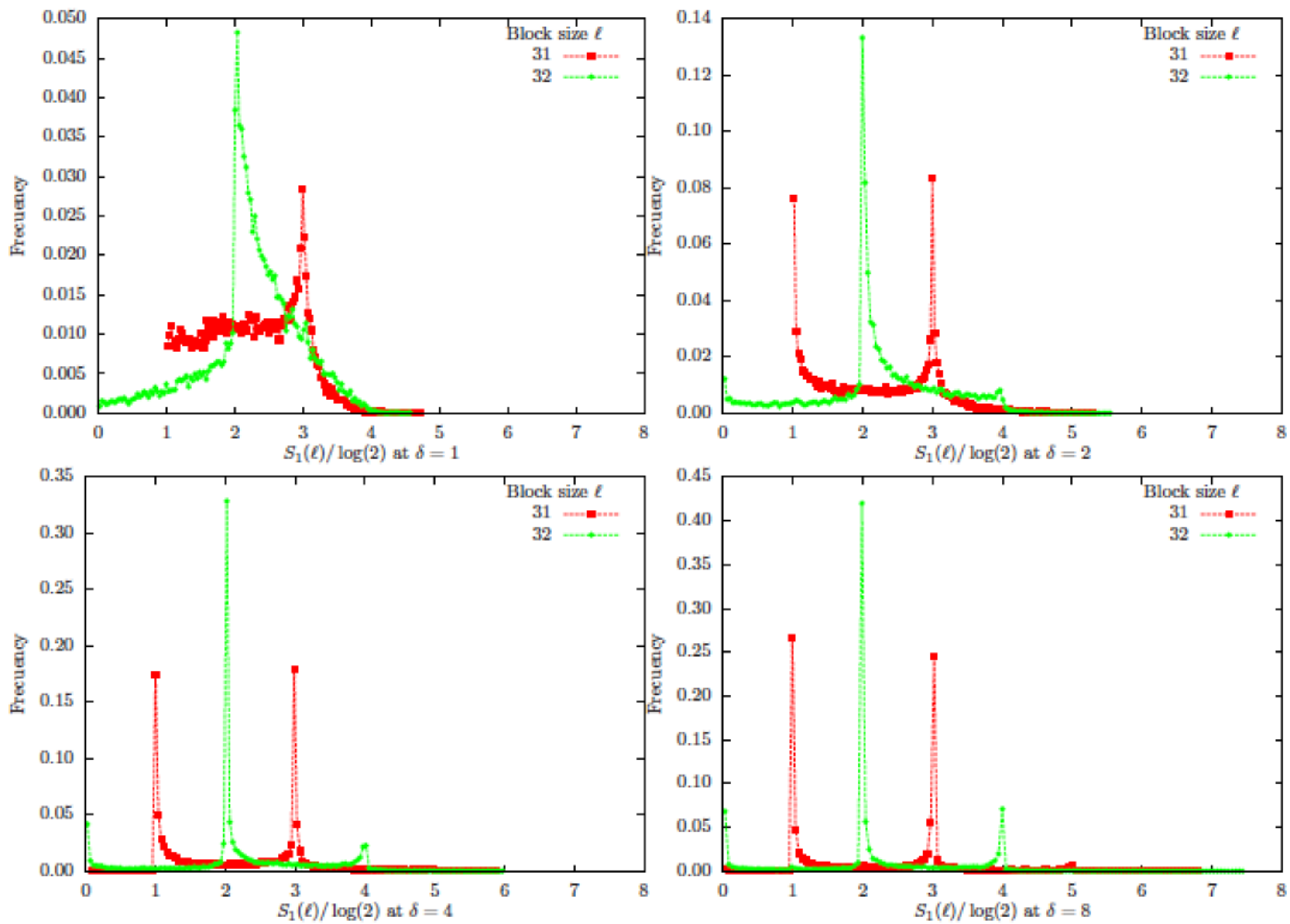
Choose random couplings with probablity distribution

$$P_\delta(J_i) = \frac{1}{\delta} J_i^{-1+1/\delta} \;, \quad 0 < J_i < 1, \quad 1 < \delta < \infty$$



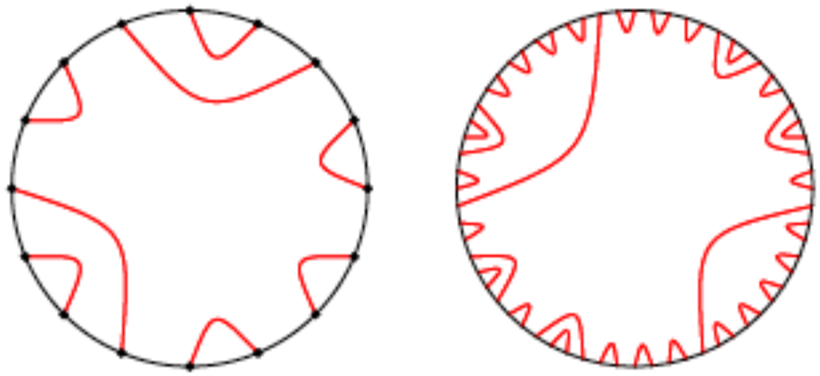
Average entanglement entropy $\langle S_\ell \rangle = \langle -tr\,\rho_\ell \log \rho_\ell \rangle$

$L = 64, \quad 2\times 10^4$ realizations $\quad \ell = 32$ (green) $\ell = 31$ (red)




Laflorencie (2005), Ramirez, …

In the limit $\delta \longrightarrow \infty$ the ground state is a valence bond



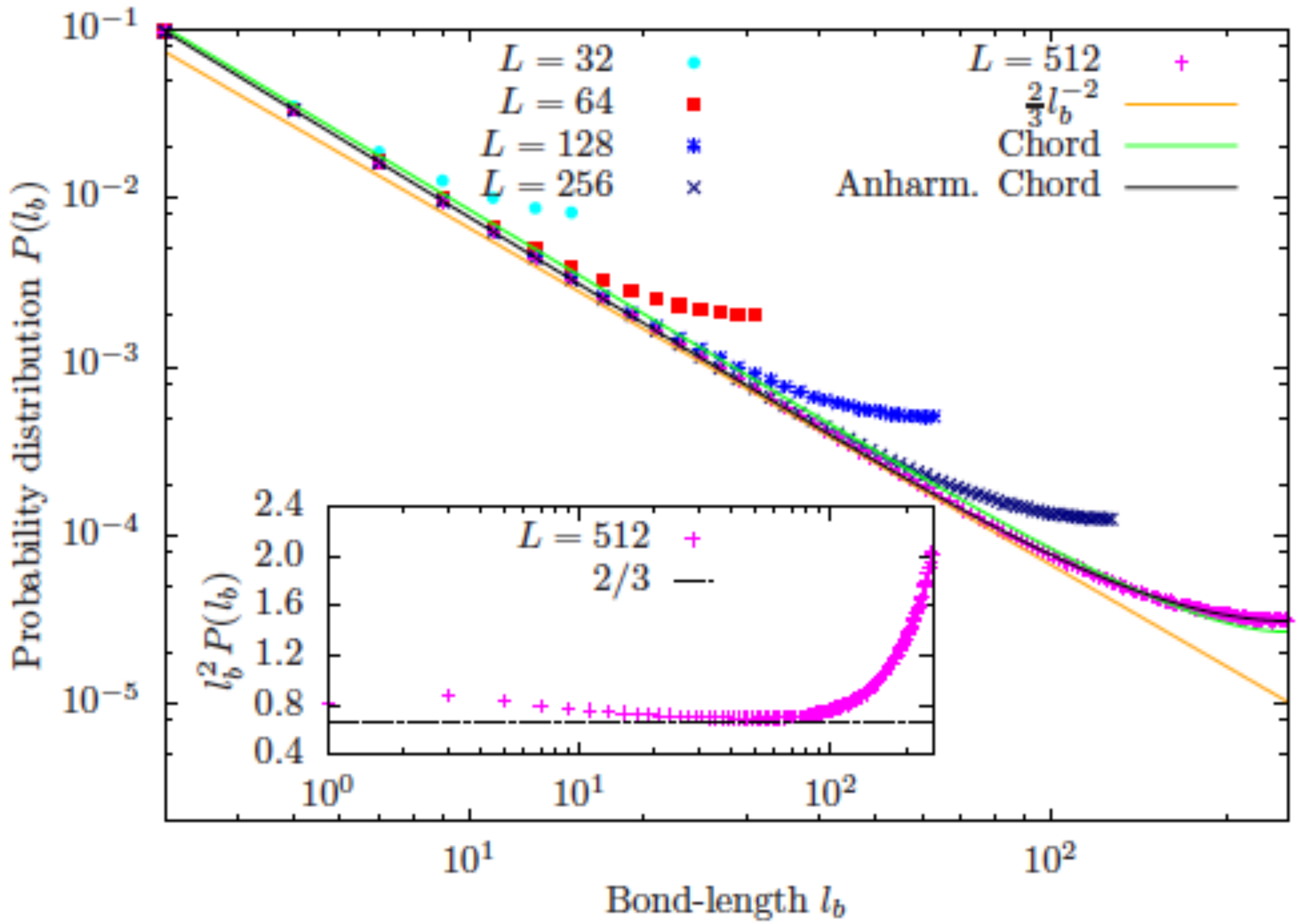
Infinite random fixed point under the Dasgupta-Ma-Fisher RG ***(IRFP)***

$$\langle S_\ell \rangle = \frac{\log 2}{3} \log\left[\frac{L}{\pi} Y\left(\frac{\pi \ell}{L}\right)\right] + c'_1$$

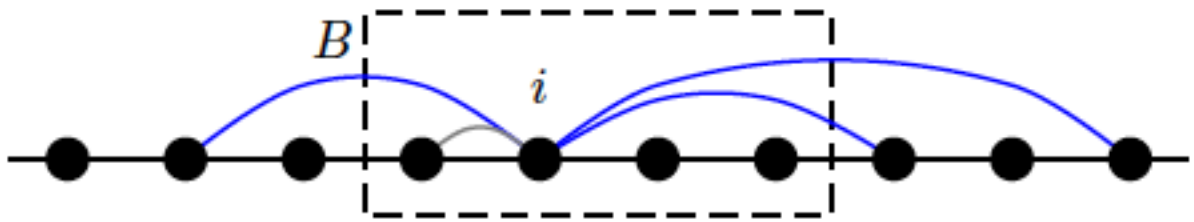
$$Y(x) \approx (1+k_1)\sin(x) - \frac{k_1}{3}\sin(3x), \quad k_1 = 0.1025$$

Fagotti, Calabrese,Moore (2011), Ramirez et al (2014)

Bond length probability $P(l_b) \approx \frac{2}{3} l_b^{-2} \; (l_b : odd)$ (Hoyos, et al 2007)



# Entanglement entropy from bond probabilities




$$\langle S_\ell \rangle = \log 2 \left[ \sum_{l_b=1}^{\ell} l_b\, P(l_b) + \ell \sum_{l_b=\ell+1}^{L/2} P(l_b) \right] \approx \frac{\log 2}{3} \log \ell$$

# Correlated random spin chains

(J. Rodríguez-Laguna, S. Santalla,G. Ramirez, GS (2007)

$$H = \sum_{i=1}^{L} J_i (S_i^x \; S_{i+1}^x + S_i^y \; S_{i+1}^y)$$

$$t_i = -\log J_i \qquad \tilde{J}_i = \frac{J_{i-1} J_{i+1}}{J_i} \longrightarrow \tilde{t}_i = t_{i+1} + t_{i+1} - t_i$$

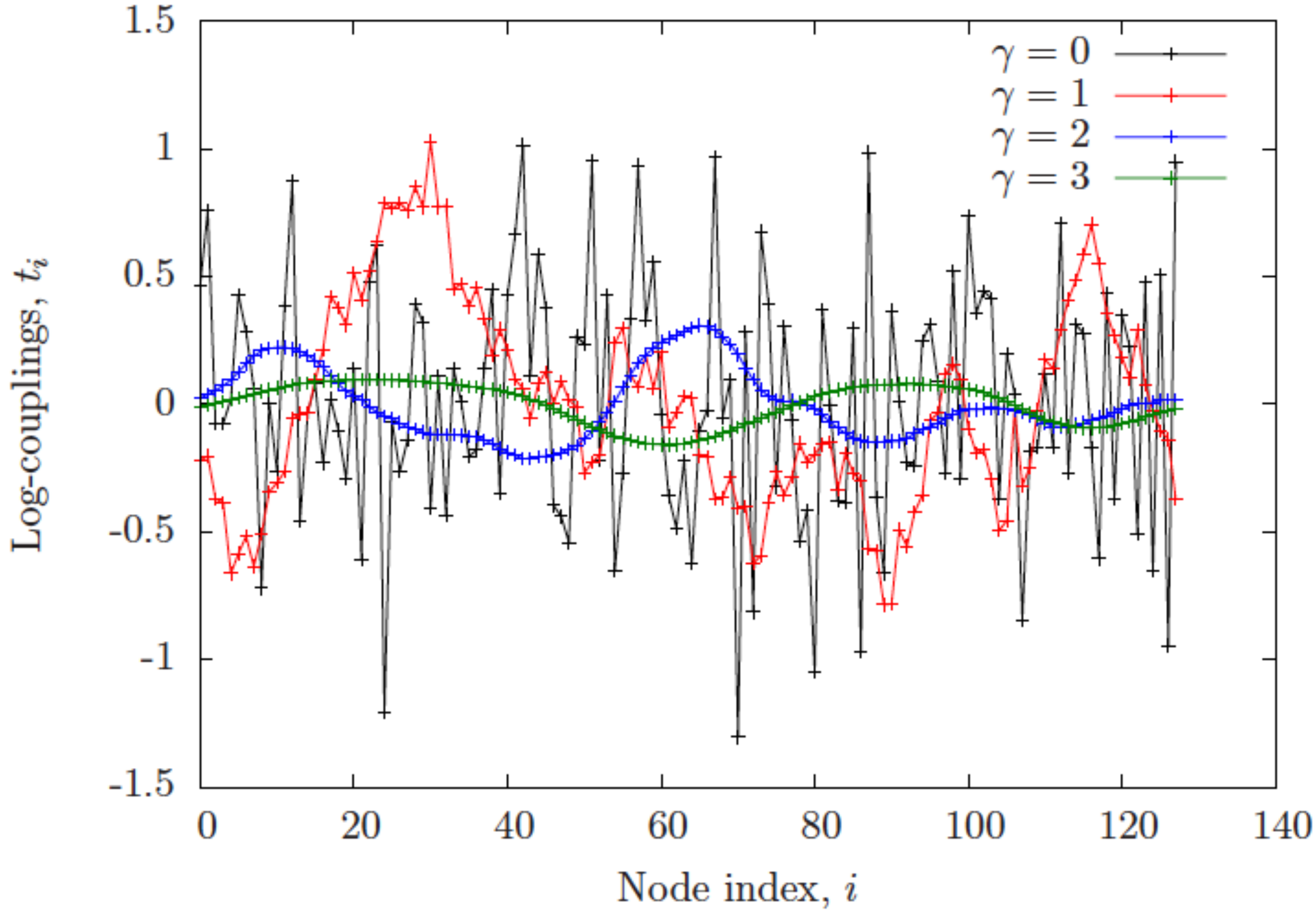
Correlated random hoppings

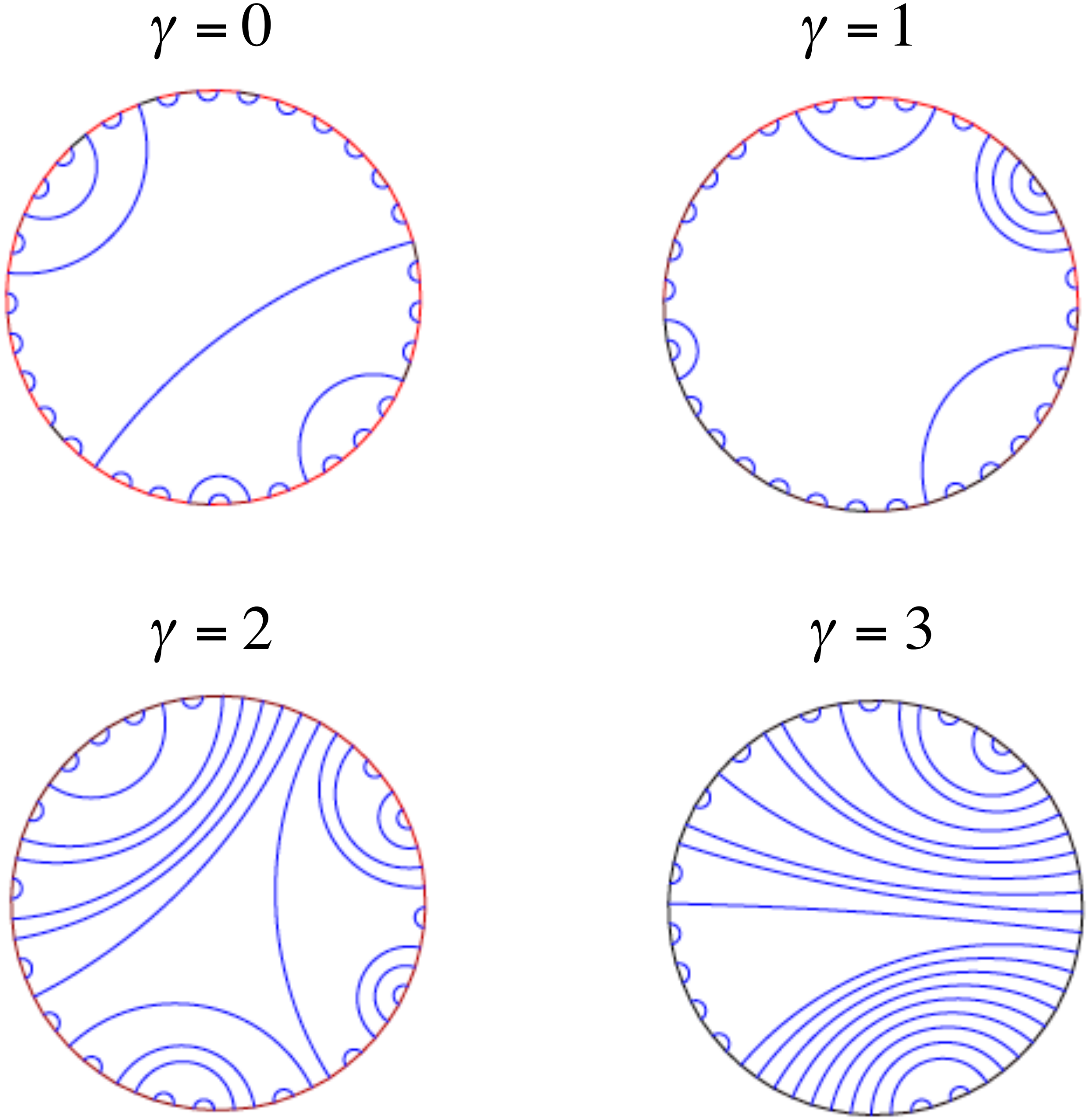
$$t_j = \sum_k A_k \sin(j k + \phi_k), \quad k = \frac{2\pi n}{N}, n = 1, \ldots, N$$

$$A_k = u_k \, k^{-\gamma}$$

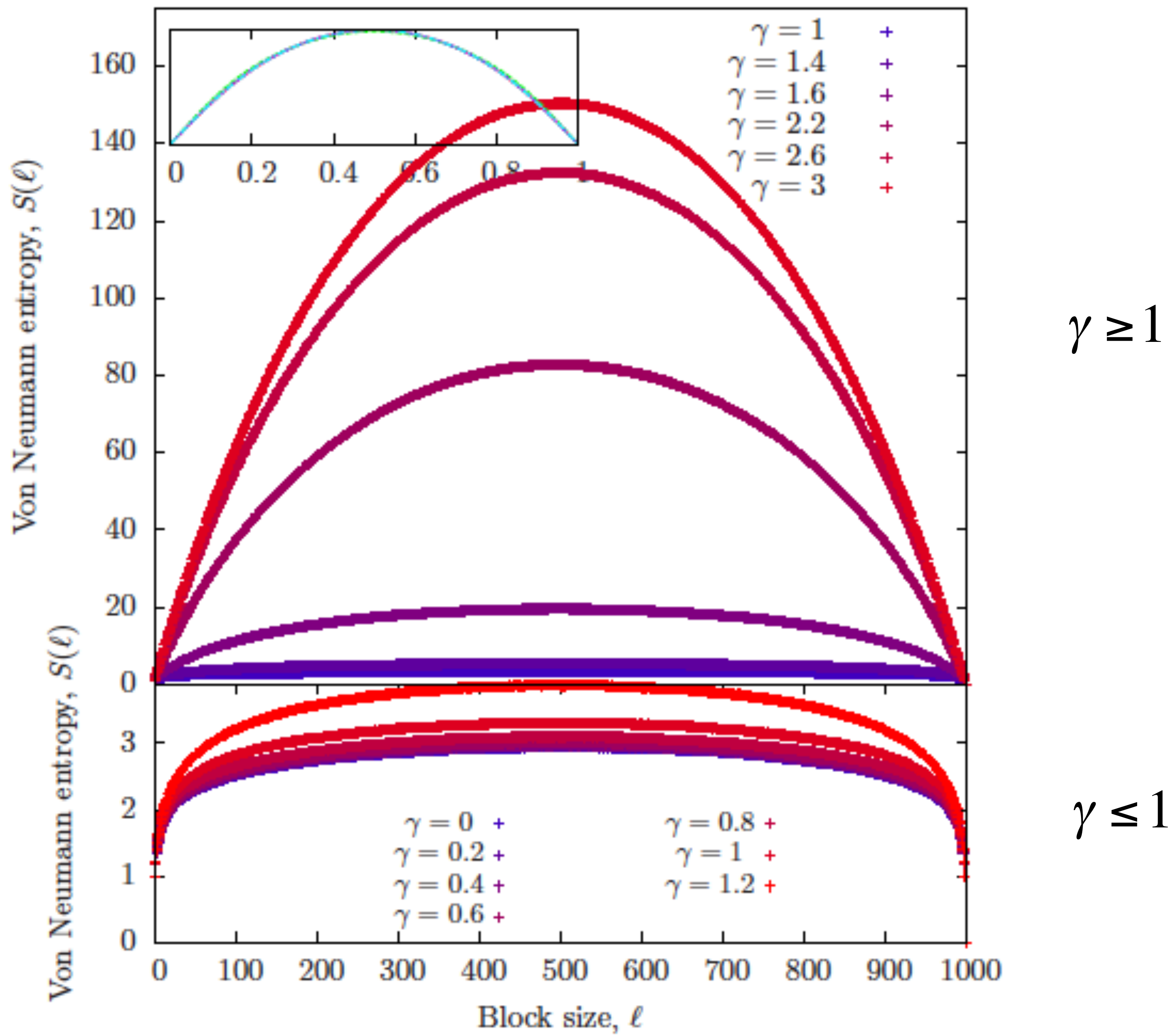
$$P(\phi_k) = \frac{1}{2\pi}, \quad \phi_k \in [0, 2\pi] \quad , \quad P(u_k) = \frac{1}{\sqrt{2\pi}} e^{-u_k^2/2}, \; u_k \in R$$

Samples of  t's for N= 128 and increasing values of  $\gamma$





$\gamma = 0$
$\gamma = 1$
$\gamma = 2$
$\gamma = 3$

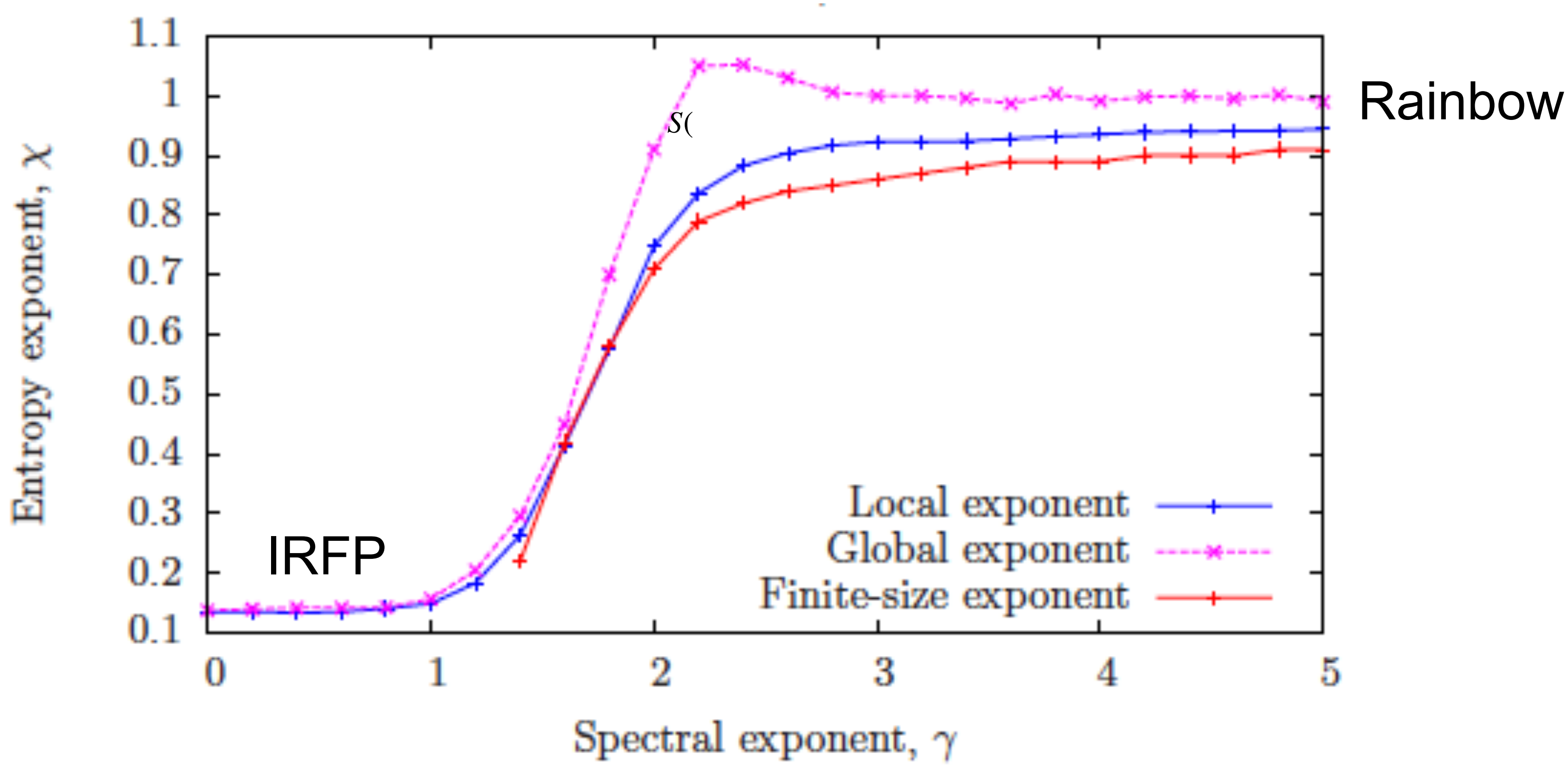


Von Neumann entropy, $S(\ell)$
Block size, $\ell$
$\gamma = 1$
$\gamma = 1.4$
$\gamma = 1.6$
$\gamma = 2.2$
$\gamma = 2.6$
$\gamma = 3$
$\gamma = 0$
$\gamma = 0.2$
$\gamma = 0.4$
$\gamma = 0.6$
$\gamma = 0.8$
$\gamma = 1$
$\gamma = 1.2$
$\gamma \geq 1$
$\gamma \leq 1$

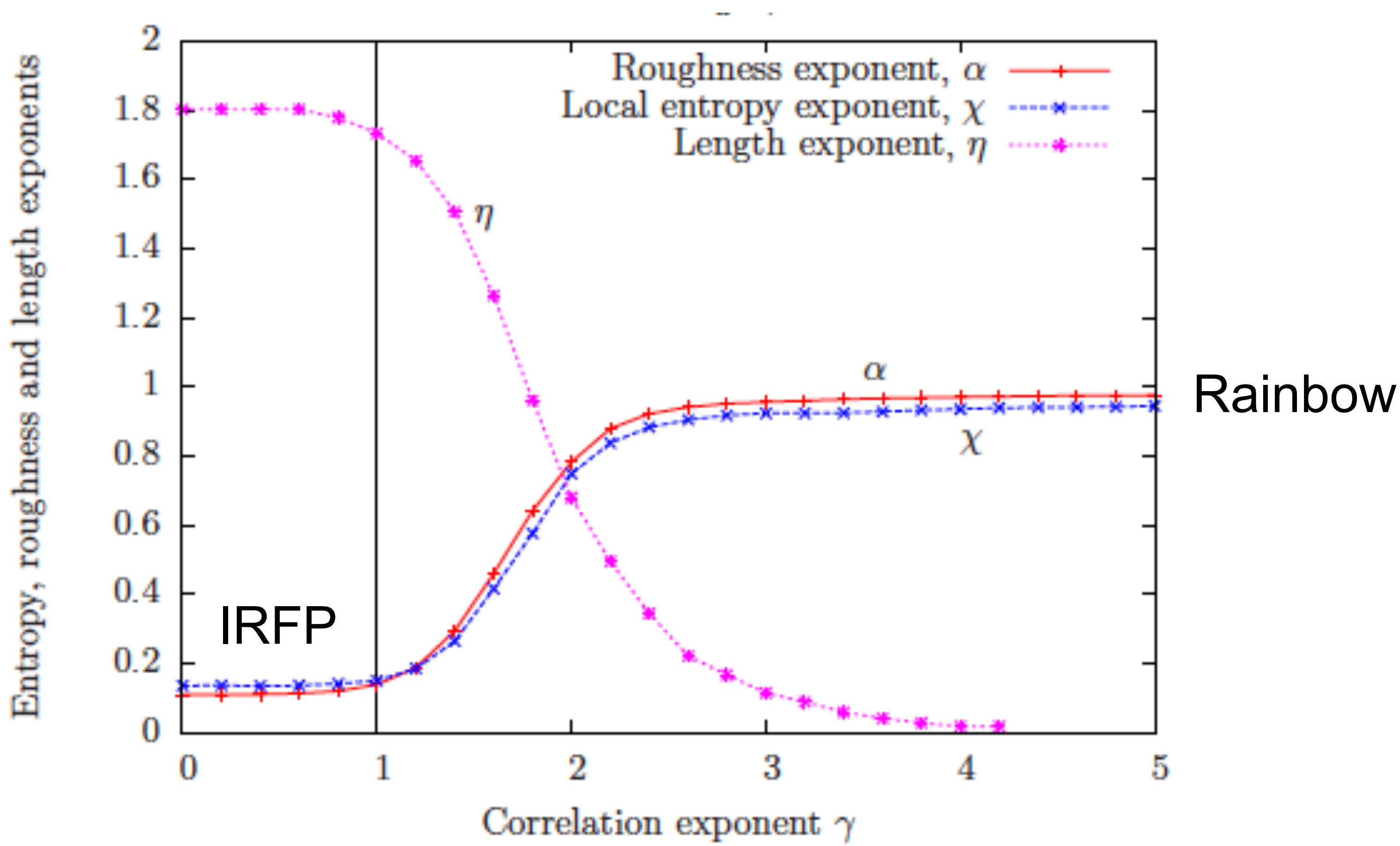
Entanglement entropy

$$\langle S_\ell \rangle \approx A\left[\frac{L}{\pi} Y\left(\frac{\pi \ell}{L}\right)\right]^{\chi}$$

$$Y(x) = \sin(x) + \sum_{n \geq 1} \alpha_n \sin((2n+1)x)$$



Bond length probability $P(l_b) \propto l_b^{-\eta}$



Use $P(l_b) \propto l_b^{-\eta}$

$$\langle S_\ell \rangle = \log 2 \left[ \sum_{l_b=1}^{\ell} l_b P(l_b) + \ell \sum_{l_b=\ell+1}^{L/2} P(l_b) \right] \propto \ell^{2-\eta}$$

$\langle S_\ell \rangle \propto \ell^{\chi}$ Suggests $\chi + \eta = 2$

But this relation does not hold for $\gamma > 1$

Reflecting that bonds are correlated

# Prospects

- Quenches: uniform <-> rainbow
- Bethe ansatz for the rainbow models (XXZ,…)
- Holographic dual of the rainbow. Connection AdS/CFT
- Random + Rainbow = Randbow

## Rainbow works

- From conformal to volume-law for the entanglement entropy in exponentially deformed critical spin 1/2 chains
 G. Ramírez, J. Rodríguez-Laguna, GS (2015)

- Entanglement over the rainbow
 G. Ramírez, J. Rodríguez-Laguna, GS (2015).

- More on the rainbow chain: entanglement, space-time geometry and thermal states
J. Rodríguez-Laguna, J. Dubail, G. Ramírez, P. Calabrese, GS (2016)

## Random works

- Entanglement in low-energy states of the random-hopping model
 G. Ramírez, J. Rodríguez-Laguna, GS (2014)

- Entanglement in correlated random spin chains, RNA folding and kinetic roughening
 J. Rodríguez-Laguna, S. Santalla, G. Ramírez, GS (2016).

# Thank You!!!