# Extracting universal data from 2d chiral topological states: a CFT perspective

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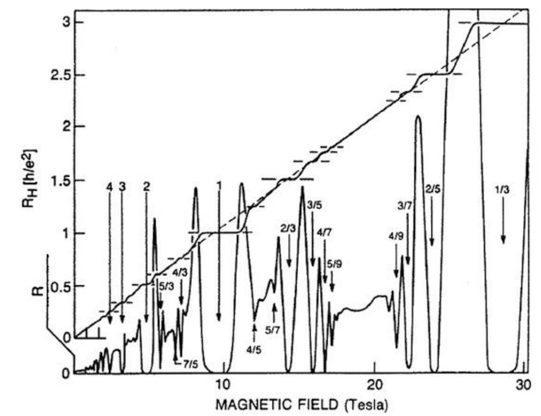
# Outline

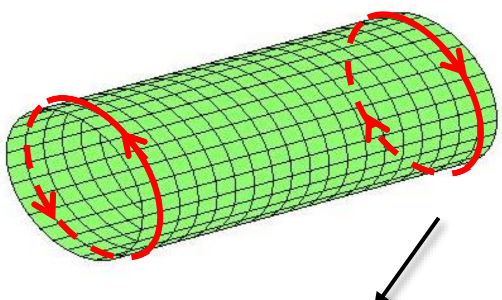
- Quick sketch of chiral conformal field theory (CFT)
- Topological entanglement entropy
- Entanglement spectrum
- Momentum polarization (and perhaps also Klein twist)
  => tomorrow's talk
- Summary

# 2d chiral topological states

- 2d topological states which break time-reversal sysmmetry
  - ➢ Integer and fractional quantum Hall states
  - ➢ p+ip superconductor
  - ➢ Chiral spin liquids
  - ➢ …



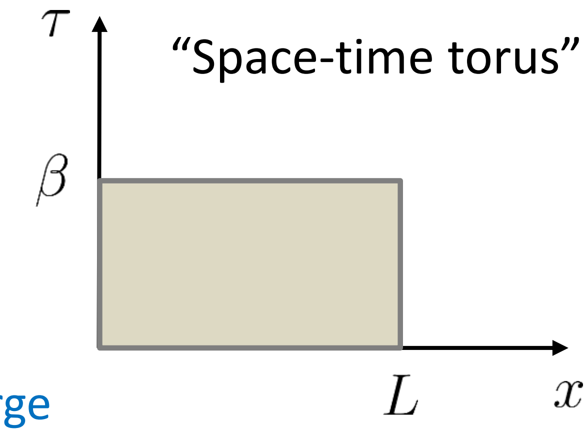

- 2d chiral topological states have deep relation with CFT
  - ➢ Gapless edge excitations
  - ➢ Trial wave functions
  - ➢ Fusion rules of anyons
  - ➢ …




- Extracting topological data is basically extracting information of the chiral CFT.

# Quick sketch of chiral CFT

- 1d chiral CFT: gapless one-way movers with linear dispersion




$$H = P$$

$$H = P = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right)$$

central charge

Eigenvectors: $L_0|\alpha\rangle = h_\alpha|\alpha\rangle$ Vacuum: $L_0|0\rangle = 0$

conformal dimension

➢ Non-chiral CFTs, describing 1d critical chains, are recovered by combining with a counter-propagating branch.

# Virasoro algebra, primary field, and Verma module

Virasoro algebra: $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$

Primary state: $L_n|\alpha\rangle = 0, \ \forall\, n > 0$ ⇒ Primary field

$$|\alpha\rangle = \lim_{z\to 0} \mathcal{V}_\alpha(z)|0\rangle$$

Verma module:

$$\begin{array}{llllll}
h_\alpha & |\alpha\rangle & & & & \\
h_\alpha + 1 & L_{-1}|\alpha\rangle & & & & \\
h_\alpha + 2 & L_{-1}^2|\alpha\rangle & L_{-2}|\alpha\rangle & & & \\
\vdots & L_{-1}^3|\alpha\rangle & L_{-1}L_{-2}|\alpha\rangle & L_{-3}|\alpha\rangle & & \\
 & L_{-1}^4|\alpha\rangle & L_{-1}^2L_{-2}|\alpha\rangle & L_{-1}L_{-3}|\alpha\rangle & L_{-2}^2|\alpha\rangle & L_{-4}|\alpha\rangle \\
 & \vdots & & & &
\end{array}$$

- Null vectors with vanishing norm should be removed, e.g. $L_{-1}|0\rangle = 0$

# Virasoro algebra, primary field, and Verma module

Further remarks:

- Additional symmetry could further unify a number of Virasoro towers into a single one [e.g. SU(2)$_1$ CFT with Kac-Moody algebra organizes them into two towers => two (Kac-Moody) primaries].

- Those appearing as edge theories of 2d chiral topological states are unitary and rational CFTs.

$c > 0,\ h \geq 0$

Finite number of primaries, *c* and *h* rational

Corresponding to anyons in 2d

Example: Ising CFT $c = 1/2$

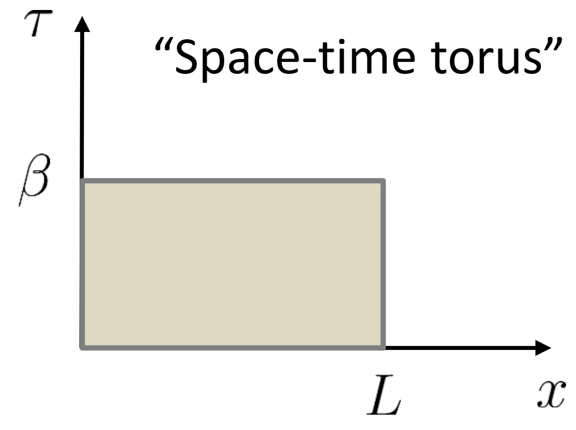
primaries $I,\ \sigma,\ \psi \quad \longrightarrow \quad h_I = 0,\ h_\sigma = \frac{1}{16},\ h_\psi = \frac{1}{2}$

# Chiral character of CFT

- Chiral character is like a partition function (restricted to each tower of states)

$$\begin{aligned}\chi_a(q) &= \mathrm{Tr}_a(e^{-\beta H}) \\ &= \mathrm{Tr}_a[e^{-2\pi\frac{\beta}{L}(L_0 - c/24)}] \\ &= \mathrm{Tr}_a(q^{L_0 - c/24})\end{aligned}$$

$$q = e^{-2\pi\frac{\beta}{L}}$$




Example: Ising CFT

$$\begin{aligned}\chi_I(q) &= q^{-1/48}(1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots) \\ \chi_\sigma(q) &= q^{1/24}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots) \\ \chi_\psi(q) &= q^{23/48}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + \ldots)\end{aligned}$$

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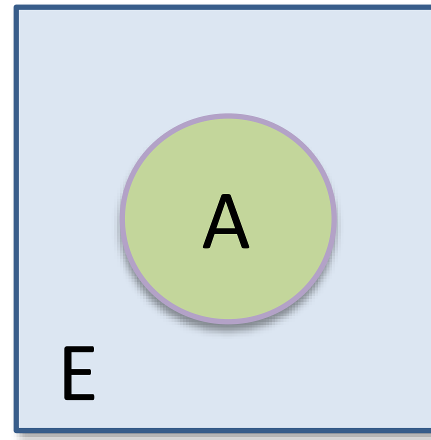
# Topological entanglement entropy

Entanglement entropy:

$$S = \alpha L - \gamma + \cdots$$

$$\gamma = \ln(D)$$

D: total quantum dimensions of anyons




$$\rho = \mathrm{Tr}_{\mathrm{E}}(|\Psi\rangle\langle\Psi|)$$

$$S = -\mathrm{Tr}(\rho \ln \rho)$$

- For 2d chiral topological states, Kitaev & Preskill also provided a heuristic "proof" based on CFT…

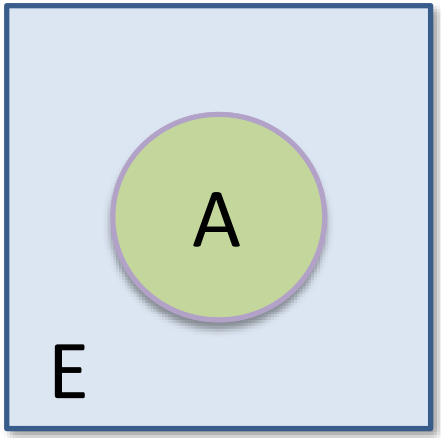
Kitaev & Preskill, '06

Levin & Wen, '06

# Topological entropy from chiral CFT

Kitaev & Preskill's assumption:

$$\rho \propto e^{-\beta H}\big|_I$$

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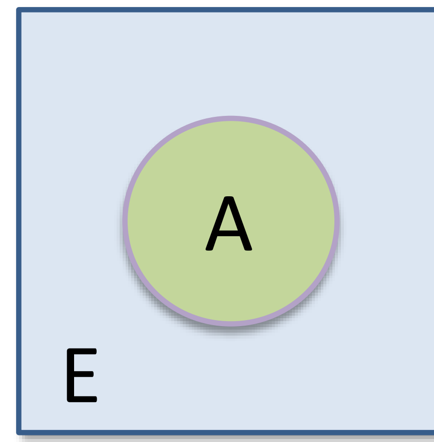
Entanglement "temperature"

Chiral CFT on the boundary of A

No anyon in A

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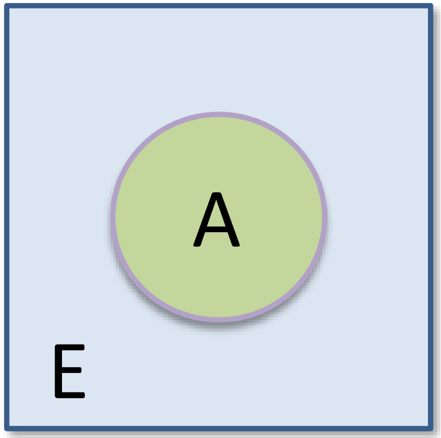



$$\rho = \mathrm{Tr}_{\mathrm{E}}(|\Psi\rangle\langle\Psi|)$$

$$S = -\mathrm{Tr}(\rho \ln \rho)$$

Renyi entropy: $$S_n = \frac{1}{1-n} \ln \mathrm{Tr}(\rho^n)$$

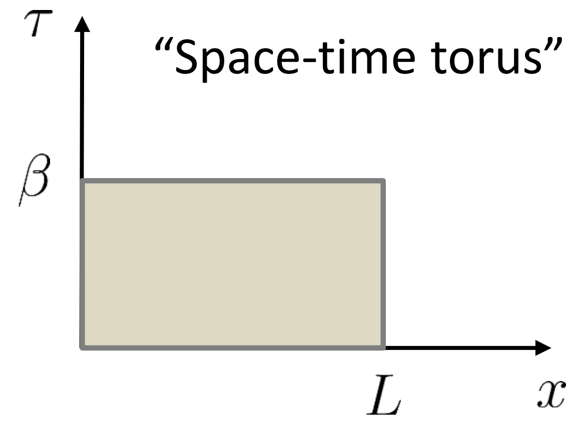
$$\mathrm{Tr}(\rho^n) = \frac{\mathrm{Tr}_I(e^{-n\beta H})}{[\mathrm{Tr}_I(e^{-\beta H})]^n} = \frac{\chi_I(q^n)}{[\chi_I(q)]^n}$$

Recall $q = e^{-2\pi\frac{\beta}{L}}$

# Calculating chiral character

$$\begin{aligned}\chi_I(q) &= \mathrm{Tr}_I(q^{L_0-c/24}) \\ &= q^{-c/24}(1+d_1 q + d_2 q^2 + \ldots)\end{aligned}$$

$$q = e^{-2\pi\frac{\beta}{L}} \to 1 \quad \text{for } L \gg \beta$$




Modular transformation:

$$q' = e^{-2\pi\frac{L}{\beta}} \to 0 \quad \text{for } L \gg \beta$$

$$\begin{aligned}\chi_I(q) &= \sum_a S_{I,a}\chi_a(q') \\ &\simeq S_{I,I}\chi_I(q' \to 0) \\ &\simeq S_{I,I}(q')^{-c/24} \\ &= \boxed{S_{I,I}e^{\frac{\pi c}{12}\frac{L}{\beta}}}\end{aligned}$$

$$\chi_a(q') = (q')^{h_a - c/24}(1 + d_1 q' + \ldots)$$

$a = I$ dominant!

# Topological entropy from chiral CFT

$$\mathrm{Tr}(\rho^n) = \frac{\chi_I(q^n)}{[\chi_I(q)]^n} = \frac{S_{I,I}\exp(\frac{\pi c}{12}\frac{L}{n\beta})}{(S_{I,I})^n\exp(\frac{\pi c}{12}\frac{nL}{\beta})}$$

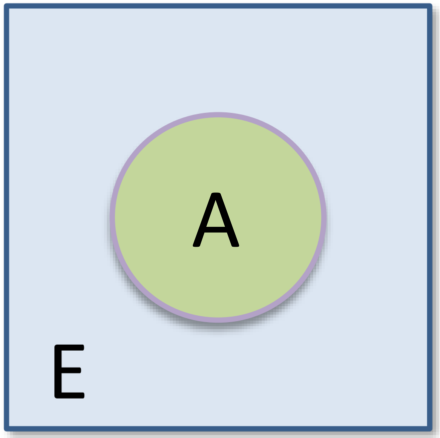
$$= (S_{I,I})^{1-n}\exp\left[\frac{\pi c}{12\beta}(\frac{1}{n}-n)L)\right]$$

$$\boxed{\chi_I(q) \simeq S_{I,I}e^{\frac{\pi c}{12}\frac{L}{\beta}}}$$

Renyi entropy:

$$\begin{aligned} S_n &= \tfrac{1}{1-n}\ln\mathrm{Tr}(\rho^n) \\ &= (1+\tfrac{1}{n})\frac{\pi c}{12\beta}L + \ln(S_{I,I}) \\ &= \boxed{\alpha_n L - \ln(D)} \end{aligned}$$

$$S_{I,I} = \frac{1}{D}$$



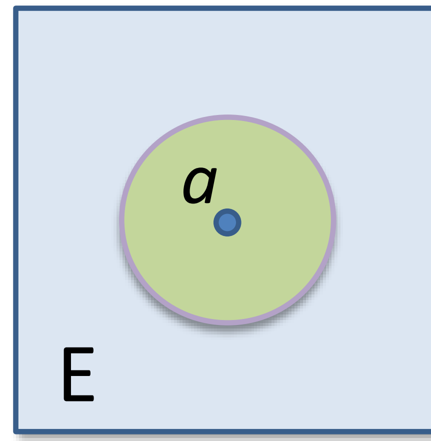

- This CFT-based derivation is consistent with the TQFT approach.

# Topological entropy in the presence of anyon

Kitaev & Preskill's assumption:

$$\rho \propto e^{-\beta H}\big|_a$$

*a* sector




$$\mathrm{Tr}(\rho^n) = \frac{\chi_a(q^n)}{[\chi_a(q)]^n} = (S_{a,I})^{1-n} \exp\left[\frac{\pi c}{12\beta}(\frac{1}{n} - n)L\right]$$

$$\boxed{\chi_a(q) \simeq S_{a,I} e^{\frac{\pi c}{12}\frac{L}{\beta}}}$$

Renyi entropy:

$$\begin{aligned} S_n &= \tfrac{1}{1-n} \ln \mathrm{Tr}(\rho^n) \\ &= (1 + \tfrac{1}{n})\frac{\pi c}{12\beta} L + \ln(S_{a,I}) \\ &= \boxed{\alpha_n L - \ln(D/d_a)} \end{aligned}$$

$$S_{a,I} = \frac{d_a}{D}$$

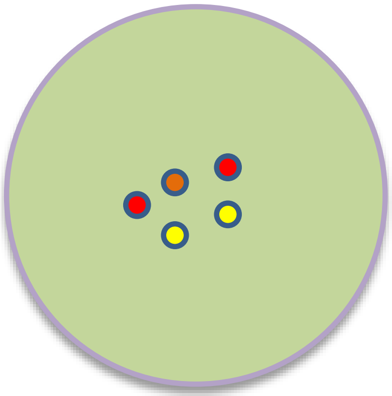
$d_a$: quantum dimension of anyon *a*

# What about many anyons?

First fuse the anyons:

$$a \times b \times c \times \ldots = I + e + f + \ldots$$

$$\rho \propto x_1 \rho_I + x_e \rho_e + x_f \rho_f + \ldots$$



- The reduced density matrix depends on which final state it is…

- If they fuse into a particular anyon sector (e.g. *f*), entanglement entropy is smaller

Such basis is called minimally entangled basis, convenient for extracting topological data

Y. Zhang, Grover, Turner, Oshikawa & Vishwanath, '12

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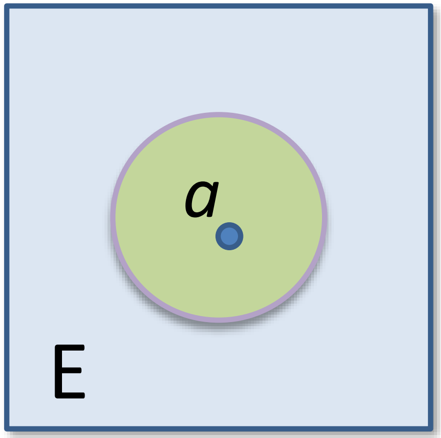
# Entanglement spectrum

- Li & Haldane: "the low-lying entanglement spectrum can be used as a fingerprint to identify topological order." H. Li & Haldane, '08

$$\rho_a = \frac{1}{\chi_a(q)} e^{-\beta H}|_a = \frac{1}{\chi_a(q)} q^{L_0 - c/24}|_a$$



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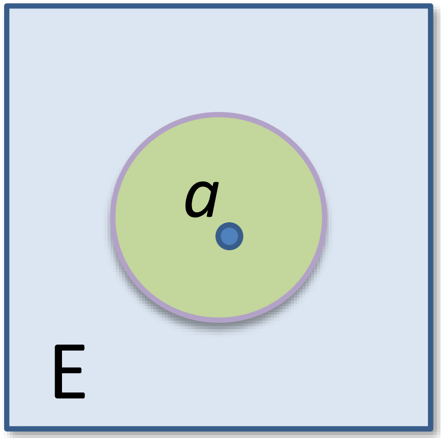


- Largest eigenvalue in sector *a:*

$$\lambda_a = \frac{1}{\chi_a(q)} q^{h_a - c/24}$$

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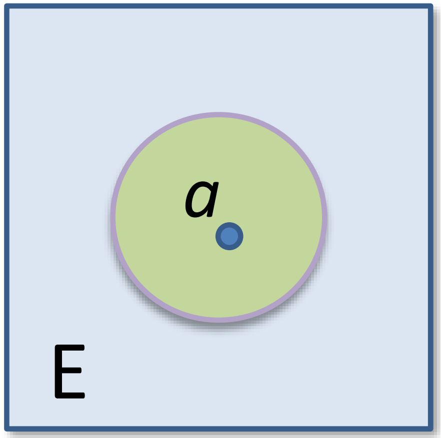



- Largest eigenvalue in sector *a:*

$$\boxed{\chi_a(q) \simeq S_{a,I} e^{\frac{\pi c}{12}\frac{L}{\beta}}}$$

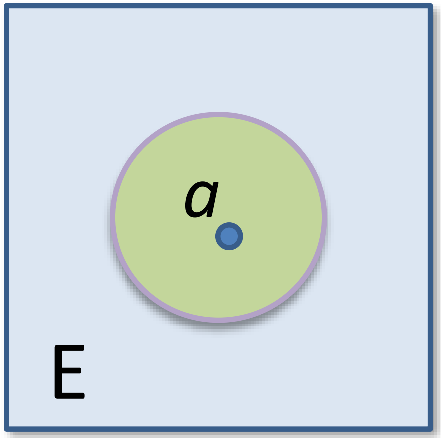
$$\lambda_a = \frac{1}{\chi_a(q)} q^{h_a - c/24}$$

$$\boxed{-\ln(\lambda_a) = \frac{\pi c}{12\beta} L - \ln(D/d_a) + 2\pi \frac{\beta}{L}(h_a - \frac{c}{24})}$$

Topological spin of anyons

# Entanglement spectrum

$$-\ln(\lambda_a) = \frac{\pi c}{12\beta} L - \ln(D/d_a) + 2\pi \frac{\beta}{L}(h_a - \frac{c}{24})$$




Further remarks:

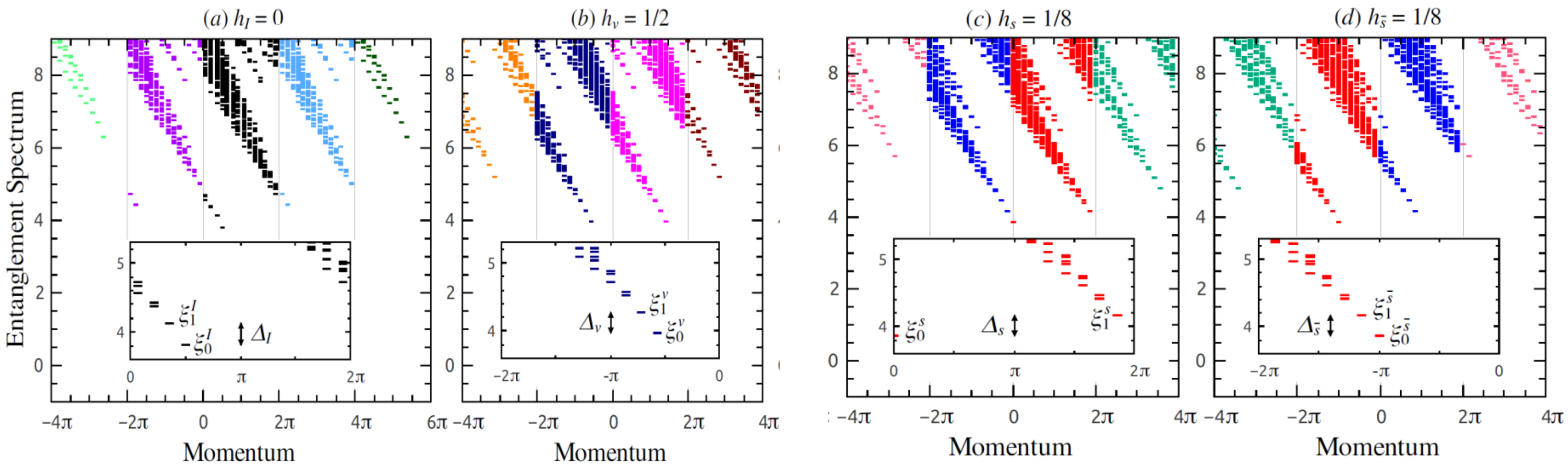
- It's consistent with Renyi entropy result with $n = \infty$, but the 1/L correction is not within TQFT framework.

- Non-CFT contributions may exist (presumably satisfy the area law), so better use

$$\boxed{-\ln(\lambda_a) = \alpha L - \ln(D/d_a) + \frac{\mu}{L}(h_a - \frac{c}{24})}$$

- If the entanglement spectrum is close to a chiral CFT, then the formula also works for low-lying entanglement "excited" states.

# Example: ¼ Laughlin-like states on the lattice




- Expect $$\frac{\xi_0^v - \xi_0^I}{\xi_1^I - \xi_0^I} \to \frac{1}{2}$$

S. Yang, Wahl, HHT, Schuch & Cirac, ’15

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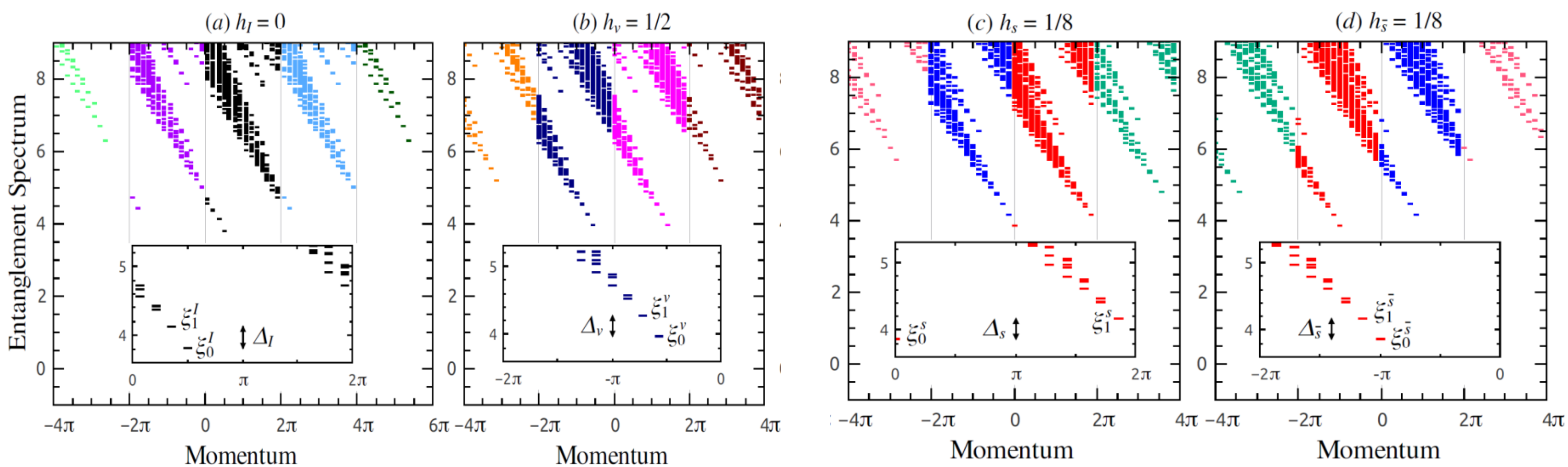
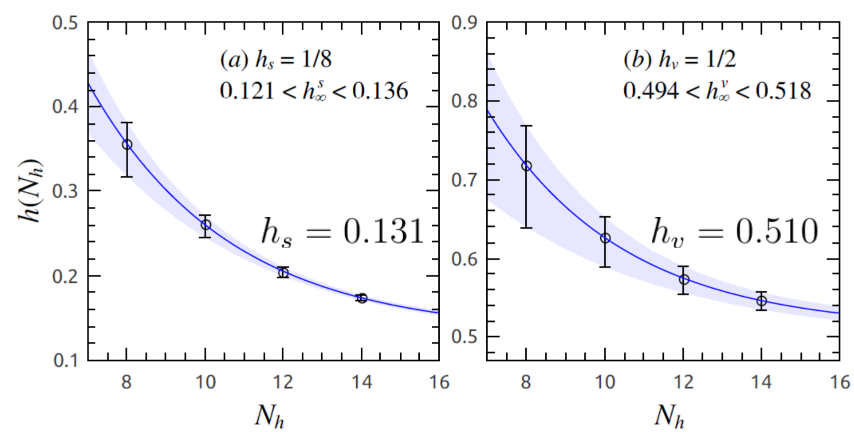



- Expect $\dfrac{\xi_0^v - \xi_0^I}{\xi_1^I - \xi_0^I} \rightarrow \dfrac{1}{2}$

- Increase *L* as much as possible

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# Summary

- For 2d chiral topological states, CFT provides a very useful tool for understanding their physical properties.

- The universal topological data from entanglement measures, such as topological entanglement entropy and entanglement spectrum, can be derived naturally from the assumption that the reduced density matrix is a thermal state of a chiral CFT.

- Other CFT-based approaches for extracting universal data available
  => momentum polarization & Klein twist (tomorrow)

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*Thank you for your attention!*