Non-local variational problems: applications to nonlinear elasticity.

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Non-linear elasticity

If we have an elastic material whose position is given by $u(x, t) \in \mathbb{R}^3$, then its deformation is described by the Couchy equation.

If what we want is to apply the calculus of Variations, we consider hyperlastic materials which allow us to find solutions as minimizers of the functional

$$\int_\Omega W(Du(x))dx - \int_\Omega f \cdot udx.$$

Where $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is called the stored energy function. The equilibrium equations of the Couchy equation would correspond with the Euler-Lagrange equations of this functional.

Non-linear elasticity

Calculus of variations

$$\text{Minimize}_{u\in\mathcal{A}}\ I(u),$$

where

$$I(u) = \int_\Omega W(x, u(x), \nabla u(x))dx,$$

and

$$\mathcal{A} = \{W^{1,p}(\Omega, \mathbb{R}^m) : \text{ u verifies boundary conditions}\},$$

$\Omega \subset \mathbb{R}^n$ a domain, and $1 < p < +\infty$.

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Peridynamics

S.A. Silling proposed a reformulation of classical continuum mechanics:

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- Absence of gradients.
- Paradigmatic example: $w = \frac{|y-y'|^p}{|x-x'|^\alpha}$ for some $p > 1$ and $0 \leq \alpha < n + p$. Also $K(x - x')|y - y'|^p$.

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Features:

- Non-local: points at a positive distance exert a force upon each other.
- Absence of gradients.
- Paradigmatic example: $w = \frac{|y-y'|^p}{|x-x'|^\alpha}$ for some $p > 1$ and $0 \leq \alpha < n + p$. Also $K(x - x')|y - y'|^p$.
- Deformations with discontinuities (fracture, dislocation...) do not require a separate treatment.

This motivates the study of functionals of the form

$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))dx'dx$$

for $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$. Fubini's theorem leads to

$$w(x, x', y, y') = w(x', x, y', y).$$

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As in the local case, we will use the Direct Method of Calculus of Variations to find conditions on w for I to have a minimizer.
The method is based on:

- **Coercivity:** $\lim_{||u|| \to \infty} I(u) = +\infty$.
- **Weak lower semi-continuity**: For any $u_j \rightharpoonup u$ weak, the inequality

$$I(u) \leq \liminf I(u_j)$$

holds.

How can we study lower semi-continuity in the local case ?

Necessary and sufficient condition for

$$\int_{\Omega} W(x, u(x))dx$$

to be wlsc in L^p is that $W(x, \cdot)$ is convex.

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Necessary and sufficient condition for

$$\int_\Omega W(x, u(x), Du(x))dx$$

to be wlsc in $W^{1,p}$ is that $W(x, y, \cdot)$ is quasiconvex.

In our non-local case, if

$$w \approx \frac{|y-y'|^p}{|x-x'|^\alpha}$$

with $\alpha > n$ we will chose the weak topology in $W^{s,p}$ (with $s + np = \alpha$).

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with $\alpha < n$ we will chose the weak topology in L^p.

P. Elbau (arXiv 2011): Necessary and sufficient condition for

$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))dx'dx$$

to be wlsc in L^p is that for $a.e. x \in \Omega$ and all $u \in L^p(\Omega, \mathbb{R}^d)$,

$$(NC) \quad y \mapsto \int_\Omega w(x, x', y, u(x'))dx' \text{ is convex in } \mathbb{R}^d.$$

A weird point is the fact that the previous condition does depend on the domain Ω, in other words, property (NC) does depend on Ω.

Proposition

$I_{\Omega'}$ is weakly lower semi-continuous in $L^p(\Omega'; R^d)$ for all $\Omega' \subset \Omega$ (equivalently, the function in (NC) is convex for all $\Omega' \subset \Omega$) iff for $a.e. x, x' \in \Omega$, the function $w(x, x', \cdot, \cdot)$ is separately convex.

Nevertheless, (NC)$\not\Rightarrow$ convexity on each variable (separately convex).

J.C.Bellido, C.Mora-Corral SIAM-JMA (2014)
J.C.Bellido, C.Mora-Corral Submitted (2017)

In $W^{s,p}$ the issue is rather simple. If $u_j \rightharpoonup u$ in $W^{s,p}$ then $u_j \to u$ in L^p and a.e hence

$$\int_\Omega \int_\Omega w(x,x',u(x),u(x'))dx'dx = \lim_{j\to\infty} \int_\Omega \int_\Omega w(x,x',u_j(x),u_j(x'))dx'dx$$

Consequently, no need of convexity conditions.

Local case

For

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we just impose $W(x, y) \geq c|y|^p$.

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Non-local case

- For
$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))$$
we can impose $w(x, x', y, y') \geq c|y|^p$. But typically I is invariant under translations: $I(u) = I(u + a)$ for all $a \in \mathbb{R}^d$ so w depends on (x, x', y, y') through $(x, x', y - y')$.

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- Functions in L^p do not have traces on the boundary. Dirichlet conditions are prescribed on $\Omega_D := \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

Non-local case

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$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))$$
we can impose $w(x, x', y, y') \geq c|y|^p$. But typically I is invariant under translations: $I(u) = I(u + a)$ for all $a \in \mathbb{R}^d$ so w depends on (x, x', y, y') through $(x, x', y - y')$.
- Functions in L^p do not have traces on the boundary. Dirichlet conditions are prescribed on $\Omega_D := \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.
- Usual assumption in engineering that $w(x, x', \cdot, \cdot) \equiv 0$ if $|x - x'| \geq \delta$.

Coercivity inequality for Dirichlet conditions:

$$\lambda \int_\Omega |u(x)|^p dx \leq \int_\Omega \int_{\Omega \cap B(x,\delta)} |u(x) - u(x')|^p dx' dx + \int_{\Omega_D} |u(x)|^p dx$$

F. Andreu, J.Mazón, J.Rossi and J.Toledo SIAM J Math Anal (2009),
B. Aksoylu and M.L. Parks Appl Math Comput (2011),
B.Hinds and P. Radu Appl Math Comput (2012).

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Coercivity inequality for Newmann condition:

$$\lambda \int_{\Omega} \left| u(x) - \frac{1}{|\Omega|} \int_{\Omega} u \right|^p dx \leq \int_{\Omega} \int_{\Omega \cap B(x,\delta)} |u(x) - u(x')|^p dx' dx$$

J. Bourgain, H. Brezis and P. Mironescu J Anal Math (2002),
A. C. Ponce JEMS (2004),
F. Andreu, J. Mazón, J. Rossi and J. Toledo J Math Pures Appl (2008),
B. Aksoylu and T. Mengesha Numer Funct Anal Optim (2010),
R. Hurri-Syrjänen and A.V. Vähäkangas J Anal Math (2013).

Existence of minimizers in L^p

$\Omega \subset \mathbb{R}^n$. $\delta > 0$. $\Omega_D = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$. $p \geq 1$.

1. $c\chi_{B(0,\delta)}(x - x')|y - y'|^p \leq w(x, x', y, y') \leq a(x, x') + C|y|^p$ with $a \in L^1(\Omega \times \Omega)$.
2. $w(x, x', y, \cdot)$ is convex.

Let $u_0 \in L^p(\Omega_D, \mathbb{R}^d)$. Then there exists a minimizer of

$$\int_\Omega \int_\Omega w(x, x', u(x), u(x'))dx'dx$$

among $u \in L^p(\Omega, \mathbb{R}^d)$ such that $u = u_0$ a.e. on Ω_D.

(Analogous statement for Neumann boundary conditions.)

Existence of minimizers in $W^{s,p}$

$\Omega \subset \mathbb{R}^n$. $\delta > 0$. $\Omega_D = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$. $0 < s < 1$. $p \geq 1$.

1. $c\chi_{B(0,\delta)}(x - x')\frac{|y-y'|^p}{|x-x'|^{n+sp}} \leq w(x, x', y, y')$.

Let $u_0 \in W^{s,p}(\Omega_D, \mathbb{R}^d)$. Then there exists a minimizer of

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nonlocal $\rightsquigarrow$ local

Think of $w(x, x', y, y') \approx \frac{|y-y'|^p}{|x-x'|^\alpha}$. Call $\beta = p - \alpha$. Ingredients:

1. Scaling:

$$I_\delta(u) := \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x - x', u(x) - u(x'))dx'dx$$

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2. Blow-up at zero (homogenization of w):

$$w^0(\tilde{x},\tilde{y}) := \lim_{t\to 0} \frac{1}{t^\beta} w(t\tilde{x}, t\tilde{y}).$$

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3. Density $\bar{w} : \mathbb{R}^{d\times n} \to \mathbb{R}$

$$\bar{w}(F) := \int_{\mathbb{S}^{n-1}} w^0(z, Fz)d\mathcal{H}^{n-1}(z).$$

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4. Quasiconvexification: $\bar{w}^{qc} : \mathbb{R}^{d \times n} \to \mathbb{R}$ of $\bar{w}$.

$$I(u) := \int_\Omega \bar{w}^{qc}(Du(x))dx$$

Pointwise limit for regular functions: If $u \in C^1(\bar{\Omega}, \mathbb{R}^d)$,

$$lim_{\delta \to 0} I_\delta(u) = \int_\Omega \bar{w}(Du(x))$$

$$\begin{cases} I_\delta(u) = \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x - x', u(x) - u(x'))dx'dx \\ \text{in } \mathcal{A}_\delta := \{u \in L^p(\Omega, \mathbb{R}^d) : u = u_0 \text{ in } \Omega_\delta := \{x \in \Omega : dist(x, \partial\Omega) < \delta\}\} \end{cases}$$

$$\begin{cases} I(u) = \int_\Omega \bar{w}^{qc}(Du(x))dx, \\ \text{in } \mathcal{A} := \{u \in W^{1,p}(\Omega, \mathbb{R}^d) : u = u_0 \text{ on } \partial\Omega\} \end{cases}$$

Theorem

$I_\delta \xrightarrow{\Gamma} I$ in $L^p(\Omega, \mathbb{R}^d)$ as $\delta \to 0$. Specifically

- ***Compactness:*** *If $u_\delta \in \mathcal{A}_\delta$ satisfy $I_\delta(u_\delta) \leq M$ then there exists $u \in \mathcal{A}$ such that $u_\delta \to u$ in $L^p(\Omega, \mathbb{R}^d)$.*
- ***Lower bound:*** *If $u_\delta \in L^p(\Omega, \mathbb{R}^d)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ satisfy $u_\delta \to u$ in $L^p(\Omega, \mathbb{R}^d)$ then $I(u) \leq \liminf_{\delta \to 0} I_\delta(u_\delta)$.*
- ***Upper bound:*** *For any $u \in \mathcal{A}$ there exists $u_\delta \in \mathcal{A}_\delta$ such that $u_\delta \to u$ in $L^P(\Omega, \mathbb{R}^d)$ and $I(u) = \lim_{\delta \to 0} I_\delta(u_\delta)$*

J.C.Bellido, C. Mora-Corral, P.Pedregal, Cal.Var,PDE (2015)

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Here is where I join my advisors as a PhD student.

A new model is needed

In Solid Mechanics, the model

$$\int_\Omega \int_\Omega w(x - x', u(x) - u(x'))dx'dx$$

is wrong. Let's see why.

Start with

$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))dx'dx$$

and apply familiar conditions on Solid Mechanics.

1. I is frame indifferent iff $w = w(x, x', |y - y'|)$.
2. I is homogeneous and isotropic iff $w = w(|x - x'|, y, y')$.

Let the material be frame indifferent, homogeneous and isotropic: $w = w(|x - x'|, |y - y'|)$. We do the nonlocal $\rightsquigarrow$ local passage. Recall the process $w \rightsquigarrow w^{o} \rightsquigarrow \bar{w} \rightsquigarrow W$. W.l.o.g., $w = w^{o}$.

$$\begin{aligned} \bar{w} &= \int_{\mathbb{S}^{n-1}} w(z, Fz) d\mathcal{H}^{n-1}(z) = \int_{\mathbb{S}^{n-1}} w(|z|, |Fz|) d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathbb{S}^{n-1}} w(1, |Fz|) d\mathcal{H}^{n-1}(z). \end{aligned}$$

Assume for simplicity that $\bar{w}$ is quasiconvex, hence $W = \bar{w}$. Thus, we would say that a quasiconvex W is retrievable in this model iff

$$W(F) = \int_{\mathbb{S}^{n-1}} W(|Fz| I) d\mathcal{H}^{n-1}(z) \qquad \forall F \in \mathbb{R}^{n\times n}_{+}.$$

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\begin{aligned}
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&= \int_{\mathbb{S}^{n-1}} w(1, |Fz|) d\mathcal{H}^{n-1}(z).
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$$

Hence W is determined by the values of diagonal matrices. There are very few retrievable W. Examples:

- $|F|^2$ is retrievable, but no other squared norm is retrievable.
- Given a convex $h, W(F) = h(|F|^2)$ is retrievable iff h is affine.
- $\det F$ is not retrievable.

R.S. Rivlin and J.L. Ericksen 1955

Theorem

The following conditions are equivalent:

1. W *is isotropic.*
2. $W(F) = h(|F|^2, |cof(F)|^2, (\det F)^2)$ *for some* $h : (0, \infty)^3 \to \mathbb{R}$.

New model

Based on T.Mengesha, D. Spector 15 and T.Mengesha and Q. Du 15, we adopt the model

$$I(u) = \int_\Omega W(\mathcal{G}u(x))dx$$

where $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a typical stored energy function in hyperelasticity, and $\mathcal{G}u(x)$ is a nonlocal gradient:

$$\mathcal{G}u(x) = \int_\Omega \frac{u(x) - u(x')}{|x - x'|} \otimes \frac{x - x'}{|x - x'|} \rho(x - x')dx'.$$

Current work: to develop a polyconvex theory in this nonlocal context.

Non-local determinant

Then, two different ways of recovering $\det \nabla u$ appear:

$$
\begin{aligned}
\det \nabla u &= G_\delta(u^1) \cdot T G_\delta(u^2) \\
\det \nabla u &= \operatorname{div}(u^1(-T)\nabla u^2)
\end{aligned}
$$

where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Omega \subset \mathbb{R}^n$

Definition

Let $u \in L^1(\Omega)$ *a scalar function, then the non-local gradient of* u *is defined:*

$$G_\rho u(x) := p.v.n \int_\Omega \frac{u(x) - u(y)}{|x - y|} \frac{(x - y)}{|x - y|} \rho(x - y) dy.$$

Definition

Let $\phi \in L^1(\Omega, \mathbb{R}^n), \phi = (\phi_1, \phi_2, \cdots, \phi_n)$ *, if we define*

$$(D_\rho)_i \phi_i(x) := -p.v.n \int_\Omega \frac{\phi_i(x) + \phi_i(y)}{|x - y|} \frac{(x_i - y_i)}{|x - y|} \rho(x - y) dy$$

then the non-local divergence of ϕ *is defined as:*

$$D_\rho \phi(x) := \sum_{i=1}^{n} (D_\rho)_i \phi_i(x) = -p.v.n \int_\Omega \frac{\phi(x) + \phi(y)}{|x - y|} \cdot \frac{(x - y)}{|x - y|} \rho(x - y) dy$$

Functional space

Definition

We define the space $S_{\rho,p}(\Omega)$ as the set of the functions $u \in L^p(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(x')|^p}{|x - x'|^p} \rho(x - x') dx' dx < \infty,$$

whose norm is

$$u^p_{S_\rho} = \int_\Omega |u(x)|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(x')|^p}{|x - x'|^p} \rho(x - x') dx' dx.$$

- $u_j \xrightarrow{S_{\rho,p}(\Omega)} u \Rightarrow G_\rho(u_j) \xrightarrow{L^p(\Omega)} G_\rho u$

Duality

$$\int_\Omega G_\rho u(x) \cdot \phi(x) dx = -\int_\Omega u(x) D_\rho \phi(x) dx.$$

Q. Du, M.D.Gunzburger, R.B. Lehoucq, K. Zhou Math. Models Met. Appl.

Sci (2013)
T. Mengesha, Q. Du, Nonlinearity (2015)
T. Mengesha, D. Spector Cal. of Var (2015)
A.Ponce JEMS (2003)

Duality

$$\int_\Omega G_\rho(u^1)\cdot TG_\rho(u^2)\varphi dx = -\int_\Omega u^1 \int_{B(x,\delta)} T\cdot G_\rho(u^2)(y)\frac{\varphi(x)-\varphi(y)}{|x-y|^2}\cdot(x-y)dydx.$$

$$+\ \textit{extra term.}$$

Thank you very much for your attention.