Constrained controllability of the semilinear heat equation

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Motivations

- Controllability of PDE has been widely investigated in the past decades;
- On the other hand, on many PDE models describing biological or physical phenomena some constraints are imposed.

Our goal: obtain some **Controllability** results under **state** and/or **control** constraints.

Existing results for the heat equation

This constrained Controllability issue for the heat equation has already been investigated in:

J. Lohéac, E. Trélat and E. Zuazua
Minimal controllability time for the heat equation under unilateral state or control constraints
Mathematical Models and Methods in Applied Sciences, Vol. 27 no. 09 (2017), pp. 1587 – 1644..

Existing results for the heat equation

Theorem (Lohéac, Trélat and Zuazua)

*Let $y_0 \in L^2(\Omega)$ be an initial datum and y_1 be a steady state. Assume $Tr(y_1) \restriction_{\partial\Omega} \geq \nu > 0$. Then, in **time large**, we can drive the system:*

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ y = u & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

*from y_0 to y_1 by means of control u satisfying the **control constraint**:*

$$u \geq 0 \qquad \text{a.e.} \quad (0,T) \times \partial\Omega.$$

*If $y_0 \geq 0$ a.e. in Ω, y fulfills the **state constraint**:*

$$y \geq 0 \qquad \text{a.e.} \quad (0,T) \times \Omega.$$

Goal of the talk

1. Generalize this result to a **semilinear** case:

$$y_t - \Delta y + f(y) = 0 \qquad \text{in } (0,T)\times\Omega;$$

2. Check how much Constrained controllability relies on the **dissipative** nature of the equation.

Outline of the talk

Dissipative case

Theorem

Assume $c \in L^\infty(\Omega)$ such that $c > -\lambda_1$. Let $y_0 \in L^2(\Omega)$ be an initial datum and y_1 be a steady state. Suppose $Tr(y_1) \restriction_{\partial\Omega} \geq \nu > 0$. Then, in time large, we can steer the system:

$$\begin{cases} y_t - \Delta y + c(x)y = 0 & \text{in}\,(0,T)\times\Omega \\ y = u & \text{on}\,(0,T)\times\partial\Omega. \end{cases}$$

from y_0 to y_1 by a control u satisfying the ***control constraint****:*

$$u \geq 0 \qquad \text{a.e.} \quad (0,T)\times\partial\Omega.$$

If $y_0 \geq 0$ a.e. on Ω, then y fulfills the ***state constraint****:*

$$y \geq 0 \qquad \text{a.e.} \quad (0,T)\times\Omega.$$

Idea of the proof-Dissipative Case

We introduce the state variable $z = y - y_1$ reducing ourselves to prove that, in time large, we can drive the system from $y_0 - y_1$ to 0 by a control $v \geq -Tr(y_1)$.
Then, the control $u = v + Tr(y_1)$ will drive the system from y_0 to y_1 and

$$u = v + Tr(y_1) \geq -Tr(y_1) + Tr(y_1) = 0.$$

Idea of the proof-Dissipative Case

Take $c \in L^\infty(\Omega)$. By the regularizing effect of the heat equation and extension-restriction arguments, we recognize that, for any initial datum $z_0 \in L^2$, we can find a control w driving the system

$$\begin{cases} z_t - \Delta z + c(x)z = 0 & \text{in } (0,\tau)\times\Omega \\ z = w & \text{on } (0,\tau)\times\partial\Omega \end{cases}$$

from z_0 to 0 in time τ and such that:

$$\|w\|_{L^\infty} \leq C(\tau)\|z_0\|_{L^2}.$$

Idea of the proof-Dissipative Case

We determine the control v as follows:

1. we let the system evolve for a long time interval $[0, T-\tau]$. Since the system **dissipative**, we have:

$$\|z(T-\tau)\|_{L^2} \leq e^{-\lambda(T-\tau)}\|y_0 - y_1\|_{L^2},$$

   where λ is the first eigenvalue of $-\Delta y + cy$.

2. we steer the system from $z(T-\tau)$ to 0 in the small time interval $[T-\tau, T]$ by a control $w \in L^\infty$ such that:

$$\|w\|_{L^\infty} \leq C(\tau)\|z(T-\tau)\|_{L^2} \leq C(\tau)e^{-\lambda(T-\tau)}\|y_0 - y_1\|_{L^2}.$$

Then, $v := w\chi_{[T-\tau,T]}$ drives our control system from $y_0 - y_1$ to 0 and, if T is large enough,

$$\|v\|_{L^\infty} \leq C(\tau)e^{-\lambda(T-\tau)}\|y_0 - y_1\|_{L^2} < \nu.$$

This implies that $v \geq -\nu \geq -Tr(y_1)$ as required.

General Case

Theorem

*Let $c \in L^\infty(\Omega)$ with **no sign assumptions**. We take two steady states y_0 and y_1 such that:*

$$Tr(y_i) \restriction_{\partial\Omega} \geq \nu > 0.$$

Then, in time large, we can steer the system:

$$\begin{cases} y_t - \Delta y + c(x)y = 0 & in\,(0,T) \times \Omega \\ y = u & on\,(0,T) \times \partial\Omega \end{cases}$$

*from y_0 to y_1 by means of a control satisfying the **control constraint**:*

$$u \geq 0 \qquad a.e. \quad (0,T) \times \partial\Omega.$$

ε-Controllability

We know that for any $z_0 \in L^2$ we can find a control w steering the control system from z_0 to 0 in time 1 and

$$\|w\|_{L^\infty} \leq C(1)\|z_0\|_{L^2}.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|z_0\|_{L^2} < \delta \qquad \Rightarrow \qquad \|w\|_{L^\infty} < \varepsilon.$$

Take $\varepsilon = \nu$. If $\|y_1 - y_0\|_{L^2} < \delta$, then, we are able to find a control $v \in L^\infty$ of size $\|v\|_{L^\infty} < \nu$ such that $u := v + Tr(y_1)$ drives the system from y_0 to y_1 in time 1. Then, $u \geq -\nu + \nu = 0$.

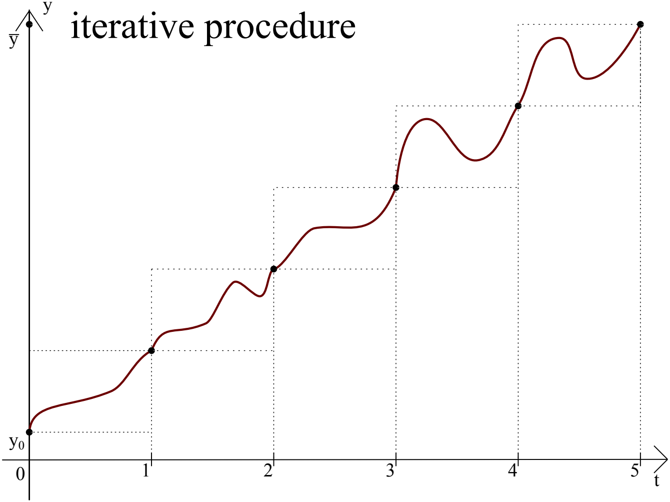
The stair-case method: from local to global

We introduce the following continuous arc joining y_0 and y_1 in the set of steady states.

$$\gamma(s) := (1-s)y_0 + sy_1.$$

The stair case method: from local to global

Then, we link y_0 by y_1 by a step by step procedure joining the steady states along γ at distance less then δ.



Dissipative case

We consider the control system:

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0,T)\times\Omega \\ y = u & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0. & \text{in } \Omega \end{cases}$$

We assume f is a C^1 **nondecreasing** function such that $f(0) = 0$. Then, thanks to the nondecreasing character of f, for any $y_0 \in L^2(\Omega)$ and $u \in L^2((0,T)\times\partial\Omega)$, there exists a unique solution

$$y \in L^2((0,T)\times\Omega) \cap C^0([0,T], H^{-1}(\Omega)).$$

Dissipative case

Theorem

Let $y_0 \in L^2(\Omega)$ be an initial datum and y_1 a bounded steady state such that $Tr(y_1) \restriction_{\partial\Omega} \geq \nu > 0$ for a constant $\nu > 0$. Then, if T is large enough, we can drive the system

$$\begin{cases} y_t - \Delta y + f(y) = 0 & in\,(0,T)\times\Omega \\ y = u & on\,(0,T)\times\partial\Omega \end{cases}$$

from y_0 to y_1 by means of a control u satisfying the ***control constraint****:*

$$u \geq 0 \qquad a.e. \quad (0,T)\times\partial\Omega.$$

If $y_0 \geq 0$ a.e. on Ω, y fulfills the ***state constraint****:*

$$y \geq 0 \qquad a.e. \quad (0,T)\times\Omega.$$

Idea of the proof-Dissipative Case

We observe that $z = y - y_1$ satisfies:

$$\begin{cases} z_t - \Delta z + f(z + y_1) - f(y_1) = 0 & \text{in } (0, T) \times \Omega \\ z = u - Tr(y_1) & \text{on } (0, T) \times \partial\Omega \\ z(0) = y_0 - y_1. & \text{in } \Omega. \end{cases}$$

We have to prove that, in time large, we can drive the above system from $y_0 - y_1$ to 0 by means of a control $v \geq -Tr(y_1)$. Then, $u = v + Tr(y_1) \geq 0$ will be the desired control.

Idea of the proof-Dissipative Case

Let $\tau > 0$ be fixed and $T > \tau$ time horizon.

1. First of all, we enjoy the **dissipative** nature of the system and its regularizing effect for a long time. Indeed, for any $\delta > 0$, taking the control to be zero in $[0, T - \tau]$, we have that the unique solution z to:

$$\begin{cases} z_t - \Delta z + f(z + y_1) - f(y_1) = 0 & \text{in } (0, T - \tau) \times \Omega \\ z = 0 & \text{on } (0, T - \tau) \times \partial\Omega \\ z(0) = y_0 - y_1 & \text{in } \Omega. \end{cases}$$

   is such that $z(T - \tau, \cdot) \in L^\infty$ and

$$\|z(T - \tau, \cdot)\|_{L^\infty} \leq \delta,$$

   whenever T is large enough;

2. to conclude, we check if we can drive $z(T - \tau, \cdot)$ to 0 by a control w of size $\|w\|_{L^\infty} < \nu$.

Idea of the proof-Dissipative Case

We have then to check the local null controllability result for the above system. We go by step. First of all, we prove a null controllability Theorem for the case f globally Lipschitz and distributed control, employing the approach of:
E. Fernández-Cara and E. Zuazua
Annales de l'Institut Henri Poincare (C) Non Linear Analysis, Vol. 17 no. 5 (2000), pp. 583 – 616.
Basically, for any $\eta \in L^\infty((0,T)\times\Omega)$, we consider the linear system

$$\begin{cases} z_t - \Delta z + \frac{f(\eta+y_1)-f(y_1)}{\eta-y_1} z = u\chi_\omega & \text{in } (0,\tau)\times\Omega \\ z = 0 & \text{on } (0,\tau)\times\partial\Omega. \end{cases}$$

and we apply Kakutani's fixed point Theorem to prove the desired result.

Idea of the proof-Dissipative Case

By extension-restriction arguments, we get the same result for the case of boundary control and f globally Lipschitz. Finally, this yields the following local null controllability result in case f is only locally Lipschitz.

Proposition

There exists $\delta > 0$ such that, for any initial datum $z_0 \in L^\infty$ such that:

$$\|z_0\|_{L^\infty} \leq \delta,$$

there exists a control $v \in L^\infty((0,T) \times \partial\Omega)$ such that:

- *v drives the system from z_0 to 0;*
- $$\|v\|_{L^\infty} \leq C \|z_0\|_{L^\infty}.$$

General case

We consider the control system:

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0,T)\times\Omega \\ y = u & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0. & \text{in } \Omega \end{cases}$$

where f is C^1 function such that $f(0) = 0$. We have then **removed the monotonicity assumption** on f. Then, for an initial datum $y_0 \in L^\infty(\Omega)$ and a boundary control $u \in L^\infty((0,T)\times\partial\Omega)$, the above system admits solution locally in time. **Blow up** phenomena in finite time may occur.

General case

Theorem

Let $\mathscr{S}$ be the set of ***bounded steady states*** *endowed with the uniform topology. Then we take two steady states y_0 and y_1 connected in $\mathscr{S}$ by a continuous arc γ such that:*

$$Tr(\gamma(s)) \restriction_{\partial\Omega} \geq \nu > 0 \qquad \forall s \in [0,1].$$

Then, in time large, we can steer the system

$$\begin{cases} y_t - \Delta y + f(y) = 0 & in\,(0,T)\times\Omega \\ y = u. & on\,(0,T)\times\partial\Omega. \end{cases}$$

from y_0 to y_1 by a control $u \in L^\infty$ satisfying the ***control constraint****:*

$$u \geq 0 \qquad a.e. \quad (0,T)\times\partial\Omega.$$

Idea of the proof-General case

By the local controllability, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any pair of bounded steady states y_0 and y_1 such that

$$\|y_1 - y_0\|_{L^\infty} < \delta,$$

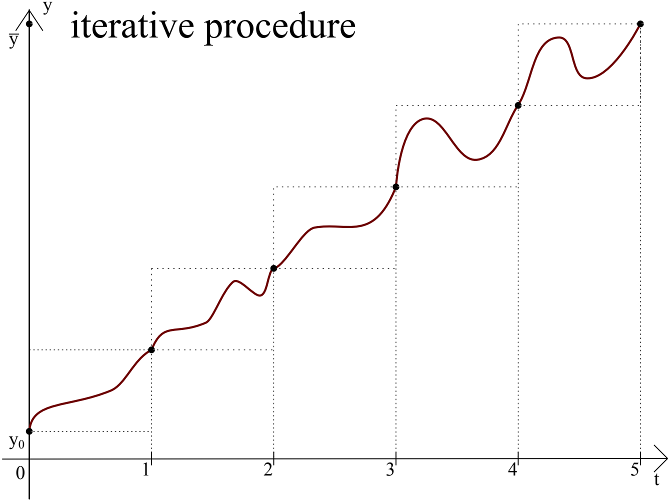
we can find a control $u = v + Tr(y_1) \in L^\infty$ driving the system from y_0 to y_1 in time 1. Moreover, we have:

$$\|v\|_{L^\infty} < \varepsilon.$$

If $\varepsilon = \nu$. Then, $u = v + Tr(y_1) \geq -\nu + \nu = 0$.

Idea of the proof-General case

We connect y_0 and y_1 stepwise joining steady states along γ at distance less than δ.



Work in progress

Our purpose is to use the same techniques to study controllability under constraints for control systems governed by:

$$y_t - div(A\nabla y) + (b, \nabla y) + cy = 0$$

and

$$y_t - div(A\nabla y) + f(x, y, \nabla y) = 0.$$

In these cases some controllability results with controls in L^∞ have been prove in:

A. Doubova, E. Fernández-Cara, M. González-Burgos and E. Zuazua
SIAM Journal on Control and Optimization, Vol. 41 no. 3 (2002), pp. 798 – 819.

Thank you for the attention!!!