Sensitivity Analysis and Uniform Regularity for the Boltzmann Equation with Uncertainty and Multiple Scales

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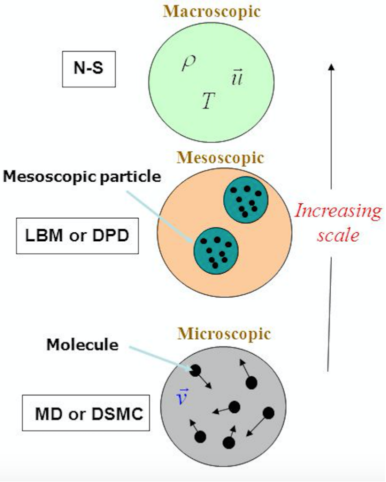
Schedule

1. The Boltzmann Equation with uncertainty

2. Sensitivity of the system under the initial perturbation

3. Stability of the system after gPC-SG

Kinetic equations with uncertainty



The Boltzmann Equation without uncertainty

Dimensionless Boltzmann Equation without uncertainty

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \underbrace{\frac{1}{\epsilon} Q(f, f)}_{\text{binary collisions of hard sphere}} , \quad \text{[Boltzmann, 1872]}$$

$$Q(f, f) = Q_{\text{gain}} - Q_{\text{loss}}$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} f(\mathbf{v}') f(\mathbf{v}'_*) \sigma(v - v_*, \omega) d\omega d\mathbf{v}_* - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} f(\mathbf{v}) f(\mathbf{v}_*) \sigma(v - v_*, \omega) d\omega d\mathbf{v}_*.$$

- $f(t, \mathbf{x}, \mathbf{v})$: the possibility of finding particles at time t and position $\mathbf{x}$ with velocity $\mathbf{v}$.
- $\epsilon = \dfrac{\text{mean free path}}{\text{macroscopic length scale}}$

ϵ describes both mesoscopic and macroscopic system

$\epsilon \to 0$:

$f \to \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|\mathbf{v}-\mathbf{u}|^2}{2T}}$ (Equilibrium), where

$\rho(t,\mathbf{x}) = \int f d\mathbf{v}, \quad \mathbf{u}(t,\mathbf{x}) = \frac{1}{\rho}\int \mathbf{v} f d\mathbf{v}, \quad T(t,\mathbf{x}) = \frac{1}{\rho}\int \frac{1}{3}(\mathbf{v}-\mathbf{u})^2 f d\mathbf{v},$

and the macroscopic quantities satisfy the Euler equation for a compressible fluid,

$$
\begin{aligned}
&\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \frac{1}{\rho} \nabla_{\mathbf{x}} (\rho T) = 0 \\
&\partial_t T + \mathbf{u} \cdot \nabla_{\mathbf{x}} T + \frac{2}{3} T \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0
\end{aligned}
\tag{1.1}
$$

The Boltzmann Equation with uncertainty

Dimensionless Boltzmann Equation with uncertainty

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\epsilon} Q(f, f), \tag{1.2}$$

where $Q(f, g)$ describes the binary collisions of hard sphere,

$$Q(f, g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f(\mathbf{v}') g(\mathbf{v}'_*) - f(\mathbf{v}) g(\mathbf{v}_*) \right) |(v - v_*) \cdot \omega| \, d\omega d\mathbf{v}_*, \tag{1.3}$$

with periodic boundary condition, and, initial data

$$f(0, \mathbf{x}, \mathbf{v}, \mathbf{z}) = f_0(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \mathbf{z} \in I_{\mathbf{z}} \subset \mathbb{R}^d, \quad \mathbf{x} \in \mathbb{T}^3, \quad \mathbf{v} \in \mathbb{R}^3. \tag{1.4}$$

$$\mathbf{v}' = \mathbf{v} - [(\mathbf{v} - \mathbf{v}_*) \cdot \omega] \, \omega, \quad \mathbf{v}'_* = \mathbf{v}_* + [(\mathbf{v} - \mathbf{v}_*) \cdot \omega] \, \omega. \tag{1.5}$$

$$\pi(\mathbf{z}) : I_{\mathbf{z}} \to \mathbb{R}^+ \text{ is the probability density function of } \mathbf{z}. \tag{1.6}$$

Other related works

- Regularity in random space for Boltzmann [Hu, Jin, 16]

$$\|f(t)\|^2_{H^m_{\mathbf{z},\mu}} = \sum_{i=0}^{m} \int \left(\partial_z^i f\right)^2 d\mu(z) d\mathbf{v} d\mathbf{x} \quad \|f(t)\|^2_{H^m_{\mathbf{z},\mu}} \lesssim e^{\frac{t}{\epsilon}} \|f(0)\|^2_{H^m_{\mathbf{z},\mu}} \qquad (2.1)$$

According to the density function $\pi(\mathbf{z})$, one has a corresponding L^2 space in the measure of $d\mu(\mathbf{z}) = \pi(\mathbf{z}) d\mathbf{z}$.

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- Uniform regularity in random space:
  - Linear kinetic equation [Jin, Liu, Ma, 16], [Li, Wang, 17]
  - Nonlinear VPFP system [Jin, Zhu, 17]

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- Uniform regularity in random space:
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  - Nonlinear VPFP system [Jin, Zhu, 17]
- Regularity result for deterministic Boltzmann equation without ϵ, [Y. Guo 2004], [R. Duan, 2007]

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f,f), \quad f(0,\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v}) \tag{2.2}$$

$$\textit{If} \qquad \left\| \frac{f(0)-M}{\sqrt{M}} \right\|_{H^N_{\mathbf{x}}(L^2_{\mathbf{v}})} \le \delta, \tag{2.3}$$

$$\textit{then} \qquad \left\| \frac{f(t)-M}{\sqrt{M}} \right\|_{H^N_{\mathbf{x}}(L^2_{\mathbf{v}})} \lesssim e^{-t} \left\| \frac{f(0)-M}{\sqrt{M}} \right\|_{H^4_{\mathbf{x}}(L^2_{\mathbf{v}})}. \tag{2.4}$$

For $N \geq 4$.

Main Theorem for the Sensitivity (Jin-Zhu, '17)

Define $\partial_{\mathbf{z}}^{\boldsymbol{\beta}} f = f_{\boldsymbol{\beta}}, \quad \|f\|_{H^m_{\mathbf{z},\mu}(H^4_{\mathbf{x}})} = \sum_{|\boldsymbol{\alpha}|\leq 4} \int (\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}})^2 \, d\mu(\mathbf{z}) d\mathbf{x} d\mathbf{v}$

Sensitivity of the system under the initial perturbation

If initially the solution to the Boltzmann equation f satisfies

$$\left\| \frac{f(0) - M}{\sqrt{M}} \right\|_{H^m_{\mathbf{z},\mu}(H^4_{\mathbf{x}}(L^2_{\mathbf{v}}))} \leq \epsilon C_0, \tag{2.5}$$

then

$$\left\| \frac{f(t) - M}{\sqrt{M}} \right\|_{H^m_{\mathbf{z},\mu}(L^2_{\mathbf{x},\mathbf{v}})} \leq \xi e^{-\epsilon\beta t} \left\| \frac{f(0) - M}{\sqrt{M}} \right\|_{H^m_{\mathbf{z},\mu}(H^4_{\mathbf{x}}(L^2_{\mathbf{v}}))}. \tag{2.6}$$

where C_0, ξ, β are constants, and C_0 depends only on m.

Where $M = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|\mathbf{v}|^2}{2}}$

Uniform regularity in random space

Under the same condition,

$$\|f\|_{H^m_{\mathbf{z},\mu}} \leq \left\|\frac{f - M + M}{\sqrt{M}}\right\|_{H^m_{\mathbf{z},\mu}} \leq \left\|\frac{f(t) - M}{\sqrt{M}}\right\|_{H^m_{\mathbf{z},\mu}} + \left\|\sqrt{M}\right\|$$

$$\leq \xi e^{-\epsilon\beta t}\left\|\frac{f(0) - M}{\sqrt{M}}\right\|_{H^m_{\mathbf{z},\mu}(H^4_{\mathbf{x}})} + |\mathbb{T}^3|. \quad (2.7)$$

Another problem we care about in UQ

- How to quantify the uncertainty?
  - Stochastic Collocation (SC) [Xiu, Hesthaven, 05]
  - Generalized Polynomial Chaos Stochastic Galarkin(gPC-SG) [Xiu, Karniadakis, 02], [Ghanem, Spanos, 91]
- The high order accuracy of gPC-SG depends on the regularity of the solution in random space and the stability of the system after gPC-SG.

The framwork of gPC-SG

For $\mu(\mathbf{z})$-measure, it has a set of corresponding orthogonal polynomial basis $\{\Phi_i\}_{i=0}^{\infty}$, s.t $\int \Phi_i \Phi_j d\mu(\mathbf{z}) = \delta_{ij}$. Find

$$\hat{f}^K = \sum_{i=0}^{K} \hat{f}_i(t, \mathbf{x}, \mathbf{v}) \Phi_i(\mathbf{z}), \tag{3.1}$$

in the K-th order subspace $\{\Phi_i\}_{i=0}^{K}$. [Hu, Jin, 2017]

$$\left\langle \partial_t \hat{f}^K + \mathbf{v} \cdot \nabla_{\mathbf{x}} \hat{f}^K, \mathbf{\Phi}^K \right\rangle_\pi = \frac{1}{\epsilon} \left\langle Q(\hat{f}^K, \hat{f}^K), \mathbf{\Phi}^K \right\rangle_\pi . \tag{3.2}$$

The deterministic system for the approximate solution after gPC-SG

$$\partial_t \hat{f}_j + \mathbf{v} \cdot \nabla_{\mathbf{x}} \hat{f}_j = \frac{1}{\epsilon} \sum_{k,l=0}^{K} E_{klj} Q(\hat{f}_k, \hat{f}_l), \quad \text{for } 0 \le j \le K$$

where $E_{klj} = \int_{I_{\mathbf{z}}} \Phi_k \Phi_l \Phi_j \, d\mu(\mathbf{z})$.

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Condition on the random basis

Condition on the random basis

The basis functions $\Phi_k(z)$ satisfy,

$$|\Phi_k| \leq (k+1)^p, \quad k \geq 0, \tag{3.3}$$

eg. Legendre ($p = 1/2$), Chebyshev ($p = 0$)

[Jin, Shu, 17] And accordingly, we define the new energy term of the approxiate solution $\hat{E}_f^N$,

$$q \geq p+2, \quad \hat{E}_f^N(t) := \sum_{k=0}^{K} \left\| (k+1)^q \hat{f}_k(t) \right\|_{H_x^N}^2 \tag{3.4}$$

Stability of the system after gPC-SG

For $N \geq 4$, if initially satisfies $\hat{E}_{\hat{f}-M}^N(0) \leq \epsilon^2 C_0$, then

$$\hat{E}_f^N(t) \leq 2\hat{E}_{\hat{f}-M}^N(0) + 2|\mathbb{T}^3|, \quad t > 0 \tag{3.5}$$

Here all the constants are independent of ϵ and K.

K dependency on initial condition

The deterministic system after gPC-SG

$$\partial_t \hat{h}_j + \mathbf{v}\cdot\nabla_{\mathbf{x}}\hat{h}_j - \frac{1}{\epsilon}\mathcal{L}\hat{h}_j = \frac{1}{\epsilon}\sum_{k,l=0}^{K} E_{klj}\Gamma(\hat{h}_k,\hat{h}_l), \quad \text{for } 0\le j\le K$$

The system for the $\{h_\beta\}_{\{|\beta|\le m\}}$

$$\partial_t h_\beta + \mathbf{v}\cdot\nabla_{\mathbf{x}} h_\beta - \frac{1}{\epsilon}\mathcal{L}h_\beta = \frac{1}{\epsilon}\sum_{\mathbf{j}\le\beta}\binom{\beta}{\mathbf{j}}\Gamma(h_{\mathbf{j}}, h_{\beta-\mathbf{j}}), \quad \text{for } 0\le|\beta|\le m$$

Recall the result for the $\sum_{|\beta|\le m}\|h_\beta(t)\|_{H^4_{\mathbf{x}}}$, we request $\sum_{|\beta|\le m}\|h_\beta(0)\|_{H^4_{\mathbf{x}}}\le\epsilon C(m)$, so if we directly adopt the energy estimate framework, we request

$$\left\|h^K(0)\right\|_{H^N_{\mathbf{x}}}\le\epsilon C(K), \tag{3.6}$$

which is bad in terms of K.

- Open Problem:
  - Get rid of the ϵ dependency on initial data.
  - Rigorous proof of the spectral convergence of the gPC-SG method
- Thanks for your attention!