External boundary control of the motion of a body in a fluid

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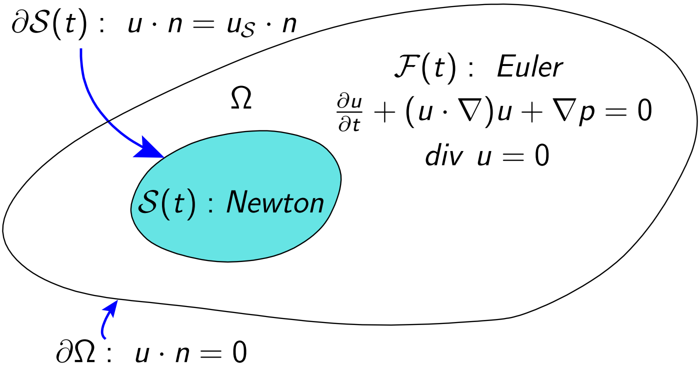
Joint work with Olivier Glass and Franck Sueur

The motion of a rigid body immersed in a perfect two-dimensional fluid

$$\Omega = \mathcal{F}(t) \cup \mathcal{S}(t) \subset \mathbb{R}^2$$



Solid: rigid movement:

$$\mathcal{S}(t) = h(t) + R(\vartheta(t))(\mathcal{S}(0) - h_0),$$
$$u_S(t,x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp},$$

where

- $h(t) \in \mathbb{R}^2$ - center of mass
- $\vartheta(t) \in \mathbb{R}$ - angle variangle
- R - matrix of rotation

(h, ϑ): Newton's law (force = fluid's pressure on the boundary)

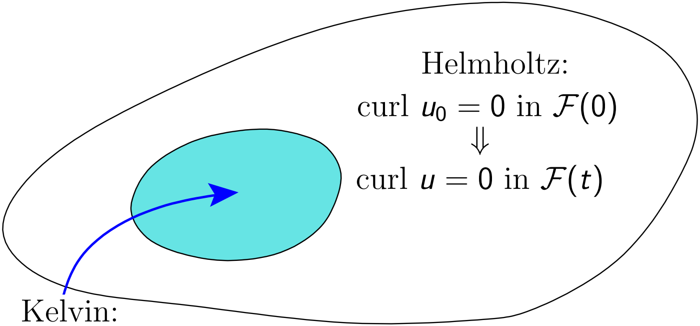
$$mh''(t) = \int_{\partial \mathcal{S}(t)} p\, n\, d\sigma_x,$$

$$\mathcal{J}\vartheta''(t) = \int_{\partial \mathcal{S}(t)} p\, (x - h(t))^{\perp} \cdot n\, d\sigma_x.$$

where

- $m > 0$ - mass
- $\mathcal{J} > 0$ moment of inertia
- fluid density = homogeneously 1

We assume that the fluid is irrotational at the initial time.



$$\int_{\partial\mathcal{S}(t)} u(t) \cdot \tau \, d\sigma = \int_{\partial\mathcal{S}(0)} u_0 \cdot \tau \, d\sigma = \gamma \in \mathbb{R}$$

The Cauchy problem for this system is now well-understood.

O. Glass, F. Sueur, Uniqueness results for weak solutions of two-dimensional fluid-solid systems, (Résultats d'unicité pour des solutions faibles de systemes fluide-structure bidimensionnels), published in Archive for Rational Mechanics and Analysis, 2015.

J.-G. Houot, J. San Martin, M. Tucsnak, Existence and uniqueness of solutions for the equations modelling the motion of rigid bodies in a perfect fluid. J. Funct. Anal., 259(11):2856–2885, 2010.

J. Ortega, L. Rosier and T. Takahashi, On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid. Ann. Inst. H. Poincaré Anal. Non Linéaire, 24(1):139–165, 2007.

There are also several papers devoted to derive simplified models for this system when the solid has a small size, or the control acts on the boundary of the solid.

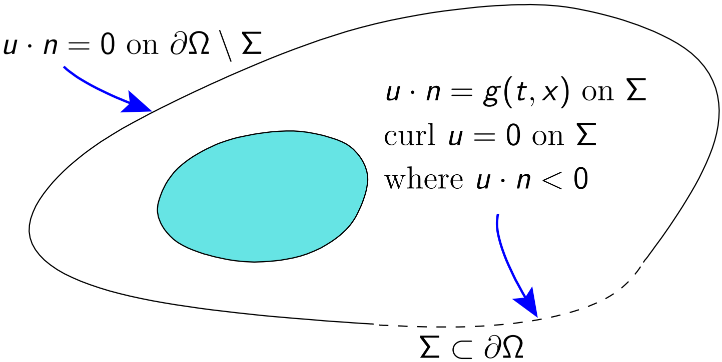
O. Glass, C. Lacave, F. Sueur, On the motion of a small body immersed in a two dimensional incompressible perfect fluid, Bull. Soc. Math. France 142 (2014), no. 2, 1–48.

O. Glass, A. Munnier, F. Sueur, Dynamics of a point vortex as limits of a shrinking solid in an irrotational fluid, preprint 2014, arXiv:1402.5387.

O. Glass, L. Rosier, On the control of the motion of a boat, (Controle du mouvement d'un bateau), published in Mathematical Models and Methods in Applied Sciences, 2013.

The control problem - Yudovich-type control

Let $T > 0$, $\Sigma \subset \partial\Omega$, find $g \in C_0^1([0,T] \times \Sigma)$ with $\int_\Sigma g = 0$ s.t.



Compatible initial and target data

$\mathcal{S}(0) \subset \Omega$ bounded, closed, simply connected with smooth boundary, $u_0 \in C^1(\overline{\mathcal{F}(0)}; \mathbb{R}^2)$, $\gamma \in \mathbb{R}$,
$h_0, h_1 \in \Omega,\ h_0', h_1' \in \mathbb{R}^2,\ \vartheta_0 = 0,\ \vartheta_1, \vartheta_0', \vartheta_1' \in \mathbb{R}$, such that

- $(h_0, 0)$ and (h_1, ϑ_1) are in the same connected component of

$$\mathcal{Q} = \{(h, \vartheta) : d\left(\partial\Omega, \mathcal{S}(h, \vartheta)\right) > 0\}$$

- $u_0 \cdot n = 0$ on $\partial\Omega,\ u_0 \cdot n = (h_0' + \vartheta_0'(x - h_0)^{\perp}) \cdot n$ on $\partial\mathcal{S}(0)$
- div $u_0 =$ curl $u_0 = 0$ in $\mathcal{F}(0)$

Main result

Theorem

Under the above conditions, there exists $g \in C_0^1([0,T] \times \Sigma)$ *and a solution* $(h, \vartheta, u) \in C^2([0,T];\mathcal{Q}) \times C^1([0,T];C^1(\overline{\mathcal{F}(t)};\mathbb{R}^2))$ *of the control system, which satisfies*

$$(h, h', \vartheta, \vartheta')(T) = (h_1, h_1', \vartheta_1, \vartheta_1').$$

Our strategy consists of three steps:

- reformulating the model as an ODE in $(h(\cdot), \vartheta(\cdot))$,
- proving that there exists a controllable simplified equation which is close to the model,
- concluding by some Brouwer-type topological arguments that the model is also controllable.

We denote

- $q = (h, \vartheta)$, $q' = (h', \vartheta')$, for any $(h, \vartheta) \in \mathcal{Q}$, $(h', \vartheta') \in \mathbb{R}^3$,
- $\mathcal{S}(q) = h + R(\vartheta)(\mathcal{S}(0) - h_0)$ and $\mathcal{F}(q) = \Omega \setminus \mathcal{S}(q)$,
- $q(t) = (h(t), \vartheta(t)), \ t \in [0, T]$.

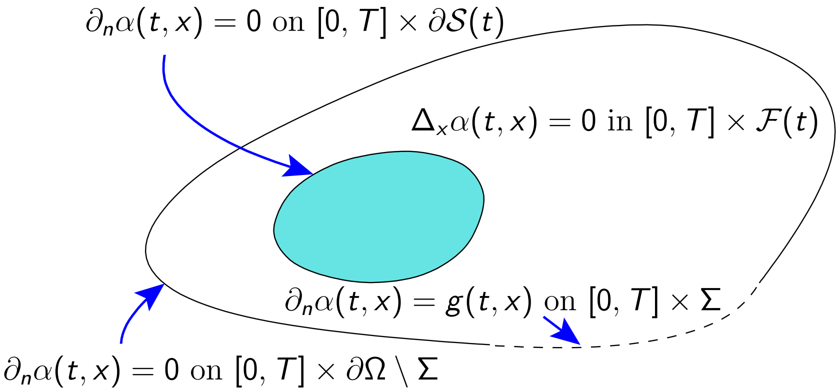
Decomposition of the fluid velocity

We know that u is the solution of the following div/curl system:

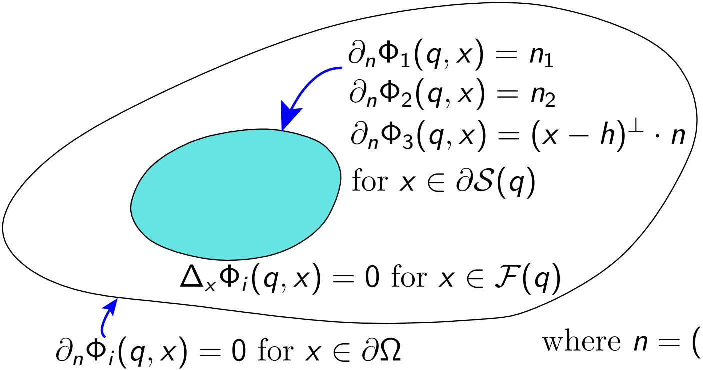
$$\begin{aligned} &\text{div } u = \text{curl } u = 0 \ \text{ in } \ \in \mathcal{F}(t), \\ &u \cdot n = \mathbb{1}_{\Sigma} g \text{ on } \partial\Omega, \\ &u \cdot n = u_S \cdot n \text{ on } \partial\mathcal{S}(t), \\ &\int_{\partial\mathcal{S}(t)} u(t) \cdot \tau \, d\sigma = \gamma. \end{aligned}$$

Since this system is linear in u, it can be uniquely decomposed with respect to the last three equations. For this decomposition, we introduce the following functions.

Let $\alpha \in C^2([0,T] \times \overline{\mathcal{F}(t)}; \mathbb{R})$ such that

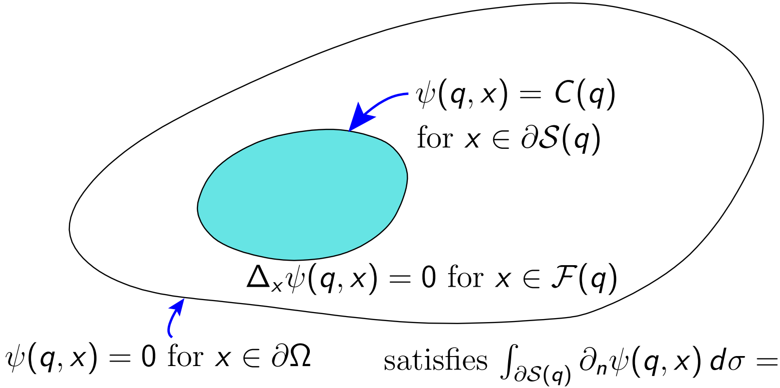


The Kirchhoff potentials $\Phi(q,\cdot) = (\Phi_1, \Phi_2, \Phi_3)(q,\cdot)$ are defined as the solution of the Neumann problem



where $n = (n_1, n_2)$

The stream function $\psi(q,\cdot)$ for the circulation term is defined in the following way. There exists a unique $C(q) \in \mathbb{R}$, such that the solution of the Dirichlet problem



satisfies $\int_{\partial\mathcal{S}(q)} \partial_n \psi(q,x)\, d\sigma = -1$

Decomposition of the fluid velocity

Lemma

Given $q \in C^2([0,T];\mathcal{Q})$, $g \in C^1_0([0,T]\times\Sigma)$, *we have that*

$$u(t,x) = \nabla\alpha(t,x) + \nabla(q'(t)\cdot\Phi(q(t),x)) + \gamma\nabla^{\perp}\psi(q(t),x),$$

is the unique solution of the Euler equation on $[0,T]\times\overline{\mathcal{F}(q(t))}$ *with* $\nabla p = -\partial_t u - \frac{\nabla_x |u|^2}{2}$.

In order to reformulate the model as an ODE, we introduce the following quantities.
We introduce the genuine and added mass matrices

$$\mathcal{M}_g = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J}, \end{pmatrix}$$

respectively,

$$\mathcal{M}_a(q) = \left(\int_{\mathcal{F}(q)} \nabla \Phi_i(q,x) \cdot \nabla \Phi_j(q,x)\, dx \right)_{i,j=1,2,3},$$

which is a symmetric and positive-semidefinite Gramian matrix, their sum

$$\mathcal{M}(q) = \mathcal{M}_g + \mathcal{M}_a(q).$$

The bilinear map $\Gamma(q)$ is defined as

$$\langle \Gamma(q), p, p \rangle = \left(\sum_{1 \leq i,j \leq 3} \Gamma^k_{i,j}(q) p_i p_j \right)_{1 \leq k \leq 3} \in \mathbb{R}^3, \ \forall p \in \mathbb{R}^3,$$

where, for each $i, j, k \in \{1, 2, 3\}$,

$$\Gamma^k_{i,j}(q) = \frac{1}{2} \left((\mathcal{M}_a)^i_{k,j} + (\mathcal{M}_a)^j_{k,i} - (\mathcal{M}_a)^k_{i,j} \right)(q),$$

with

$$(\mathcal{M}_a)^k_{i,j} := \frac{\partial (\mathcal{M}_a)_{i,j}}{\partial q_k}.$$

Finally, we introduce the quantities

$$B(q) = \int_{\partial\mathcal{S}(q)} \partial_n \psi(q,x) \left(\partial_n \Phi(q,x) \times \partial_\tau \Phi(q,x)\right) \, d\sigma_x,$$

and

$$E(q) = -\frac{1}{2} \int_{\partial\mathcal{S}(q)} |\partial_n \psi(q,x)|^2 \partial_n \Phi(q,x) \, d\sigma_x.$$

We define

- $F_\Gamma(q, q') = -(\mathcal{M}(q))^{-1}\langle \Gamma(q), q', q' \rangle$
- $G_{\overline{\Omega}}(q, q', \gamma) = (\mathcal{M}(q))^{-1}(\gamma^2 E(q) + \gamma q' \times B(q))$
- $F(\alpha, q, q', \gamma) =$ sum of all integral terms containing α

A reformulation of the model as an ODE

Proposition

We have that (q, u) is a solution of the fluid-solid system if and only if q satisfies an ODE of the form

$$q''(t) - F_\Gamma(q(t), q'(t)) = F(\alpha, q(t), q'(t), \gamma) + G_{\overline{\Omega}}(q(t), q'(t), \gamma),$$

$t \in [0, T]$, *with initial data*

$$q(0) = q_0 := (h_0, 0), \;\; q'(0) = q'_0 := (h'_0, \vartheta'_0).$$

A simplified equation

We claim that

- for small values of $\gamma \Rightarrow G_{\overline{\Omega}}(\cdot,\cdot,\gamma)$ has little impact on the dynamics,
- for good choices of $\alpha \Rightarrow F(\alpha,\cdot,\cdot,\gamma)$ will behave like the sum of two Dirac approximations in time.

A simplified equation

Therefore, we consider the following impulsive control system where the control is given by $k_0, k_1 \in \mathbb{R}^3$, which we can control by geodesic arguments.

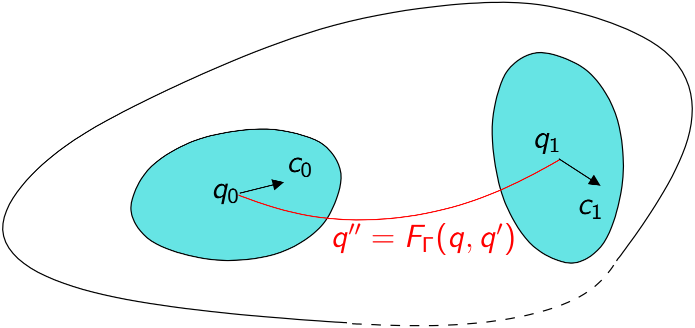
$$q''(t) - F_\Gamma(q(t), q'(t)) = k_0 \delta_0(t) + k_1 \delta_T(t), \;\; t \in [0, T],$$
$$q(0) = q_0, \;\; q'(0) = q'_0,$$

where δ_0 and δ_T denote the Dirac distributions at time 0 and T. We further set, for $\delta > 0$ small enough,

$$\mathcal{Q}_\delta = \{(h, \vartheta) : d\,(\partial\Omega, \mathcal{S}(h, \vartheta)) \geq \delta\}.$$

Geodesic arguments for the simplified equation

There exists c_0, c_1 and a geodesic of the normal form generated by F_Γ connecting (q_0, c_0) with (q_1, c_1) in $\mathcal{Q}_\delta$ in time T. We may pick our control $k_0 := c_0 - q_0'$, $k_1 := q_1' - c_1$.



Approximation via impulsive control

In order to be close to this simplified equation, we will use impulsive controls at the beginning and at the end of our time interval, to replicate the behaviour of the Dirac distributions. Namely, we will use controls of the form

$$g(t,x) = g_\varepsilon(t,x) := \beta_\varepsilon(t) g_1(x),$$

where β_ε^2 is a Dirac approximator, as $\varepsilon \to 0^+$.

Therefore, in the case of small γ and small $\varepsilon > 0$, the dominant term in $F(\alpha, q(t), q'(t), \gamma)$ will be

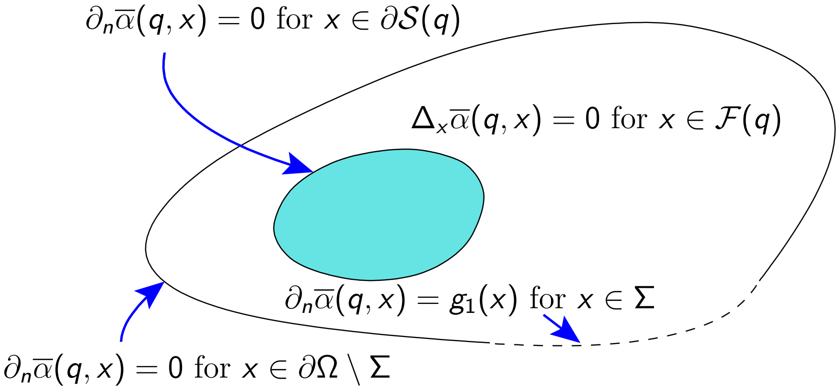
$$\mathcal{M}(q(t))^{-1} \int_{\partial \mathcal{S}(q(t))} \frac{|\nabla \alpha(t,x)|^2}{2} \, \partial_n \Phi(q(t), x) \, d\sigma.$$

For given circulation $\gamma \in \mathbb{R}$, let $q_{\varepsilon,\gamma} = (h_{\varepsilon,\gamma}, \vartheta_{\varepsilon,\gamma})$ be the associated trajectory of the solid.

Observe that

$$\alpha_\varepsilon(t,x) = \beta_\varepsilon(t)\overline{\alpha}(q_{\varepsilon,\gamma}(t),x),$$

where $\overline{\alpha}(q,\cdot)$ as the solution of



Difficulty: $F(\alpha, \cdot, \cdot, \cdot)$ contains terms involving $\partial_t \alpha$, β_ε is compactly supported in time $\Rightarrow$ we will need to handle carefully any terms containing β'_ε.

On the other hand, the dominant term in $F(\alpha, q, q', \gamma)$ becomes

$$\overline{F}(\overline{\alpha}, q) = \mathcal{M}(q)^{-1} \int_{\partial \mathcal{S}(q)} \frac{|\nabla \overline{\alpha}(q, x)|^2}{2} \partial_n \Phi(q, x) \, d\sigma.$$

Given $q \in \mathcal{Q}$, we claim that we can choose g_1 such that $\overline{F}(\overline{\alpha}, q)$ can attain any given direction, up to an arbitrarily small error.

Choosing the impulsive control

Proposition

For any $\nu > 0$, $v \in \mathbb{R}^3$, $q = (h, \vartheta) \in \mathcal{Q}$, *there exists* $\overline{\alpha}(q, \cdot) \in C^8(\mathcal{F}(q))$ *such that*

$$\Delta_x \overline{\alpha}(q, x) = 0 \text{ in } \mathcal{F}(q), \ \partial_n \overline{\alpha}(q, x) = 0 \text{ on } \partial\mathcal{F}(q) \setminus \Sigma,$$

and

$$\left| \overline{F}(\overline{\alpha}, q) - v \right| \leq \nu,$$

$$\int_{\partial\mathcal{S}(q)} \overline{\alpha}(q, x)\, \partial_n \Phi(q, x)\, d\sigma_x = (0, 0, 0).$$

The limit trajectory

Consider $\bar{q} = (\bar{h}, \bar{\vartheta}) : [0, T] \to \mathcal{Q}$ defined by

$$\bar{q}''(t) = F_\Gamma(\bar{q}(t), \bar{q}'(t)), \ \forall t \in [0, T],$$
$$\bar{q}(0) = q_0,$$
$$\bar{q}'(0) = q_0' + \overline{F}(\overline{\alpha}, q_0).$$

Approximation result

Proposition

Assume that

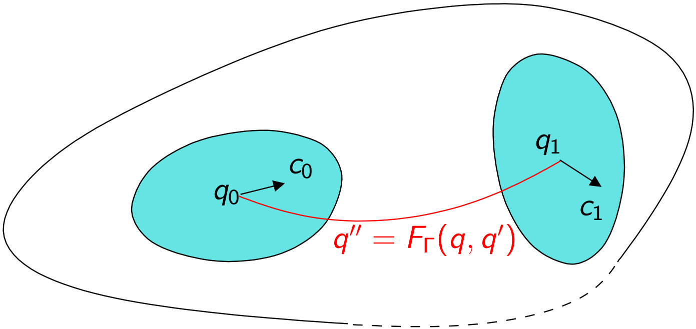
$$\int_{\partial \mathcal{S}(q_0)} \overline{\alpha}(q_0, x)\, \partial_n \Phi(q_0, x)\, d\sigma_x = (0,0,0).$$

(key assumption in order to tackle the terms containing β'_ε)
Then, we have

$$\lim_{(\varepsilon,\gamma)\to 0} \|q_{\varepsilon,\gamma} - \bar{q}\|_{C^1((0,T])} = 0.$$

Concluding ideas

- In the case $\gamma = 0$, for any $q_0, q_1 \in \overline{\Omega}_0$, there exists c_0, c_1 and a geodesic associated with F_Γ connecting (q_0, c_0) with (q_1, c_1) in time T.



- We may pick our control such that the initial jump in velocity made by the trajectory $\bar{q}$ is exactly a jump from q_0' to $c_0 + \mathcal{O}(\nu)$. Using the stability of the geodesic, we deduce that $|(\bar{q}(T), \bar{q}'(T)) - (q_1, c_1)| = \mathcal{O}(\nu)$.

- We may use similar arguments to add another impulsive control at the end of the time interval which creates a jump in velocity from c_1 to $q_1' + \mathcal{O}(\nu)$.

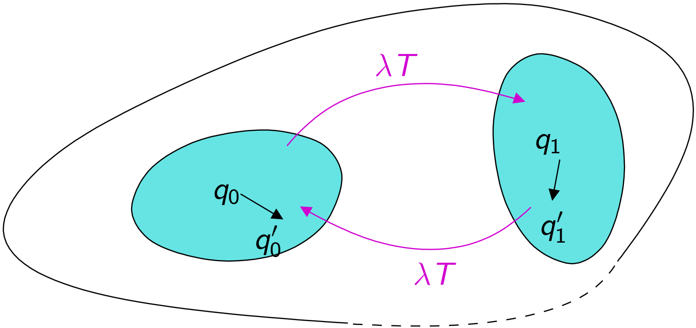
- For small γ and ε, we end up with a trajectory $q_{\varepsilon,\gamma}$ which at time T is close to (q_1, q_1'), with respect to ν, γ and ε.
- We conclude that in fact we may attain exactly (q_1, q_1'), by using a Brouwer-type topological argument .

A time-rescale argument (Coron)

Let $\gamma \in \mathbb{R}$, $\lambda \in (0, 1)$.

- Finding a solution of our ODE transporting (q_0, q_0') to (q_1, q_1') in time λT with circulation γ is equivalent to finding a solution transporting $(q_0, \lambda q_0')$ to $(q_1, \lambda q_1')$ in time T with circulation $\lambda\gamma$.
- We may construct the latter using our previous strategy, for small enough λ.
- Therefore, we have managed to arrive to our target (q_1, q_1'), but in shorter time, λT. To fix this, we pick λ such that it also satisfies $\frac{1-\lambda}{2\lambda} \in \mathbb{N}$.

- We may simply keep jumping back and forth between the states (q_0, q_0') and (q_1, q_1') until time T, by using the reversibility of the equation.



Thank you for your attention!