Optimal Actuator Location for Norm Optimal Control of Null Controllable Heat Equation and Related Problems

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VII Partial differential equations, optimal design and numerics

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Original problems

Consider the following system

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega u, & \text{in } \Omega \times (0, T), \\ y = 0, & \text{on } \partial\Omega \times (0, T), \\ y(0) = y_0, & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open domain, $\omega \subset \Omega$ is a measurable subset with $|\omega| = \alpha|\Omega|$ ($0 < \alpha < 1$ is a given constant and $|\cdot|$ is the Lebesgue measure), $y_0 \in L^2(\Omega)$ is the initial data, $u \in L^2(\Omega \times (0, T))$ is the control.

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Denote $y(\cdot; \chi_\omega u)$ the solution of (1).

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It is well known that for each $y_0 \in L^2(\Omega)$ there exists $u \in L^2(\Omega \times (0, T))$ such that $y(T; \chi_\omega u) = 0$. i.e., the system (1) is null controllable.

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$$N(\omega, y_0) = \inf\{\|\chi_\omega u\|_{L^2(\Omega\times(0,T))} \bigm| y(T; \chi_\omega u) = 0\}. \tag{2}$$

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It is also well known that there exists a unique $u^* \in L^2(\Omega \times (0, T))$ such that $y(T; \chi_\omega u^*) = 0$ and $\|\chi_\omega u^*\|_{L^2(\Omega\times(0,T))} = N(\omega, y_0)$.

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We call u^* the norm optimal control. Note that u^* steer y_0 to 0 with the minimal norm in $L^2(\Omega \times (0,T))$.

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- The problem

$$\sup_{\|y_0\|\le 1} N(y_0)$$

  is the best observability constant problem, please see [Y. Privat, E. Trélat and E. Zuazua]'s work.

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Main theorem

Theorem

There exists $\omega^* \subset \Omega$ *with* $|\omega^*| = \alpha|\Omega|$ *such that*

$$N(y_0) = N(\omega^*, y_0) = \inf_{|\omega|=\alpha|\Omega|} N(\omega, y_0). \tag{4}$$

In particular,

$$\omega^* = \{x \in \Omega \mid \Phi(x) > \lambda\} \cup E, \tag{5}$$

where $E \subset \{x \in \Omega \mid \Phi(x) = \lambda\}$ *satisfying*

$$\big|\{x \in \Omega \mid \Phi(x) > \lambda\}\big| + |E| = \alpha|\Omega| \tag{6}$$

and $\Phi(x) = \int_0^T \varphi^2(x,t)\mathrm{d}t$, *here* $\lambda > 0$ *is a constant and* $\varphi \in L^2(\Omega \times (0,T))$ *satisfies the following system without initial data*

$$\begin{cases} \partial_t\varphi + \Delta\varphi = 0, & \text{in } \Omega \times (0,T), \\ y = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{7}$$

Outline

Outline of proof: Relax the control problem

Relax the original problem to a relaxed problem

$$\inf_{\beta \in \mathcal{U}_\alpha} N(\beta, y_0),$$

where

$$\mathcal{U}_\alpha = \left\{ \beta \in L^\infty(\Omega; [0,1]) \;\Big|\; \int_\Omega \beta^2 \mathrm{d}x = \alpha|\Omega| \right\},$$

and

$$N(\beta, y_0) = \inf\{\|\beta u\|_{L^2(\Omega \times (0,T))} \;\big|\; y(\cdot; \beta u) = 0\} \tag{8}$$

such that $y(\cdot; \beta u)$ is the solution to the following system

$$\begin{cases} \partial_t y - \Delta y = \beta u, & \text{in } \Omega \times (0, T), \\ y = 0, & \text{on } \partial\Omega \times (0, T), \\ y(0) = y_0, & \text{in } \Omega. \end{cases} \tag{9}$$

Outline of proof: Characterize optimal control

Let

$$\mathcal{J}(\beta,\varphi_T)=\frac{1}{2}\iint_{\Omega\times(0,T)}\beta^2\varphi^2\mathrm{d}x\mathrm{d}t+(y_0,\varphi(0))_{L^2(\Omega)}, \tag{10}$$

where φ is the solution to the following system

$$\begin{cases}\partial_t\varphi+\Delta\varphi=0, & \text{in } \Omega\times(0,T),\\ \varphi=0, & \text{on } \partial\Omega\times(0,T),\\ \varphi(T)=\varphi_T, & \text{in } \Omega.\end{cases} \tag{11}$$

Usually, we wish $u_\beta^*\in L^2(\Omega\times(0,T))$ can be characterize by

$$u_\beta^*=\beta\varphi^* \tag{12}$$

and

$$\mathcal{J}(\beta,\varphi_T^*)=\inf_{\varphi_T\in L^2(\Omega)}\mathcal{J}(\beta,\varphi_T), \tag{13}$$

where φ^* is the solution of (11) with initial data φ_T^*.

Outline of proof: Difficulty

Unfortunately, it is impossible to get (12) since the problem (13) cannot attain the infimum. i.e., for each $\beta \in \mathcal{U}_\alpha$, extreme value problem

$$\inf_{\varphi_T \in L^2(\Omega)} \mathcal{J}(\beta, \varphi_T) \tag{14}$$

do not have minimizer. [Please see Zuazua's book, page 135.]

Outline of proof: Overcome the difficulty

In order to get the formula (12), we rewrite the functional (10) by

$$\mathcal{J}(\beta, \varphi) = \frac{1}{2} \iint_{\Omega \times (0,T)} \beta^2 \varphi^2 \mathrm{d}x \mathrm{d}t + (y_0, \varphi(0))_{L^2(\Omega)}, \tag{15}$$

and denote

$$\mathcal{X} = \{\varphi \in L^2(\Omega \times (0, T)) \mid \varphi(T) = \varphi_T \in L^2(\Omega)\}, \tag{16}$$

it is obviously that $\mathcal{X} \subset C((0, T); L^2(\Omega)) \cap L^2(\Omega \times (0, T))$, now, we define

$$\|\varphi\|_* = \sup_{\beta \in \mathcal{U}_\alpha} \left[\iint_{\Omega \times (0,T)} \beta^2 \varphi^2 \mathrm{d}x \mathrm{d}t \right]^{\frac{1}{2}},$$

thanks to observability inequality on measurable set, we get $(\mathcal{X}, \| \cdot \|_*)$ is a norm space. Denote X_* the complement of X under the norm $\| \cdot \|_*$.

Outline of proof: Characterize the space X_*

Lemma

Let $\beta \in \mathcal{U}_\alpha$, and let X_* be the complement of X under the norm $\|\cdot\|_*$. Then under an isometric isomorphism, any element of X_* can be expressed as a function $\varphi \in C((0,T); L^2(\Omega)) \cap L^2(\Omega \times (0,T))$ which satisfies (in the sense of weak solution)

$$\begin{cases} \partial_t \varphi + \Delta \varphi = 0, & \text{in } \Omega \times (0,T), \\ \varphi = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

and $\beta\varphi = \lim_{n\to\infty} \beta\varphi(\cdot; z_n)$ in $L^2(\Omega \times (0,T))$ for some sequence $\{z_n\}_{n=1}^{\infty} \subset L^2(\Omega)$, where $\varphi(\cdot; z_n)$ is the solution of (11) with initial value $\varphi_T = z_n$.

Remark: $\|\cdot\|_*$ is indeed an equivalent norm of the norm of $L^2(\Omega \times (0,T))$.

Outline of proof: Characterize the norm optimal control

Lemma

For each $\beta \in \mathcal{U}_\alpha$, the optimal control u_β^* of (2) can be characterize by (12), i.e.,

$$u_\beta^* = \beta\varphi^*,$$

where $\varphi^* \in C((0, T); L^2(\Omega)) \cap L^2(\Omega \times (0, T))$ satisfies

$$\begin{cases} \partial_t \varphi^* + \Delta\varphi^* = 0, & \text{in } \Omega \times (0, T), \\ \varphi^* = 0, & \text{on } \partial\Omega \times (0, T). \end{cases}$$

and

$$\mathcal{J}(\beta, \varphi^*) = \inf_{\varphi \in X_*} \mathcal{J}(\beta, \varphi).$$

Moreover,

$$\mathcal{J}(\beta, \varphi^*) = -\frac{1}{2} N(\beta, y_0)^2. \tag{17}$$

Outline of proof: Transform problem (3) to a minimax problem

From above Lemmas, we rewrite the relaxed case of problem (3) as the following problem

$$\begin{aligned} N_\alpha(y_0)^2 &= \inf_{\beta\in\mathcal{U}_\alpha} N(\beta, y_0)^2 \\ &= 2 \inf_{\beta\in\mathcal{U}_\alpha}\left[-\inf_{\varphi\in X_*}\mathcal{J}(\beta,\varphi)\right] \\ &= -2 \sup_{\beta\in\mathcal{U}_\alpha}\inf_{\varphi\in X_*}\mathcal{J}(\beta,\varphi). \end{aligned} \tag{18}$$

Remarks:

1. (18) is obviously a minimax problem.
2. In general, $N(y_0) \geq N_\alpha(y_0)$.

Outline of proof: Existence of the minimax problem

Since $\mathcal{U}_\alpha$ is weakly star compact and $X_* \subset L^2(\Omega \times (0, T))$, check the conditions of von Neumann Minimax Theorem directly, there exists $\beta^*, \varphi^* \in X_*$ so that

$$\mathcal{J}(\beta, \varphi^*) \leq \mathcal{J}(\beta^*, \varphi^*) \leq \mathcal{J}(\beta^*, \varphi), \forall \beta \in \mathcal{U}_\alpha, \varphi \in X_*. \tag{19}$$

This means that (β^*, φ^*) is the saddle point of $\mathcal{J}$.

Outline of proof: Characterize the relaxed problem

There are two ways to characterize the relaxed problem:

1. By the symmetric method that used in **[Y. Privat, E. Trélat and E. Zuazua, Complexity and regularity of maximal energy domains for the wave equation with fixed initial data,** *Discrete Contin. Dyn. Syst.*, **35(2015), 6133-6153.]**

2. By real analysis method that used in **[B.Z. Guo, Y. Xu and D.H. Yang, Optimal actuator location of minimum norm controls for heat equation with general controlled domain,** *J. Diff. Equ.*, **261 (2016), 3588-3614.]**

Choose a method as above we can get our main theorem.

Outline of proof: Open problem

Let y be a solution to the following system

$$\begin{cases} \partial_t y - \Delta y = 0, & \text{in } \Omega \times (0,T), \\ y = 0, & \text{on } \partial\Omega \times (0,T), \\ y(0) = y_0, & \text{in } \Omega, \end{cases}$$

and $Y(x) = \int_0^T y^2(x,t)\mathrm{d}t \ (x \in \Omega)$.

Problem: for any $A \in \mathbb{R}$, does the following equality hold

$$\Big|\{x \in \Omega \mid Y(x) = A\}\Big| = 0? \tag{20}$$

For Further Reading

- Observability inequality on measurable set for heat equation
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Thank you for your attention!