The approximation of boundary controls for the one-dimensional wave equation

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The control for the continuous wave equation

Given $T \geq 2$ and $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$ there exists a control function $v \in C^0([0,T],\mathbb{C})$ such that the solution of the wave equation

$$\begin{cases} u''(t,x) - u_{xx}(t,x) = 0 & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0 & t \in (0,T), \\ u(t,1) = v(t) & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

satisfies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)).$$

The discrete model of the wave equation

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. For $T > 0$, we consider the following semi-discrete space approximation of the wave equation by the explicit finite-differences method:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} = 0 & 1 \leq j \leq N,\ t > 0, \\ u_0(t) = 0 & t \in (0, T), \\ u_{N+1}(t) = v_h(t) & t \in (0, T), \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \leq j \leq N. \end{cases} \tag{2}$$

The discrete controllability problem

Given $T \geq 2$, $h > 0$ and $((u_j^0, u_j^1))_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in C^0([0, T])$ such that the solution of the equation (2) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

Bad news

The main problem

The sequence of discrete controls $(v_h)_{h>0}$ converges to a control v of the continuous wave equation?

In general, there exist high-frequency spurious solutions generated by the discretization process that make the discrete controls diverge when the mesh-size h goes to zero. (Glowinski - Li - Lions ('90), Infante - Zuazua ('99), Micu ('03), Zuazua ('05), etc...)

Basically, this difficulty can be overcame by using an appropriate filtering technique to eliminate the short wave length components of the solutions of the discrete system, i.e. the large frequencies (of order $|n| = N$) of the discretized problem.

Our aim

- To filtering the initial data in an optimal range in order to restore the uniform controllability property.

- To obtain a relation between the range of filtration and the minimal time of control, recovering in many cases the usual minimal time to control for the (continuous) wave equation.

How we can control such a special "wave"?

Now, we can control everything!

Back to the math reality: The adjoint problem

Let us consider the corresponding homogeneous adjoint problem:

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t) - 2w_j(t) + w_{j-1}(t)}{h^2} = 0 & 1 \le j \le N,\ t > 0, \\ w_0(t) = 0 & t \in (0,T), \\ w_{N+1}(t) = 0 & t \in (0,T), \\ w_j(0) = w_j^0, \quad w_j' = w_j^1 & 1 \le j \le N. \end{cases} \tag{3}$$

The discretisation matrix

We define the matrix $A_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ as follows:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The adjoint problem (3) can be rewritten in a matricial form as follows:

$$\begin{cases} W''(t) + A_h W(t) = 0 & t > 0, \\ W(0) = W^0, \quad W'(0) = W^1, \end{cases} \tag{4}$$

where $W(t) = (w_1(t), ..., w_N(t))^T \in \mathbb{C}^N$ and the initial data is $\begin{pmatrix} W^0 \\ W^1 \end{pmatrix} = \begin{pmatrix} (w_j^0)_{1\le j\le N} \\ (w_j^1)_{1\le j\le N} \end{pmatrix} \in \mathbb{C}^{2N}$.

The operator $\mathcal{A}_h$

Now, if we set $Z(t) = \begin{pmatrix} W(t) \\ W'(t) \end{pmatrix}$ and $Z^0 = \begin{pmatrix} W^0 \\ W^1 \end{pmatrix}$, then (4) has the following equivalent vectorial form

$$\begin{cases} Z'(t) + \mathcal{A}_h Z(t) = 0 \\ Z(0) = Z^0, \end{cases}$$

where the operator $\mathcal{A}_h$ is given by $\mathcal{A}_h = \begin{pmatrix} 0 & -I_N \\ A_h & 0 \end{pmatrix}$ and I_N is the identity matrix of size N.

The eigenvalues and eigenvectors of the operator $\mathcal{A}_h$

The eigenvalues of $\mathcal{A}_h$ are given by the family $(i\,\lambda_n)_{1\le|n|\le N}$, where

$$\lambda_n = \frac{2}{h}\sin\left(\frac{n\pi h}{2}\right), \qquad 1\le|n|\le N, \tag{5}$$

and the corresponding eigenvectors are

$$\Phi_h^n = \begin{pmatrix} \frac{1}{i\lambda_n}\varphi_h^n \\ -\varphi_h^n \end{pmatrix} \qquad (1\le|n|\le N),$$

where

$$(\varphi_h^n)_{1\le|n|\le N} = \begin{pmatrix} \sin(n\pi h) \\ \sin(2n\pi h) \\ \dots \\ \sin(n\pi h N) \end{pmatrix} \in \mathbb{C}^{2N}.$$

Note that $(\Phi_h^n)_{1\le|n|\le N}$ forms an orthonormal basis in $\mathbb{C}^{2N}$.

Fourier decomposition of the initial condition

In general, for the semi-discrete equation (2), we will consider the following discretization of the initial condition (u^0, u^1) given by

$$U_h^0 = \begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1 \leq j \leq N} = \sum_{1 \leqslant |n| \leqslant M} a_{hn}^0 \Phi_h^n(x), \tag{6}$$

where $M = f(N) \leqslant N$ represents the range of filtration.

Previous result

S. Micu [Numer. Math. ('02)] proved that if the initial data are given by

$$\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} = \sum_{1\leq |n|\leq M} a_{hn}^0 \Phi_h^n,$$

with $M = \sqrt{N}$, then there exists a sequence of bounded controls $(v_h)_{h>0}$ for (2) provided that the initial condition verifies some conditions on its Fourier coefficients and that the time is large enough (but no quantitative estimate of this minimal time is given).

Moreover, S. Micu [Numer. Math. ('02)] proved that there exists regular initial data (with $M = N$) for which there exists no sequence of discrete controls uniformly bounded in $L^2(0, T)$.

The main results

Open problem

What about the range between $\sqrt{N}$ and N?

Optimal filtration for the approximation of boundary controls for the wave equation

- By filtering the initial data in an optimal range, we restore the uniform controllability property.
- We obtain a relation between the range of filtration and the minimal time of control, recovering in many cases the usual minimal time to control for the (continuous) wave equation.

A sufficient and necessary condition for the null-controllability

Lemma

Given $T > 0$, system (2) is null-controllable at time T if, and only if, for any initial data $U^0 = (u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, there exists $v_h \in C^0([0,T],\mathbb{C})$ which verifies

$$\int_0^T v_h(t)\,\frac{\overline{w_N(t)}}{h}\,dt = h\sum_{1\le j\le N}(u_j^0\overline{w}_j^1 - u_j^1\overline{w}_j^0)\ \left(\left((w_j^0, w_j^1)_{1\le j\le N}\right) \in \mathbb{C}^{2N}\right),$$

where $W = (w_1(t), ..., w_N(t))^T$ is the solution of (4).

The moment problem

Lemma

Given $T > 0$, system (2) is null-controllable at time T if, and only if, for any initial data $(u_j^0, u_j^1)_{1\leq j\leq N} = \sum_{1\leq |n|\leq N} a_n \Phi_h^n$ there exists $v_h \in C^0([0,T],\mathbb{C})$ which verifies

$$\int_0^T v_h(t)\, e^{-i\lambda_n t}\, dt = \frac{(-1)^n h}{\sin(n\pi h)} a_n \qquad (1 \leq |n| \leq N)\,. \tag{7}$$

The moment problems have been, from the very beginning, one of the most successful method for controllability problems (see the books of Avdonin and Ivanov ('95), Coron ('07), Komornic and Loretti ('05), Russel ('78), Tucsnak and Weiss ('09)).

The biorthogonals

A sequence $(\theta_m)_{1\leq|m|\leq N} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(e^{i\lambda_n t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t)e^{i\overline{\lambda}_n t}dt = \delta_{mn} \qquad (1\leq|m|,\,|n|\leq N)\,. \tag{8}$$

An explicit formula for the discrete control

Once we have a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ we can construct a control thanks to the following formula :

$$v_h(t)=\sum_{1\leq|n|\leq N}\frac{(-1)^{n+1}h}{\sin(n\pi h)}e^{-i\lambda_n\frac{T}{2}}a_n(h)\theta_n\left(t-\frac{T}{2}\right),$$

where $a_n(h)$ is related to a_n by the following relations:

$$a_n(h):=\begin{cases}0, & |n|>f(N),\\ \frac{1}{2}\left(\frac{\lambda_n}{n\pi}+1\right)a_n+\frac{1}{2}\left(\frac{\lambda_n}{n\pi}-1\right)a_{-n}, & |n|\leqslant f(N).\end{cases}\tag{9}$$

Construction of the biorthogonals

We construct and evaluate an explicit biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ in the following way:

1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.

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1. We construct an entire function P_m, with the property that $P_m(\lambda_n) = \delta_{mn}$.
2. We obtain an optimal estimate of the product P_m on the real axis.

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1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.
2. We obtain an optimal estimate of the product P_m on the real axis.
3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m \in L^1(R)\cap L^2(R)$ and $M_m(\lambda_n)=\delta_{mn}$.

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3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m \in L^1(R)\cap L^2(R)$ and $M_m(\lambda_n)=\delta_{mn}$.
4. The Fourier transform of the entire function $\psi_m(z) := P_m(z)M_m(z)$ gives the element θ_m of a biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$.

The Weierstrass product

$$P_m(z) = \prod_{\substack{1 \le |n| \le N \\ n \ne m}} \left(\frac{z}{\lambda_n} - 1 \right) \prod_{\substack{1 \le |n| \le N \\ n \ne m}} \frac{\lambda_n}{\lambda_m - \lambda_n} := P_m^1(z) S_m \qquad (z \in \mathbb{C}). \tag{10}$$

Lemma

For every $1 \le |m| \le N$, we have that

$$|S_m| = \cos^2 \frac{m\pi h}{2}. \tag{11}$$

Optimal estimates of the Weierstrass product

Proposition

There exists a constant $C > 0$ such that for any $1 \leq |m| \leq N$ we have that

$$|P_m(x)| \leq \begin{cases} C & \left(|x| < \frac{2}{h}\right) \\ C\exp\left(\varphi(x)\right) & \left(|x| \geq \frac{2}{h}\right), \end{cases} \tag{12}$$

where

$$\varphi(x) = \frac{2}{h}\ln\left(\frac{xh}{2} + \sqrt{\frac{x^2h^2}{4} - 1}\right).$$

The multiplicator

For every $b > 0$ we set

$$H_b := \frac{1_{[-b,b]}}{2b}.$$

We remark that

$$\int_{\mathbb{R}} H_b = \int_{-b}^{b} H_b = 1.$$

We introduce a small parameter $\eta \in (0,1)$ and we define the sequence $b_0 = b_1 = \frac{\eta}{2}$ and $b_2 = \ldots = b_{2/h+2} = \frac{h(1-\eta)}{2+h}$. We consider the convolution product

$$u := H_{b_0} * \ldots * H_{b_{\frac{2}{h}+2}}.$$

The multiplicator

Lemma

u is of class $C^{2/h+1}$ and is compactly supported in $[-1,1]$. Moreover, one has $u^{(2/h+1)} \in W^{1,\infty}(\mathbb{R})$, and the following estimates hold:

$$\int_{-1}^{1} u = ||u||_1 = 1,$$

$$||u^{(2)}||_1 \leqslant \frac{4}{\eta^2} \tag{13}$$

and

$$||u^{2/h+2}||_1 \leqslant \frac{4}{\eta^2}\left(\frac{2+h}{h(1-\eta)}\right)^{\frac{2}{h}}. \tag{14}$$

We use some estimates from Hormander ('03).

The estimates of the multiplicator

We define

$$M_m(z) := \int_{-1}^{1} u(t) e^{-i\frac{T^-}{2}(z-\lambda_m)t} dt, \qquad (15)$$

where $T^- := T(1-\delta)$, $\delta \in (0,1)$ is some sufficiently small constant.

Lemma

One has

$$M_m(\lambda_m) = 1 \qquad (16)$$

and for every $x \in \mathbb{R}$,

$$|M_m(x)| \leqslant 1. \qquad (17)$$

Moreover, for every $x \in \mathbb{R}$,

$$|M_m(x)| \leqslant \frac{16}{\left(\eta T^- |x-\lambda_m|\right)^2}, \qquad (18)$$

$$|M_m(x)| \leqslant \frac{16}{\left(\eta T^- |x-\lambda_m|\right)^2} \left(\frac{2+h}{h(1-\eta)|x-\lambda_m|T^-} \right)^{\frac{2}{h}}. \qquad (19)$$

The sequence of biorthogonals

Now, we consider

$$\psi_m(z) := P_m(z)M_m(z), \qquad z \in \mathbb{C}, \tag{20}$$

$$\Gamma(f) := \limsup_{N\to\infty} \frac{f(N)}{N} \in [0,1], \tag{21}$$

where $f : \mathbb{N}^* \to \mathbb{N}^*$ is the range of filtration.

The optimal time

If $T > \frac{2}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$, then $\psi_m \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and we have

$$||\psi_m||_{L^1(\mathbb{R})} \leqslant C(T,\Gamma(f)). \tag{22}$$

The biorthogonal θ_n is defined as the inverse Fourier transform of ψ_n.

The main result

Theorem

Let $(u^0, u^1) \in L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$*. Then, for any* $T > \frac{2}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$*, there exists a control* $v_h \in C^0([0,T],\mathbb{C})$ *bringing the solution of* (2) *(with initial condition* U_h^0*) to (0,0) such that the sequence* $(v_h)_{h>0}$ *is bounded in* $C^0([0,T],\mathbb{C})$*.*

The main ideas of the proof

We define our control as follows:

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_n \frac{T}{2}} a_n(h)\theta_n\left(t-\frac{T}{2}\right),$$

where $a_n(h)$ was defined in (9).

The key estimate

For any $T > \frac{2}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$ we are able to prove that

$$||\theta_m||_\infty \leqslant C. \tag{23}$$

Conclusions

- By filtering in an optimal way the initial condition we obtain that the minimal time of control is optimal ($T = 2$).
- The optimal range of filtration is localized in the area where the gap between the eigenvalues of the discrete model becomes small.
- Beyond this range, the gap is altered by the numerical discretization and we lose the optimal time of control.

Open problems

- To use similar estimates if we add a vanishing viscosity term (Micu, SICON ('08))
- What about the fractional Laplacian case?


Thank you