Null controllability for the heat equation in one dimension via backstepping approach

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joint work with Jean-Michel Coron

Null-controllability for the heat equation

Known methods:

- Fundamental solutions: Jones (77), Littman (78)
- Carleman estimates: Fursikov & Imanuvikov (96), Lebeau & Robbiano (95).
- Transmutation method (null-controllability of the heat equation vs the exact controllability of the wave equation): Miller (06).
- Flatness approach (consider x as a time variable): Martin, Rosier & Rouchon (14).

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With Jean-Michel Coron, we propose the backstepping method !

Backstepping approach

Backstepping: is a technique to stabilise the system.

Finite dimensional system: see e.g., Coron's book 07.

Partial differential equations:

- Initiated by Coron & Andréa-Novel (98), Liu & Krstic (00).
- Heat equations: Liu (03), Smyshlyaev & Krstic (04).
- Other equations: wave equations (Krstic et al. 08), hyperbolic equations (Krstic & Smyshlyaev 08, Coron et al. 13, Hu & Meglio 15, Auriol & Di Meglio 16, Coron, Hu & Olive 17), KdV equations (Cerpa & Coron 13) . . .
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Setting

Consider the following control system, for $T > 0$ given:

$$\begin{cases} u_t(t,x) = \big(a(x)u_x(t,x)\big)_x & \text{in } (0,T)\times[0,1], \\ u(t,0) = 0, \quad u(t,1) = U(t) & \text{for } t \in (0,T), \\ u(t=0,\cdot) = u_0 & \text{for } x \in [0,1]. \end{cases} \tag{1}$$

Here the state is $u(t,\cdot) \in L^2(0,1)$ and the control is $U(t) \in \mathbb{R}$. We assume that $a \in H^2(0,1)$ and a is uniformly elliptic, i.e., for some $\Lambda \geqslant 1$,

$$1/\Lambda \leqslant a \leqslant \Lambda \text{ in } [0,1]. \tag{2}$$

Null-controllability: U is chosen such that $u(T,\cdot) = 0$.

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Result

Theorem (Coron & Ng., ARMA 17)

Let $T > 0$. *There exists a piecewise constant functional* $\mathcal{K} : [0, T) \to L^2(0, 1)^*$ *s.t., for every* $u_0 \in L^2(0, 1)$, *if* $u \in C^0\big([0, T); L^2(0, 1)\big)$ *is the solution of* (1) *with* $U(t)$ *defined by*

$$U(t) := \mathcal{K}(t)u(t, \cdot),$$

then

$$u(t, \cdot) \to 0 \text{ in } L^2(0, 1) \text{ as } t \to T_-,$$
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$L^2(0, 1)^*$ is the set of continuous linear maps from $L^2(0, 1)$ into $\mathbb{R}$.

1. The feedback system is non-local.
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Idea of the proof

$\mathcal{K}$ is constructed via backstepping technique where the kernel depends on time. Let $(\lambda_n) \nearrow \infty$, $(t_n) \nearrow T$ with $t_0 = 0$ (which will be precise later!). The form of $\mathcal{K}$, for $t_n \leqslant t < t_{n+1}$,

$$\mathcal{K}(t)v := \int_0^1 k_n(1, y)v(y)\,dy \text{ for } v \in L^2(0,1),$$

where k_n is designed by backstepping as follows. Set, for $t_n \leqslant t < t_{n+1}$,

$$w(t,x) = u(t,x) - \int_0^x k_n(x,y)u(t,y)\,dy.$$

Then k_n defined in $D := \{(x,y) \in (0,1)^2;\ x \geqslant y\}$ is chosen s.t., if $u_t - (au_x)_x = 0$ for $x \in (0,1)$ and $u(t,0) = 0$, for $t_n \leqslant t < t_{n+1}$, then

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$$\|w(t,\cdot)\| \leqslant e^{-\lambda_n (t-t_n)} \|w(t_n,\cdot)\|_{L^2} \text{ for } t \in [t_n, t_{n+1}).$$

Next goals:

- Find k_n.
- Find l_n such that

$$u(t,x) = w(t,x) + \int_0^x l_n(x,y) w(t,y)\, dy \text{ for } t \in [t_n, t_{n+1}).$$

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Construction of k_n

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$$w_t - \left(aw_x\right)_x + \lambda w = \Big(2a(x)\big(k_x(x,x) + k_y(x,x)\big) + a_x(x)k(x,x) + \lambda\Big)u(t,x)$$

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$$w_t - (aw_x)_x + \lambda w = \Big(2a(x)\big(k_x(x,x) + k_y(x,x)\big) + a_x(x)k(x,x) + \lambda\Big)u(t,x)$$

$$+ \int_0^x \Big(\big(a(x)k_x(x,y) - (a(y)k_y(x,y)\big)_y - \lambda k(x,y)\Big)u(t,y)\,dy + a(0)k(x,0)u_x(t,0).$$

Construction of k_n

Recall $u_t - (au_x)_x = 0,\ u(t,0) = 0,$ and $w(t,x) := u(t,x) - \int_0^x k(x,y)u(t,y)\,dy.$

We have

$$w_t(t,x) = u_t(t,x) - \int_0^x k(x,y)u_t(t,y)\,dy = u_t(t,x) - \int_0^x k(x,y)\big(a(y)u_y(t,y)\big)_y\,dy$$

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Construction of k_n-contd - 1

Take the subindex back. Here is the system of k_n, with the notation
$\frac{d}{dx}k_n(x,x) = \partial_x k_n(x,x) + \partial_y k_n(x,x)$,

$$\begin{cases} 2a(x)\frac{d}{dx}k_n(x,x) + a_x(x)k_n(x,x) + \lambda_n = 0 & \text{for } x \in [0,1], \\ k_n(x,0) = 0 & \text{for } x \in [0,1], \\ \big(a(x)k_{n,x}(x,y)\big)_x - \big(a(y)k_{n,y}(x,y)\big)_y - \lambda_n k_n(x,y) = 0 & \text{in } D. \end{cases}$$

Recall $D := \{(x,y) \in (0,1)^2;\ x \geqslant y\}$. Solving the first equation with $k_n(0,0) = 0$, the system of k_n can be rewritten under the form

$$\begin{cases} k_n(x,x) = g_n(x) & \text{for } x \in [0,1], \\ k_n(x,0) = 0 & \text{for } x \in [0,1], \\ \big(a(x)k_{n,x}(x,y)\big)_x - \big(a(y)k_{n,y}(x,y)\big)_y - \lambda_n k_n(x,y) = 0 & \text{in } D. \end{cases}$$

The system is non-standard !

Key points: Well-posedness and good estimates for k_n !

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The inverse tranformation

One next wants to know how to compute u from w. Define

$$v(t,x) = w(t,x) + \int_0^x l_n(x,y) w(t,y)\,dy \text{ for } t_n \leqslant t < t_{n+1},$$

and search l_n s.t. if $w_t - \left(a w_x\right)_x + \lambda_n w = 0$ and $w(t,0) = 0$, for $t_n \leqslant t < t_{n+1}$, then $v_t - \left(a v_x\right)_x = 0$. Recall that

$$w_t - \left(a w_x\right)_x + \lambda w = \Big(2a(x)\big(k_x(x,x) + k_y(x,x)\big) + a_x(x) + \lambda\Big) u(t,x)$$

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Similarly, we have

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The inverse tranformation-contd 1

We then require

$$\begin{cases} 2a(x)\dfrac{d}{dx}l_n(x,x) + a_x(x)l_n(x,x) + \lambda_n = 0 & \text{for } x \in [0,1], \\ l_n(x,0) = 0 & \text{for } x \in [0,1], \\ \big(a(x)l_{n,x}(x,y)\big)_x - \big(a(y)l_{n,y}(x,y)\big)_y + \lambda_n l_n(x,y) = 0 & \text{in } D. \end{cases}$$

Solving the first equation with $l_n(0,0) = 0$, this system can be rewritten as

$$\begin{cases} l_n(x,x) = g_n(x) & \text{for } x \in [0,1], \\ l_n(x,0) = 0 & \text{for } x \in [0,1], \\ \big(a(x)l_{n,x}(x,y)\big)_x - \big(a(y)l_{n,y}(x,y)\big)_y + \lambda_n k_n(x,y) = 0 & \text{in } D. \end{cases}$$

Key points: Well-posedness and good estimates for l_n!

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The inverse tranformation-contd 2

In fact, we can prove

Lemma

We have, for $t_n \leqslant t < t_{n+1}$,

$$u(t,x) = v(t,x) := w(t,x) + \int_0^x l_n(x,y)w(t,y)\,dy. \tag{3}$$

Recall that $w(t,x) := u(t,x) - \int_0^x k_n(x,y)w(t,y)\,dy$.

Sketch of the proof. We claim that

$$l_n(x,y) = k_n(x,y) + \int_y^x l_n(x,\xi)k_n(\xi,y)\,d\xi.$$

Admitting this claim, we have (ignore the t variable)

$$v(x) = u(x) - \int_0^x k_n(x,y)u(y) + \int_0^x l_n(x,y)\left[u(y) - \int_0^y k_n(y,\xi)u(\xi)\,d\xi\right]dy$$

$$= u(x) + \int_0^x \left[l_n(x,y) - k_n(x,y) - \int_y^x l_n(x,\xi)k_n(\xi,y)\,d\xi\right]u(y)\,dy = u(x) : \ (3)$$

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Admitting this claim, we have (ignore the t variable)

$$\begin{aligned} v(x) &= u(x) - \int_0^x k_n(x,y) u(y) + \int_0^x l_n(x,y) \left[u(y) - \int_0^y k_n(y,\xi) u(\xi)\, d\xi \right] dy \\ &= u(x) + \int_0^x \left[l_n(x,y) - k_n(x,y) - \int_y^x l_n(x,\xi) k_n(\xi,y)\, d\xi \right] u(y)\, dy = u(x) : \ (3) \end{aligned}$$

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Sketch of the Proof-contd

We now prove the claim. Define

$$\hat{l}_n(x,y) = k_n(x,y) + \int_y^x l_n(x,\xi) k_n(\xi,y)\, d\xi \quad \text{in } D.$$

The idea is to show that $\hat{l}_n$ and l_n satisfy the same system. Recall

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We are ready to prove

Theorem (Coron & Ng., ARMA 17)

Let $T > 0$, $(\lambda_n) \nearrow \infty$, $(t_n) \nearrow T$ *with* $t_0 = 0$. *Set*

$$s_0 = 0 \quad \textit{and} \quad s_n = \sum_{k=0}^{n-1} \lambda_k (t_{k+1} - t_k) \textit{ for } n \geqslant 1.$$

If $\lim_{n \to +\infty} (t_{n+1} - t_n)\lambda_n / \sqrt{\lambda_{n+1}} = +\infty$ *and* $\lim_{n \to +\infty} s_n / n = +\infty$, *then*

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Proof

We have

$$\|w(t,\cdot)\|_{L^2}^2 \leqslant Ce^{C\sqrt{\lambda_n}}\|u(t,\cdot)\|_{L^2}^2, \quad t_n < t < t_{n+1}, \tag{4}$$

$$\|u(t,\cdot)\|_{L^2}^2 \leqslant C\lambda_n^2\|w(t,\cdot)\|_{L^2}^2, \quad t_n \leqslant t < t_{n+1}, \tag{5}$$

$$\|w(\xi_2,\cdot)\|_{L^2}^2 \leqslant \|w(\xi_1,\cdot)\|_{L^2}^2 e^{-2\lambda_n(\xi_2-\xi_1)}, \quad t_n \leqslant \xi_1 < \xi_2 < t_{n+1}. \tag{6}$$

This implies $\|u(t_{n+1},\cdot)\|_{L^2}^2 \leqslant C\lambda_n^2 e^{-2\lambda_n(t_{n+1}-t_n)+C\sqrt{\lambda_n}}\|u(t_n,\cdot)\|_{L^2}^2$. Since $(t_{n+1}-t_n)\lambda_n/\sqrt{\lambda_{n+1}} \to +\infty$, it follows that

$$\|u(t_{n+1},\cdot)\|_{L^2}^2 \leqslant Ce^{-\lambda_n(t_{n+1}-t_n)}\|u(t_n,\cdot)\|_{L^2}^2. \tag{7}$$

Recall that $s_n = \sum_{k=0}^{n-1}\lambda_k(t_{k+1}-t_k)$. We derive that

$$\|u(t_{n+1},\cdot)\|_{L^2}^2 \leqslant e^{-s_{n+1}+Cn}\|u(0,\cdot)\|_{L^2}^2. \tag{8}$$

We have, for $t_n \leqslant t < t_{n+1}$,

$$\|u(t,\cdot)\|_{L^2}^2 \overset{(5)}{\lesssim} \lambda_n^2\|w(t,\cdot)\|_{L^2}^2 \overset{(6)}{\lesssim} \lambda_n^2 e^{-2\lambda_n(t-t_n)}\|w(t_n+,\cdot)\|_{L^2}^2$$

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$$\overset{(8)}{\leqslant} e^{-s_{n-1}/2+C(n-2)}\|u(0,\cdot)\|_{L^2}^2 :\Longrightarrow$$ the conclusion by the choice of (t_n,λ_n). □

Proof

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Recall that $s_n = \sum_{k=0}^{n-1}\lambda_k(t_{k+1}-t_k)$. We derive that

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On the properties of k_n and l_n

We used the fact that k_n and l_n are well-defined and the following estimates hold

$$\|k_n\|_{L^\infty_x(L^2_y)} \leqslant Ce^{C\sqrt{\lambda_n}} \quad \text{and} \quad \|l_n\|_{L^\infty_x(L^2_y)} \leqslant C\lambda_n.$$

Here are the systems of k_n and l_n:

$$k_n(x,x) = g_n(x) = l_n(x,x) \quad \big(2a(x)g_n'(x) + a_x(x)g_n(x) + \lambda_n = 0,\ g_n(0) = 0\big),$$

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Some comments:

- Well-posedness: known methods are based on special functions or a fixed point arguments. Both methods are based on the case a is constant (then $\xi = x + y, \eta = x - y$, $a\partial^2_{\xi\eta}k_n + \lambda_n k_n = 0$, Krstic & Smyshlyaev's book 08).
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Motivation of our approach

Recall (ignore the subindex n)

$$\begin{cases} k(x,x) = g(x) & \text{for } x \in [0,1], \\ k(x,0) = 0 & \text{for } x \in [0,1], \\ \big(a(x)k_x(x,y)\big)_x - \big(a(y)k_y(x,y)\big)_y - \lambda k(x,y) = 0 & \text{in } D. \end{cases}$$

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Goal: 1) To establish a finite speed propagation property for K :if $\operatorname{supp} f \subset D$ then $K = 0$ in $[0,1]^2 \setminus D$. 2) To obtain a good estimate for K.

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A key lemma

Lemma

Let $\lambda > 1$, $f \in L^2\big((0,1)^2\big)$, *and let* a_1, a_2 *be elliptic and Lipschitz. There exists a unique solution* $K \in L^2\big((0,1); H^1_0(0,1)\big) \cap H^1\big((0,1)^2\big)$ *of*

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such that $K(x,0) = K(x,1) = 0$, $K(0,y) = K_x(0,y) = 0$ *in* $(0,1)$. *Moreover,*

$$\int_0^1 |\nabla K(x,y)|^2 \, dy \leqslant C e^{C\sqrt{\lambda}} \int_0^1 \int_0^1 |f(x,y)|^2 \, dy \, dx \quad \text{for } x \in [0,1]. \tag{9}$$

Assume in addition that $a_1(x,x) \geqslant a_2(x,x)$ *in* $(0,1)$ *and* $\operatorname{supp} f \subset D$. *We have*

$$K(x,y) = 0 \text{ in } [0,1]^2 \setminus D : \textit{finite speed propagation.}$$

- The standard energy method gives (9) with the power λ; this is not good enough for our approach.
- The method works well for equations with lower oder terms.

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Assume in addition that $a_1(x,x) \geqslant a_2(x,x)$ *in* $(0,1)$ *and* $\operatorname{supp} f \subset D$. *We have*

$$K(x,y) = 0 \text{ in } [0,1]^2 \setminus D : \text{finite speed propagation.}$$

- The standard energy method gives (9) with the power λ; this is not good enough for our approach.
- The method works well for equations with lower oder terms.

A key lemma

Lemma

Let $\lambda > 1$, $f \in L^2\big((0,1)^2\big)$, *and let* a_1, a_2 *be elliptic and Lipschitz. There exists a unique solution* $K \in L^2\big((0,1); H^1_0(0,1)\big) \cap H^1\big((0,1)^2\big)$ *of*

$$\big(a_1(x,y)K_x(x,y)\big)_x - \big(a_2(x,y)K_y(x,y)\big)_y - \lambda K(x,y) = f(x,y) \text{ in } [0,1]^2,$$

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Proof.

Multiplying the equation of K by $K_x(x, y)$, integrating with respect to y from 0 to 1, and using an integration by parts, we have

$$\int_0^1 \frac{1}{2}\Big[\frac{d}{dx}\big(a_1(x,y)K_x^2(x,y)\big) + a_{1,x}(x,y)K_x^2(x,y) + \frac{d}{dx}\big(a_2(x,y)K_y^2(x,y)\big) - a_{2,x}(x,y)K_y^2(x,y) - \lambda\frac{d}{dx}K^2(x,y)\Big]\,dy = \int_0^1 f(x,y)K_x(x,y)\,dy.$$

This implies

$$\frac{d}{dx}\int_0^1 \Big[a_1(x,y)K_x^2(x,y) + a_2(x,y)K_y^2(x,y) - \lambda K^2(x,y)\Big]\,dy = 2\int_0^1 f(x,y)K_x(x,y)\,dy - \int_0^1 \Big[a_{1,x}(x)K_x^2(x,y) - a_{2,x}(x,y)K_y^2(x,y)\Big]\,dy. \quad (10)$$

Integrating (10) from 0 to x, using the properties of a_1 and a_2, we obtain

$$\int_0^1 \big[K_x^2(x,y) + K_y^2(x,y)\big]\,dy \leqslant C\int_0^1 \lambda K^2(x,y)\,dy + C\int_0^x\int_0^1 \big[K_x^2(s,y) + K_y^2(s,y)\big]\,dy\,ds + \|f\|^2_{L^2(0,1)^2} \quad (11)$$

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Proof contd 1.

Set

$$\hat{K}(x,y) = K(\lambda^{-1/2}x, y) \text{ for } (x,y) \in [0, \lambda^{1/2}] \times [0,1].$$

We derive from (11) that, for $x \in [0, \lambda^{1/2}]$,

$$\int_0^1 \left[\hat{K}_x^2(x,y) + \lambda^{-1}\hat{K}_y^2(x,y)\right] dy$$

$$\leqslant C\int_0^1 \hat{K}^2(x,y)\,dy + C\int_0^x \int_0^1 \left[\hat{K}_x^2(s,y) + \lambda^{-1}\hat{K}_y^2(s,y)\right] dy\,ds + \|f\|_{L^2}^2. \quad (12)$$

Define

$$V_1(x) = \int_0^1 \left[\hat{K}_x^2(x,y) + \lambda^{-1}\hat{K}_y^2(x,y)\right] dy \quad \text{and} \quad V_2(x) = \int_0^1 \hat{K}^2(x,y)\,dy.$$

We have

$$V_2'(x) = 2\int_0^1 \hat{K}_x(x,y)\hat{K}(x,y)\,dy \leqslant 2V_1^{1/2}(x)V_2^{1/2}(x), \quad (13)$$

and from (12) we obtain

$$V_1(x) \leqslant C\Big(V_2(x) + \int_0^x V_1(s)\,ds + \|f\|_{L^2}^2\Big). \quad (14)$$

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Proof contd 2.

A combination of (13) and (14) yields

$$V_1(x) + V_2'(x) \leqslant C\Big(V_2(x) + \int_0^x V_1(s)\,ds + \|f\|_{L^2}^2\Big). \qquad (15)$$

We derive that

$$\int_0^x V_1(s)\,ds + V_2(x) \leqslant C\|f\|_{L^2}^2 e^{Cx};$$

which, together with (12), implies that

$$\int_0^1 \Big[\hat{K}_x^2(x,y) + \lambda^{-1}\hat{K}_y^2(x,y)\Big]\,dy \leqslant C\|f\|_{L^2}^2 e^{Cx}.$$

Estimate (9) now follows by a change of variables and the definition of $\hat{K}$. We next establish that $K(x,y) = 0$ in $[0,1]^2 \setminus D$. Define

$$E(x) = \frac{1}{2}\int_x^1 \Big(a_1(x,y)K_x^2(x,y) + a_2(x,y)K_y^2(x,y)\Big)\,dy.$$

We can show that, using the fact $a_1(x,x) \geqslant a_2(x,x)$,

$$E'(x) \leqslant C(\lambda)E(x).$$

Since $E(0) = 0$, it follows that $E = 0$. The proof is complete. □

Conclusion

We propose a new method to obtain the null controllability of the heat equation in one dimension via back stepping approach:

- We choose the kernels depending on time.
- New methods are implemented to obtain the well-posedness of the kernels and reach their optimal estimates w.r.t. damping coefficients.
- Using this new approach, we can also semi-globally stabilize the heat equations in arbitrary time.

Thank you for your attention!