Uniform observability of the semi-discrete wave equation obtained from a mixed finite element method

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The observability property for the continuous wave equation

The solutions of the wave equation with an L^∞ potential $a(x)$,

$$\begin{cases} u_{tt} - u_{xx} + a(x)u = 0 & \text{for } x \in (0,1), \quad t > 0 \\ u(t,0) = u(t,1) = 0 & \text{for } t > 0, \\ u(0,x) = u^0(x) & \text{for } x \in (0,1) \\ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases} \tag{1}$$

satisfy the following property (E. Zuazua-93):

$$E(0) \leq C_1 e^{C_2 \sqrt{\|a(x)\|_{L^\infty}}} \int_0^T |u_x(0,t)|^2 dt,$$

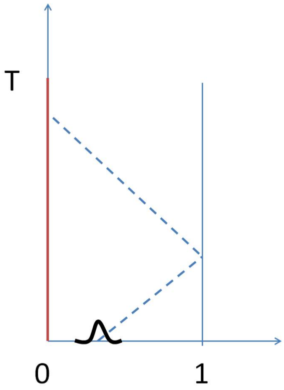
where $T > 2$ and

$$E(t) = \int_0^1 |u_t(x,t)|^2 dx + \int_0^1 |u_x(x,t)|^2 dx.$$

Main question: Is there a space semidiscretization of the wave equation that preserve this property?

The case $a(x) = 0$

Observability is related with the speed of propagation. To observe at $x = 0$ we have to be aware of all disturbances induced by the initial data.



Computing the velocity of propagation ($a(x) = 0$)

When considering solutions of the form

$$u(x,t) = exp(i\xi x - \omega(\xi)t), \quad \xi \in (-\pi/2, \pi/2),$$

we obtain the dispersion relation

$$\omega(\xi) = \pm|\xi|, \quad \xi \in (-\pi, \pi),$$

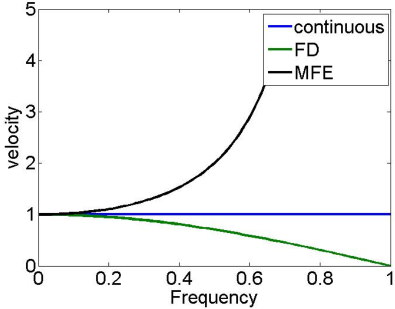
and the group velocity of waves

$$v(\xi) = \frac{d\omega}{d\xi} = \pm 1, \quad \xi \in (-\pi/2, \pi/2),$$

This explains why the time for the observability must be greater than 2.

Understanding the case $a(x) = 0$

This group velocity can be also computed for discrete approximations



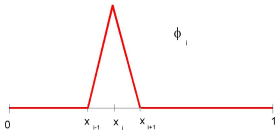
Question: Is this Mixed finite elements approach robust enough to deal with a potential?

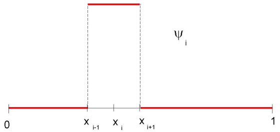
Main idea: Large frequencies should be OK ...

The mixed finite element method

Main idea:

$$u = \sum u_h^k \phi_k, \qquad u_t = \sum v_h^k \psi_k$$





See F. Brezzi and M. Fortin - 91

Matrix formulation:

$$\left\{ \begin{array}{l} M_h U_h'' + K_h U_h + L_h U_h = 0, \quad t > 0, \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1. \end{array} \right.$$

$$K_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix}, \quad M_h = \frac{h}{4} \begin{pmatrix} 2 & 1 & 0 & \ldots & 0 \\ 1 & 2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix},$$

$$L_h = h \begin{pmatrix} a_1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_2 & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & a_N \end{pmatrix}, \qquad U_h = \begin{pmatrix} U_{1,h} \\ U_{2,h} \\ \ldots \\ U_{N,h} \end{pmatrix}.$$

Main result: Uniform observability

Theorem

Assume $a_j \geq 0$ (equivalent to $a(x) \geq 0$). There exist constants $C, T > 0$, independent of h, such that

$$E_h(U_h(0)) \leq C \int_0^T \left| \frac{U_{1,h(t)}}{h} \right|^2 dt$$

where

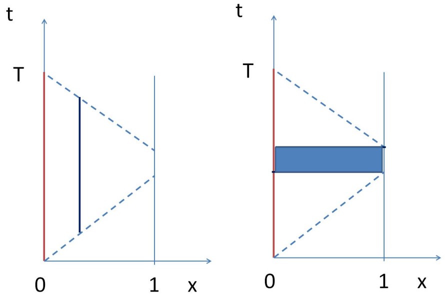
$$E_h(U_h) = (M_h U_h', U_h') + (K_h U_h, U_h)$$

Idea of the proof

Try to mimic the following proof for the continuous wave equation (Zuazua, 1993). Consider for $\tau > 2$ and $1 < \beta < \tau/2$

$$F(x) = \int_{\beta x}^{\tau - \beta x} \mathcal{E}(x, s)\, ds,$$

where $\mathcal{E}(x,t) = \frac{1}{2}(|u_t|^2 + |u_x|^2 + \|a\|_{L^\infty}|u|^2)$.



Analytically,

1. Consider for $\tau > 2$ and $1 < \beta < \tau/2$

$$F(x) = \int_{\beta x}^{\tau - \beta x} \mathcal{E}(x,s)\, ds,$$

   where $\mathcal{E}(x,t) = \frac{1}{2}(|u_t|^2 + |u_x|^2 + \|a\|_{L^\infty}|u|^2)$.

2. Prove $F'(x) \leq CF(x)$, for some constant $C > 0$.
3. By Gronwall's inequality $F(x) \leq cF(0)$
4. Using the conservation of the energy prove that

$$\mathcal{E}(0) \leq C_1 \int_0^1 F(x)\, dx \leq C_2 F(0).$$

At the discrete level, define

$$\mathcal{E}_j^h(s) = \left|\frac{U_{j+1}-U_j}{h}\right|^2 + \left|\frac{U'_{j+1}+U'_j}{2}\right|^2 + a_M\left|\frac{U_{j+1}+U_j}{2}\right|^2,$$

Consider also $\tau > 2$, $1 < \beta < \tau/2$ and the discrete version of $F(x)$:

$$F_j^h = \frac{1}{2}\int_{\beta x_j}^{\tau-\beta x_j} \mathcal{E}_j^h(s)ds.$$

Lemma

The following discrete version of $F'(x) \le cF(x)$ holds

$$\frac{F_j^h - F_{j-1}^h}{h} \le c(a_M)\left(\frac{F_j^h + F_{j-1}^h}{2} + R_j^h(\beta x_j) + R_j^h(\tau - \beta x_j)\right),$$

$$R_j^h(s) = \frac{1}{h}\int_{s-\beta h}^{s} \mathcal{E}_j^h(r)dr - \frac{\mathcal{E}_j^h(s-\beta h) + \mathcal{E}_j^h(s)}{2},$$

The proof does not work!!

Remark. For particular solutions having only two frequencies λ_n^h, λ_m^h with

$$|\lambda_n^h - \lambda_m^h| < 1,$$

we have

$$\frac{F_j^h - F_{j-1}^h}{h} \leq c(a_M)\frac{F_j^h + F_{j-1}^h}{2}$$

and the proof works!

The main ingredient is this lemma:

Lemma

Let $r > 0$, $t \geq 0$ and ν_1, ν_2 be two different real numbers such that,

$$r|\nu_2 - \nu_1| \leq \frac{2\pi}{3}.$$

Then, the following estimate holds

$$\frac{f(t) + f(t+r)}{2} \leq \frac{5}{r} \int_t^{t+r} f(s)\, ds,$$

for any function $f(t)$ of the form

$$f(t) = \left| b_1 e^{i\nu_1 t} + b_2 e^{i\nu_2 t} \right|^2,$$

with $b_1, b_2 \in \mathbb{C}$.

The situation so far...

Proposition

A uniform observability inequality holds but for particular solutions having only two frequencies λ_n^h, λ_m^h with

$$|\lambda_n^h - \lambda_m^h| < 1.$$

Here we change the strategy of the proof.

It is well known that the observability inequality can be obtained from two main properties:

1. A uniform spectral gap

$$\inf_{n \neq m} |\lambda_n^h - \lambda_m^h| > \gamma > 0,$$

   where $\{\lambda_n^h\}_n$ are the frequencies.

2. A uniform observability inequality for the eigenfunctions.

- The uniform observability of the eigenfunctions is obtained from the discrete version of the continuous proof, that works fine for solutions having only one frequency.
- The espectral gap follows from the combination of the previous proposition and the following one:

Proposition

Assume that the uniform observability inequality holds for particular solutions having two frequencies λ_h^n, λ_h^m, then there exists a constant $C(T)$, uniform in h, such that

$$|\lambda_n^h - \lambda_m^h| \geq C(T).$$

Idea of the proof:

$$\int_0^T \left| a_1 e^{i\lambda t} + a_2 e^{i\mu t} \right|^2 \geq C_1(|a_1|^2 + |a_2|^2) \Rightarrow |\lambda - \mu| > C_2(T, C_1)$$

The situation is as follows: For solutions having only two frequencies we have

$$|\lambda_n^h - \lambda_m^h| < 1 \Rightarrow \text{ Uniform observability}$$

$$\text{Uniform observability} \Rightarrow |\lambda_n^h - \lambda_m^h| > C(T)$$

Therefore,

$$\inf_{n\neq m} |\lambda_n^h - \lambda_m^h| > \min(1, C(T)).$$

and the spectral gap condition holds.

Final comments

1. The only condition for the discrete potential a_j^h is $0 < a_j^h < a_M$ for all $j = 1, ..., N$.
2. The proof can be adapted to non-positive potentials. This case is more technical.
3. The time for observability is larger than the continuous one and probably not optimal.
4. The proof cannot be adapted to higher dimensions.