Approximation of the controls for the wave equation with a potential

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Controlled wave equation

Given any $T > 0$ and initial data $\left(u^0, u^1\right) \in \mathcal{H} := L^2(0,1) \times H^{-1}(0,1)$, the exact controllability in time T of the linear wave equation with a potential,

$$\begin{cases} u''(t,x) - u_{xx}(t,x) + a(x)u(t,x) = 0 & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = 0 & t \in (0,T) \\ u(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x),\ u'(0,x) = u^1(x) & x \in (0,1) \end{cases} \tag{1}$$

where a is a real potential function, consists of finding a scalar function $v \in L^2(0,T)$, called control, such that the corresponding solution (u, u') of (1) verifies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)). \tag{2}$$

It is know that, if $T \geq 2$ this property holds.

Variational result

The function $v \in L^2(0,T)$ is a control which drives to zero the solution of (1) in time T if and only if, the following relation holds

$$\int_0^T v(t)\overline{\varphi}_x(t,1)dt = \langle u^1, \varphi(0,\cdot)\rangle_{H^{-1},H_0^1} - \int_0^1 u^0(x)\overline{\varphi_t}(0,x)dx \qquad (3)$$

for every $\begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$, where $\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix} \in H_0^1(0,1) \times L^2(0,T)$ is the solution of the following adjoint backward problem

$$\begin{cases} \varphi_{tt}(t,x) - \varphi_{xx}(t,x) + a(x)\varphi(t,x) = 0 & t > 0,\ x \in (0,1) \\ \varphi(t,0) = \varphi(t,1) = 0 & t > 0 \\ \varphi(T,x) = \varphi^0(x) & x \in (0,1) \\ \varphi_t(T,x) = \varphi^1(x) & x \in (0,1). \end{cases} \qquad (4)$$

Spectral analysis

By denoting $W = \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}$, equation (4) is equivalent with

$$\begin{cases} W_t + AW = 0 \\ W(T) = W^0 = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \end{cases} \tag{5}$$

where $$A = \begin{pmatrix} 0 & -1 \\ L & 0 \end{pmatrix}, \quad Lu = -u_{xx} + au.$$

Eigenvalues of L are $(\nu_n)_{n\in\mathbb{N}^*}$ and the corresponding eigenfunctions are $(\varphi_n)_{n\in\mathbb{N}^*}$.
If $a \equiv 0 \Rightarrow \nu_n = n^2\pi^2$ and $\varphi_n = \sin(n\pi x)$.
Eigenvalues of A: $(i\lambda_n)_{n\in\mathbb{Z}^*}$, $\lambda_n = \text{sgn}(n)\sqrt{\nu_{|n|}}$, $\|\nu_n - n^2\pi^2\| \leq \|a\|_{L^\infty}$.
Eigenfunctions of A form an orthogonal basis in $H_0^1(0,1) \times L^2(0,1)$:

$$\Phi^n = \frac{sgn(n)}{\sqrt{2}\lambda_n} \begin{pmatrix} 1 \\ -i\lambda_n \end{pmatrix} \varphi_{|n|}. \tag{6}$$

Moment problem for the wave equation

The null-controllability of the wave equation is equivalent to solve the following moment problem:

For any $(u^0, u^1) = \sum_{n\in\mathbb{Z}^*} a_n^0 i\,\mathrm{sgn}(n)\lambda_n \Phi^n$, find $v \in L^2(0,T)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(t+\frac{T}{2}\right)e^{-i\lambda_n t}dt = \frac{\sqrt{2}\lambda_n e^{-i\frac{T}{2}\lambda_n}}{(\varphi_{|n|})_x(1)} \qquad (n\in\mathbb{Z}^*). \tag{7}$$

A solution v of the moment problem may be constructed by means of a biorthogonal sequence to the family $(e^{i\lambda_n t})_{n\in\mathbb{Z}^*}$.

Biorthogonal sequence

Definition

A family of functions $(\theta_m)_{m\in\mathbb{Z}^*} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ with the property

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t)e^{-i\lambda_n t}dt = \delta_{mn} \qquad (m,\, n\in\mathbb{Z}^*), \tag{8}$$

is called a biorthogonal sequence to $(e^{i\lambda_n t})_n$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$.

Once we have a biorthogonal sequence to $(e^{i\lambda_n\, t})_{n\in\mathbb{Z}^*}$, a "formal" solution of the moment problem is given by

$$v(t) = \sqrt{2}\sum_{n\in\mathbb{Z}^*} \frac{e^{i\lambda_n \frac{T}{2}}}{(\varphi_{|n|})_x(1)} a_n^0 \theta_n\left(t-\frac{T}{2}\right) \qquad (t\in(0,T)). \tag{9}$$

Main problems

- the existence of a biorthogonal sequence $(\theta_m)_m$ to the family $(e^{i\lambda_n t})_n$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$
- evaluation of the norm of $(\theta_m)_m$

This estimates are needed to show the convergence of the series in (9) and to have a bound of the norm of v.

A constructive way to obtain a biorthgonal sequence

- $(\Psi_m)_{m\in\mathbb{Z}^*}$ entire functions.

  H1 ▷ $|\Psi_m(z)| \leq A e^{\frac{T}{2}|z|}$,

  H2 ▷ $\Psi_m \in L^2(\mathbb{R})$,

  H3 ▷ $\Psi_m\left(i\overline{\lambda}_n\right) = \delta_{mn}$.

Paley–Wiener Theorem (1934)

$$\theta_m \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right) \text{ such that } \Psi_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{-izt}\, dt.$$

Plancherel's Theorem (1910)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |\theta_m(t)|^2\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\Psi_m(x)|^2\, dx.$$

Finite differences for the wave equation

Let $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$, $x_j = jh$, $0 \le j \le N+1$, $a_j = a(x_j)$

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{h^2} + a_j u_j(t) = 0 & 1 \le j \le N,\ t > 0 \\ u_0(t) = 0 & t \in (0,T) \\ u_{N+1}(t) = v_h(t) & t \in (0,T) \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \le j \le N. \end{cases} \tag{10}$$

Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \le j \le N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution $(u_j)_{1 \le j \le N}$ of (10) satisfies

$$u_j(T) = u_j'(T) = 0,\ \forall j = 1, 2, .., N. \tag{11}$$

$u_j(t) \approx u(t, x_j)$ if $(U_h^0, U_h^1) \approx (u^0, u^1)$.

Non uniformly observability and controllability

Let $T > 0$. For any $h > 0$, there exists a constant $C = C(T, h)$ such that

$$\left\| \begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix}_{1\leq j\leq N} (0) \right\|_{1,0}^2 \leq C \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt, \tag{12}$$

for any $\begin{pmatrix} \varphi_j^0 \\ \varphi_j^1 \end{pmatrix}_{1\leq j\leq N} \in \mathbb{C}^{2N}$ and $\begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix}_{1\leq j\leq N}$ solution of the corresponding backward equation, but there exists $a \in L^\infty(0,1)$ (Infante and Zuazua (MMAN, 1999)) such that

$$\lim_{h\to 0} \sup_{(\varphi,\, \varphi')\ \text{solution}} \frac{\left\| \begin{pmatrix} \varphi_j \\ \varphi_j' \end{pmatrix} (0) \right\|_{1,0}^2}{\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt} = \infty. \tag{13}$$

(13) shows that the system (10) is not uniformly controllable. This is equivalent with the existence of initial data $(u^0, u^1) \in \mathcal{H}$ to which corresponds an unbounded sequence of controls $(v_h)_{h>0}$.

Regularity and filtration of the initial data

If $a \neq 0$, we prove that we can restore the uniform controllability property if:

- Initial data (u^0, u^1) are sufficiently smooth and discretized by points

$$U_h^0 = (u^0(jh))_{1 \leq j \leq N}, \quad U_h^1 = (u^1(jh))_{1 \leq j \leq N};$$

- Initial data (u^0, u^1) are in the energy space $\mathcal{H}$ and the high frequencies of their discretization are filtered out,

$$(U_h^0, U_h^1) = \sum_{1 \leq |n| \leq \sqrt{N}} a_{nh} \Phi_h^n;$$

These results are similar with the ones obtained in *(Micu, Numer. Math, 2002)*, but the proof is more difficult since the eigenvalues and eigenvectors are not explicit.

The matriceal form of the equation

We write (10) as an abstract Cauchy form

$$\begin{cases} U_h''(t) + A_h U_h(t) + D_h U_h(t) = B_h v(t) & t \in (0,T) \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1, \end{cases} \tag{14}$$

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$D_h = \begin{pmatrix} a_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & \dots & 0 & a_N \end{pmatrix}, \quad B_h v(t) = \frac{1}{h^2}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_h(t) \end{pmatrix}.$$

If $a \geq 0$ and $\|a\|_{L^\infty(0,1)} \leq \delta$ the eigenvalues are given by the family $(i\,\lambda_m)_{1\leq |m|\leq N}$, where

$$\lambda_m = \eta_m + \epsilon_m,$$

$$\eta_m = \frac{2}{h}\sin\left(\frac{m\pi h}{2}\right) \qquad 0 \leq \epsilon_m \leq \frac{\delta}{|\eta_m|}. \tag{15}$$

- To obtain the asymptotic relation (15) we apply an argument based on the smoothing method and the Rouche's Theorem.

$$\begin{cases} v_{n+1} - (2 + \eta^2 h^2 + a_n h^2) v_n + v_{n-1} = 0, & (n \geq 1) \\ v_0 = 0,\ v_1 = u_1. \end{cases} \tag{16}$$

$i\lambda_m \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_h \Leftrightarrow u_1 \neq 0$ and

$$v_{N+1}(i\lambda_m) = 0. \tag{17}$$

Localization of the eigenvalues

$$\begin{cases} u_{n+1} - (2+\eta^2 h^2)u_n + u_{n-1} = 0, & (n \geq 1) \\ u_0 = 0,\ u_1 \in \mathbb{C}. \end{cases} \tag{18}$$

The equation $u_{N+1}(\eta) = 0$ has the roots $(i\eta_m)_{1\leq|m|\leq N}$ given by (15). If $\|(a_n)_n\|_\infty < \delta$ we have that

$$|u_{N+1}(\eta) - v_{N+1}(\eta)| < |u_{N+1}(\eta)| \quad \forall \eta \in \partial B_{i\,\eta_m}\left(\frac{\delta}{|\eta_m|}\right)$$

$\left(B_{i\,\eta_m}\left(\frac{\delta}{|\eta_m|}\right)\right)_{1\leq m\leq N}$ are disjoint if δ is small enough.

- In the continuous case the localization of eigenvalues can be done with balls of radius $\frac{\delta}{|\eta_m|^2}$, which are of order δh^2 in the high frequencies.
- But, in the discrete case we can localize the eigenvalues only with balls of radius $\frac{\delta}{|\eta_m|}$, which are of order δh in the high frequencies.

A biorthogonal sequence to $\Lambda = (e^{\lambda_n t})_{1\leq|n|\leq N}$

Theorem

There exist $T_0 > 0$ and $h_0 > 0$ such that for any $T > T_0$ and $h < h_0$ there exists a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, such that, for any finite sequence $(a_m)_{1\leq|m|\leq N}$ we have that

$$\left\| \sum_{1\leq|m|\leq N} a_m\theta_m \right\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C \sum_{1\leq|m|\leq N} |a_m|^2 e^{2\omega m^2 h}, \qquad (19)$$

where ω and C are two positive constants independent of m and h.

The construction of a biorthogonal sequence to a family of exponentials $\Lambda = \left(e^{i\,\lambda_n t}\right)_{n\geq 1}$ in $L^2(-T/2, T/2)$

- A Weierstrass product

  (P1) $P_m(z) = \prod_{n\neq m} \frac{z-\lambda_n}{\lambda_m - \lambda_n}$, (P2) $|P_m(x)| \leq C_1 \exp(\varphi(x))$,

$$\varphi(x) = \begin{cases} C & |x| \leq \frac{2}{h} \\ \frac{C}{\sqrt{h}}\sqrt{|x| - \frac{2}{h}} & |x| > \frac{2}{h} \end{cases}$$

- A multiplier

  (M1) $|M_m(x)| \leq C_2 \exp(-\varphi(x))$, (M2) $|M_m(\lambda_m)| \geq C_3 \exp(-\omega m^2 h)$.

- **The entire function**

  (E1) $\Psi_m(z) = P_m(z) \frac{M_m(z)}{M_m(\lambda_m)} \frac{\sin(\varepsilon(z-\lambda_m))}{\varepsilon(z-\lambda_m)}$.

- **Th. Paley-Wienner** $\Rightarrow$ $(\theta_m)_m = (\widehat{\Psi}_m)_m$ **biorthogonal**

Uniformly boundedness of the sequence of controls

Theorem

Let $T > T_0$ and $h < h_0$. For any $\left(U_h^0, U_h^1\right) \in \mathbb{C}^{2N}$ of the form

$$\left(U_h^0, U_h^1\right) = \sum_{1 \le |n| \le N} \varrho_n a_{hn}^0 \Phi_h^n, \tag{20}$$

with

$$\varrho_n = \begin{cases} 1 & \text{if } |n| \le \sqrt{N} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \varrho_n = \exp(-2\omega h n^2), \tag{21}$$

and $(a_{hn}^0)_{1 \le |n| \le N}$ uniformly bounded in l^2, there exists a control $v_h \in L^2(0, T)$ for problem (10) such that the family $(v_h)_{h>0}$ is uniformly bounded in $L^2(0, T)$.

Uniformly boundedness of the sequence of controls

Theorem

Let $T > T_0$, $h < h_0$ *and* $\left(u^0\, u^1\right) \in L^2(0,1) \times H^{-1}(0,1)$ *of the form* $\left(u^0, u^1\right) = \sum_{n\in\mathbb{Z}^*} a_n^0 \Phi^n$ *with the property*

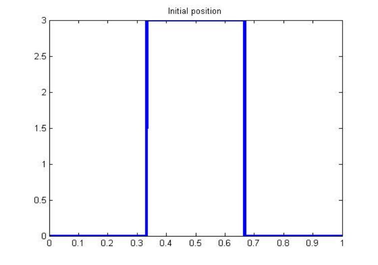
$$\sum_{n\in\mathbb{Z}^*} |a_n^0|^2 n^2 e^{3\omega h n^2} < +\infty. \tag{22}$$

Given $\left(U_h^0, U_h^1\right) \in \mathbb{C}^{2N}$ *of the form*

$$\left(U_h^0, U_h^1\right) = \left(u^0(jh), u^1(jh)\right), \tag{23}$$

there exists a uniformly bounded family of controls $(v_h)_{h>0}$ *in* $L^2(0,T)$ *for problem* (14).

Numerical results



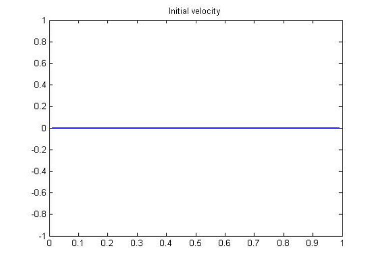


Figure: Initial data to be controlled.

$$u^0(x) = \begin{cases} 3 & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0 & \text{if } x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) \end{cases}, \qquad u^1(x) = 0 \qquad (x \in (0,1)).$$

$$N = 100; T = 4.77;\ a(x) = 1 + \sin(3\pi x)$$

A conjugate gradient method for the corresponding discrete optimization approach.

Numerical results

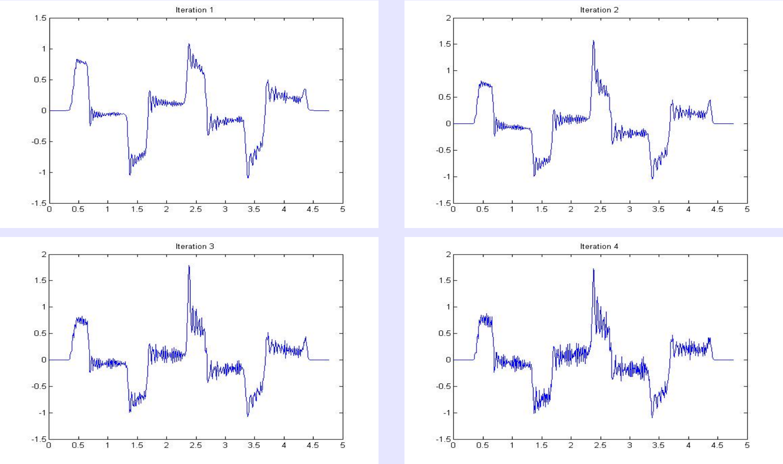


Figure: The first four iterations of the conjugate gradient method for the approximation of $\widehat{v}_h$ with $N = 100$ without filtration.

Numerical results

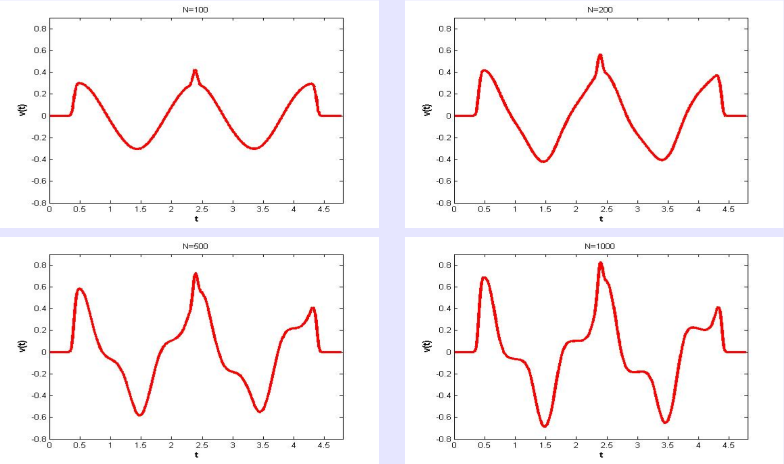


Figure: The approximation of the control $\widehat{v}_h$ with $N = 100$, 200, 500 and 1000 by using filtration of the initial data.

Comments and open problems

- We have asked the potential a verifies: there exist $\alpha \in \mathbb{R}$ and $\delta > 0$ such that
$$\|a - \alpha\|_{L^\infty(0,1)} \leq \delta. \tag{24}$$

- We do not have obtained the optimal control time.
- The range of filtration is $\sqrt{N}$. In the case $a = 0$, *(Lissy, Roventa, 2017)* shows that range of filtration may be δN.
- In *(Allonsius, Boyer, Morancey, 2017)* it was considered a similar problem with a non uniform grid. They have proved similar results for a system of parabolic equations using different techniques. However, our strategy allow us to obtain a better localization of the eigenvalues.

THANK YOU!!!!