Finite time stabilization : some particular examples

Vincent Perrollaz

LMPT, Université François Rabelais, Tours

Benasque, August 28, 2017

Outline

Control Theory

General control system:

$$\begin{cases} \dot{X}(t) = F(X(t), U(t)), \\ X(0) = X_0, \end{cases} \tag{1}$$

state of the system: $X \in \mathcal{X}$, control: $U \in \mathcal{U}$. Two classical problems (among others):

- Exact controllability: for $T > 0$, $X_0, X_1 \in \mathcal{X}$ being given, find $U : [0, T] \mapsto \mathcal{U}$, such that:

$$X \text{ solution of (1)} \Rightarrow X(T) = X_1.$$

- Asymptotic stabilization: let $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ be an equilibrium, find $\mathbb{U} : \mathcal{X} \mapsto \mathcal{U}$, such that X_e is asymptotically stable for:

$$\dot{X}(t) = F(X(t), \mathbb{U}(X(t))).$$

Why feedback stabilization?

Robustness with respect to 4 kinds of errors

- Actuators.
- Observation.
- Delay.
- Modeling.

(Even more so when we have a Lyapunov function)

Finite time Stabilization (or the best of both)

Let $(X_e, U_e) \in \mathcal{X} \times \mathcal{U}$ be an equilibrium.
Find $\mathbb{U} : \mathcal{X} \mapsto \mathcal{U}$, such that $\mathbb{U}(X_e) = U_e$ and for any X_0 any solutions

$$\dot{X}(t) = F(X(t), \mathbb{U}(X(t))).$$

satisfy

$$\exists T > 0, \qquad X(T) = X_e.$$

Remarks/Questions :

- No backward uniqueness.
- Feedback not smooth.
- T depends on X_0?

1. Finite time stabilization on a toy model
   - The different kinds of problems
   - Spectral Analysis
   - Lyapunov Functionals

2. Quasilinear Hyperbolic systems
   - Network of Canals
   - Abstract Problem

An abstract result

Theorem (Xu)

Let H be a separate Hilbert space. Let B be the infinitesimal generator of $S(t)$ a C_0 semigroup. If $R(\lambda, B)$ $(:= (\lambda Id - B)^{-1})$ is an entire function of finite exponential type i.e.

$$\exists \gamma, C > 0, \qquad s.t. \qquad \forall \lambda \in \mathbb{C}, \quad |||R(\lambda, B)||| \leq Ce^{\gamma|\lambda|},$$

then we have :

$$\forall t > \gamma, \qquad S(t) = 0.$$

Case of transport equation

$$\partial_t y + c\partial_x y = 0, \quad x \in (0, L), \qquad y(t, 0) = 0,$$

then $(\lambda Id - B)u = f$ becomes

$$\lambda u - c\dot{u} = f, \qquad u(0) = 0,$$

and so

$$u(x) = -\int_0^x e^{\frac{\lambda(x-r)}{c}} \frac{f(r)}{c} dr,$$

from which we get :

$$||u||_{L^2}^2 \leq \frac{L^2}{c^2} e^{2\frac{L|\lambda|}{c}} ||f||_{L^2}^2.$$

The case of the transport equation

$$\begin{aligned} \partial_t y + c\partial_x y = 0, &\qquad (t,x) \in (0,T)\times(0,L) \\ y(t,0) = 0, &\qquad t \in (0,T). \end{aligned}$$

Using the method of characteristics :

$$y(t,x) = \begin{cases} y_0(x-ct) & \text{if } x > ct, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \geq \frac{L}{c}$, $y(t,.) = 0$.

A Family of Lyapunov Functionals

- For $\nu > 0$:

$$J_\nu(t) := \int_0^L y^2(t,x)e^{-\nu x}dx.$$

- Formally at least :

$$\begin{aligned} \dot{J}_\nu(t) &= \int_0^L 2y_t(t,x)y(t,x)e^{-\nu x}dx \\ &= \int_0^L -2cy_x(t,x)y(t,x)e^{-\nu x}dx \\ &= [-cy^2(t,x)e^{-\nu x}]_0^L - c\nu J_\nu(t) \\ &\leq -c\nu J_\nu(t). \end{aligned}$$

- Using Gronwall :

$$J_\nu(t) \leq e^{-c\nu t}J_\nu(0).$$

Return to the L^2 norm

- Norm equivalence

$$\forall t \geq 0, \qquad e^{-\nu L}||y(t,.)||^2_{L^2(0,L)} \leq J_\nu(t) \leq ||y(t,.)||^2_{L^2(0,L)}.$$

- Inequality on L^2

$$||y(t,.)||^2_{L^2(0,L)} \leq e^{-\nu c(t-\frac{L}{c})}||y_0||^2_{L^2(0,L)},$$

- For $t \geq \frac{L}{c}$, letting $\nu \to +\infty$ we get $y(t,.) = 0$.

Remarks

- Can be adapted to general "transport" type equations.
- Good for robustness estimate and perturbation :

$$y_t + cy_x = \epsilon g(y),$$

$$y_t + cy_x = \epsilon y_{xx}.$$

- In certain cases, useful for exact controllability to trajectory.

Flow control in networks of canals

- Consider first **one canal**. Canal: rectangular cross section, with slope and friction. An appropriate fluid model is given by the shallow water system

$$H_t + (HV)_x = 0,$$
$$V_t + (\frac{V^2}{2} + g\cos(\theta)H)_x = g\sin(\theta) - c_f\frac{V^2}{2H}$$

- H: water depth, V: water velocity, and g the gravitation constant.
- θ slope angle, c_f friction term.
- Physically, input controlled: flow rate

$$Q(t,x) = H(t,x)V(t,x).$$

- Hypothesis : slope and friction (almost) negligible...
- Objective : stabilize the system around (an almost) constant equilibrium state (H^*, V^*). Set $Q^* = H^*V^*$.

Literature about the control of Saint-Venant equations

- **Stabilization**: Greenberg-Li '84, Coron-d'Andréa Novel-Bastin '99, Xu-Sallet '02, Leugering-Schmidt' 02, de Halleux-Prieur-Coron-d'Andréa Novel-Bastin '03,..., Bastin-Coron '11,...
- **Controllability**: Gugat-Leugering '03,..., Gugat-Leugering '09, Li '10, Li-Rao-Wang '10...
- Actually applied on Sambre and Meuse rivers.
- For an actual up to date litterature see the book by Bastin Coron.

Characteristic velocities

- Characteristic velocities:

$$\mu = V - \sqrt{g\cos(\theta)H}$$
$$\lambda = V + \sqrt{g\cos(\theta)H}$$

- **subcritical** (or **fluvial**) flow:

$$\mu < 0 < \lambda$$

- Equivalent to $0 < V^* < \sqrt{g\cos(\theta)H^*}$ and $V \sim V^*$, $H \sim H^*$.
- Pick $c > 0$ s.t.

$$\sqrt{gH^*} - V^* > 2c.$$

Riemann invariants

Defined as:

$$u = V + 2\sqrt{g\cos(\theta)H}$$
$$v = V - 2\sqrt{g\cos(\theta)H}$$

Inverted as

$$H = \left(\frac{u - v}{4\sqrt{g\cos(\theta)}}\right)^2$$
$$V = \frac{u + v}{2}$$

μ and λ expressed in terms of u, v:

$$\mu = \frac{1}{4}(u + 3v)$$
$$\lambda = \frac{1}{4}(3u + v)$$

Diagonal form

- Shallow water transformed into the diagonal system

$$
\begin{aligned}
u_t + \lambda(u,v)u_x &= \epsilon f(u,v) \\
v_t - \mu(u,v)v_x &= \epsilon g(u,v)
\end{aligned}
$$

  where

$$
0 < c < \lambda(u,v), \mu(u,v)
$$

- $\epsilon > 0 \Rightarrow$ there exist equilibrium state $(u^*_\epsilon, v^*_\epsilon)$ close to (u^*, v^*) in $\mathcal{C}^\infty$.

Boundary conditions

- At $x = 0$

$$u(t,0) = y_g(t)$$

with y_g an integrator s.t.

$$\frac{\mathrm{d}y_g}{\mathrm{d}t} = -K\frac{(y_g - u^*)}{|y_g - u^*|^\gamma} \tag{2}$$

$K > 0$ to be chosen later on and $\gamma \in (0,1)$.

- At $x = 1$

$$v(t,0) = y_d(t)$$

y_d an integrator s.t.

$$\frac{\mathrm{d}y_d}{\mathrm{d}t} = -K\frac{(y_d - v^*)}{|y_d - v^*|^\gamma} \tag{3}$$

- The system on (u, v, y_g, y_d) is autonomous.

Result

Theorem (Gugat, Rosier, P.)

There exist $\epsilon_0 > 0$ and $\delta > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, any $\gamma \in]0, 1[$, and any initial data (u_0, v_0) which are Lipschitz if we suppose

$$\begin{aligned} ||u_0 - u^*||_{W^{1,\infty}} \leq \delta, \qquad & ||v_0 - v^*||_{W^{1,\infty}} \leq \delta, \\ y_g(0) = u_0(0), \qquad & y_d(0) = v_0(L). \end{aligned}$$

then the full system has a unique solution $(u_\epsilon, v_\epsilon) \in \mathrm{Lip}([0, +\infty) \times [0, L])$ satifying the original system almost everywhere and for positive time t

$$||(u_\epsilon - u^*_\epsilon, v_\epsilon - v^*_\epsilon)(t)||_{L^\infty(0,L)} \leq M \inf(1, \frac{e^{-\frac{C_\epsilon t}{3}}}{\epsilon^{\frac{1+\kappa}{3}}})||(u_0 - u^*_\epsilon, v_0 - v^*_\epsilon)||^{\frac{2}{3}}_{L^\infty(0,L)},$$

where $M = M(\delta)$, $\kappa = \frac{c\delta^\gamma}{KL\gamma}$ and $C_\epsilon \underset{\epsilon \to 0^+}{\sim} -\frac{c}{L}\ln(\epsilon)$.

- Existence through global in time Schauder fixed point in Frechet space.
- Stabilization through Lyapunov functionnal. (cf transport case)

$$L_\theta(u, v, y_g, y_d) = \int_0^L u^2(x)e^{-\theta x} + v^2(x)e^{-\theta(L-x)}dx + \frac{\tilde{C}|y_g|^{\gamma+2}}{K(\gamma+2)}e^{\theta\frac{c}{K\gamma}|y_g|^\gamma} + \frac{\tilde{C}|y_d|^{\gamma+2}}{K(\gamma+2)}e^{\theta\frac{c}{K\gamma}|y_d|^\gamma}$$

Cut of on θ depending on ϵ.

- When $\epsilon \to 0$: finite time stabilization.
- Results actually hold for tree shaped graph. (coupling much "easier" than for wave equation)
- Question : robustness with respect to observation/actuation error + sampled control
  $\Rightarrow$ Entropy solutions
  $\Rightarrow$ no linearization, boundary layers, few a posteriori technique, some generalizations are false (cf Bressan Coclite).

THANK YOU FOR YOUR ATTENTION