Nonlocal isoperimetric problems

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Experiment

Experimentally it has been observed that a charged droplet is stable until its charge reaches a specific threshold.

When the charge (or the voltage) is too big, a Taylor cone appears and a liquid jet is developed, which takes away a small fraction of the volume but a large fraction of the charge (see vanden-Broeck, Keller 1980; Miksis 1981, etc...).

Experiment

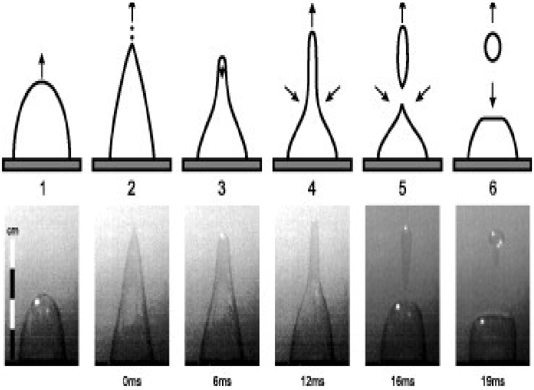

Figure: creation of a liquid jet

The model problem

In 1882 Lord Rayleight proposed a model for charged droplets at equilibrium.

Up to constants depending on the material, the associated energy is:

$$\mathcal{F}_Q(E) = P(E) + Q^2 \inf_{\mu:\, \mu(E)=1} \int_E \int_E \frac{d\mu(x)\, d\mu(y)}{|x-y|}$$

where $E \subset \mathbb{R}^3$ is the droplet and Q is the total charge.

Notice that the nonlocal functional

$$\mathcal{I}(E) = \inf_{\mu(E)=1} \int_E \int_E \frac{d\mu(x)\, d\mu(y)}{|x-y|}$$

is the inverse of the capacity of E, that is

$$\frac{1}{\mathcal{I}(E)} = \inf \left\{ \int_{\mathbb{R}^3} |\nabla f|^2 \, dx : \ f \in C^1_c(\mathbb{R}^3), \ f \geq 0, f \geq 1 \text{ on } E \right\}.$$

The Euler-Lagrange equation of $\mathcal{F}_Q$, with volume constraint, reads

$$\begin{aligned} -\Delta v &= 0 && \text{in } \mathbb{R}^3 \setminus E \\ v &= \inf_{\mu:\, \mu(E)=1} \int_E \int_E \frac{d\mu(x)\, d\mu(y)}{|x-y|} && \text{in } E \\ \lim_{|x|\to+\infty} v(x) &= 0 \\ \kappa &= Q^2 |\nabla v|^2 + \lambda && \text{on } \partial E \end{aligned}$$

where λ is a Lagrange multiplier and

$$v(x) = \int_E \frac{d\mu_{\min}(y)}{|x-y|}$$

is the associated Coulombic potential.

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The functional $\mathcal{F}_Q$ has an attractive/repulsive character, with the two terms competing with eachother. One expects that the first term wins only when the charge Q is small enough.

Linear stability of the ball

An important stability result is:

Theorem (Fontelos, Friedman 2004)

For all $m > 0$ there exists $\overline{Q}(m)$ such that the ball is a linearly stable critical point of $\mathcal{F}_Q$ if $Q < \overline{Q}(m)$, while it is unstable if $Q > \overline{Q}(m)$.

Nonexistence of minimizers

Note that linear stability does not imply nonlinear stability. Indeed,

Theorem (Goldman,N.,Ruffini 2015)

For all m, Q positive, the minimum problem

$$\min_{E:\,|E|=m} \mathcal{F}_Q(E)$$

has no solution (not even local minimizers).

This means that the ball of volume m, even if is linearly stable when Q is not too large, is always linearly unstable.

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It is not clear why this nonlinear instability is not observed in experiments. Maybe there are some extra terms in the energy which have been neglected (entropy, elastic energy).

Proof

The idea behind this result is that one can concentrate all the charge in little droplets with very small volume and perimeter, so that the two terms in the energy $\mathcal{F}_Q$ essentially decouple.

It is enough to observe that a ball of radius R and charge Q has energy $\mathcal{F}_Q(B_R) = 4\pi R^2 + cQ^2/R$, where c is an absolut constant. Hence, N balls of radius R, charge Q/N and mutual distance at least D have energy

$$\mathcal{F}_Q(N\, B_R) = 4\pi N\, R^2 + c\frac{Q^2}{N\, R} + \epsilon(D),$$

with $\epsilon(D) \to 0$ as $D \to \infty$.

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Choosing $R = 1/N^\alpha$, with $\alpha \in (1/2, 1)$, it then follows

$$\lim_{N\to\infty} \mathcal{F}_Q(N\,B_R) = 0.$$

Riesz potentials

A similar result holds for the more general energy

$$\mathcal{F}_{Q,\alpha}(E) = P(E) + Q^2 \mathcal{I}_\alpha(E) \qquad \alpha \in (0, n)$$

where now $E \subset \mathbb{R}^n$ and

$$\mathcal{I}_\alpha(E) = \inf_{\mu:\, \mu(E)=1} \int_E \int_E \frac{d\mu(x)\, d\mu(y)}{|x-y|^{n-\alpha}}$$

is the α-Riesz energy of E.

Theorem (Goldman,N.,Ruffini 2015)

Let $\alpha < n - 1$. For all m, Q positive, the minimum problem

$$\min_{E:\,|E|=m} \mathcal{F}_{Q,\alpha}(E)$$

has no solution (not even local minimizers).

Notice that we require $\alpha < n - 1$. Indeed, minimizers may exists, if the charge Q is not too large, when $\alpha \geq n - 1$.

Existence of minimizers

Existence can be proved under additional regularity assumptions. Given $\delta > 0$, we let $\mathcal{K}_\delta$ be the class of sets satisfying the δ-ball condition.

Theorem (Goldman,N.,Ruffini 2015)

There exist $0 < Q_0 \leq Q_1$ such that the minimum problem

$$\min_{E \in \mathcal{K}_\delta :\, |E|=m} \mathcal{F}_Q(E)$$

has a solution for $Q \leq Q_1 \delta^3/\sqrt{m}$. Moreover, if $Q \leq Q_0 \delta^3/\sqrt{m}$, then the ball is the unique minimizer.

We don't know if $Q_0 = Q_1$, that is, if the ball is the unique possible minimizer.

Existence of minimizers

Existence can be also proved under a convexity assumption.

Theorem (Goldman,N.,Ruffini 2016)

The minimum problem

$$\min_{E \text{ convex}, |E|=m} \mathcal{F}_{Q,\alpha}(E)$$

has a solution for all $\alpha < n$ *and* $Q \geq 0$.
Moreover, if $\alpha = 0$ *and* $n = 2$ *the minimizers are of class* $C^{1,1}$.

Here we set

$$\mathcal{I}_0(E) = \inf_{\mu:\, \mu(E)=1} \int_E \int_E \log\left(\frac{1}{|x-y|}\right) d\mu(x)\, d\mu(y).$$

Existence of minimizers

A alternative regularization, leading to existence of minimizers for small charges, consists in requiring that the measure μ is absolutely continuous w.r.t. the Lebesgue measure, that is, $\mu = \rho dx$, and adding to the energy an extra term proportional to

$$\int_E \rho^2 \, dx.$$

Such a term is related to the dielectric response of the fluid (Muratov, N. 2016).

Flat droplets

When the droplet is very flat, and the charge is proportionally small, we can study the behavior of the functional $\mathcal{F}_Q$ in dimension $n = 2$. In this case, existence can be proved for sufficiently small charges.

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Theorem (Muratov,N.,Ruffini 2016)

Let $n = 2$. For any $Q > 0$, the ball of radius $Q/2$ is the unique (unconstrained) minimizer of $\mathcal{F}_Q$.
For $Q \leq 2\sqrt{m/\pi}$ the ball is the unique minimizer of the problem

$$\min_{E:\,|E|=m} \mathcal{F}_Q(E).$$

For $Q > 2\sqrt{m/\pi}$ there are no minimizers.

Notice that, in this case, we have a complete characterization of minimizers.